Journal of
Marine Science and Technology

Volume 31 | Issue 2 Article 9

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Recommended Citation
Tseng, Wei-Kuo; Tsai, King-Cheng; Liou, Chian; Guo, Jiunn-Liang; and Sun, Yi-Jia (2023) "Algorithms for the Generalized Inverse Solution and Direct Solution: Using an Algebra Computer-Based System to Obtain Meridian Arc Length," *Journal of Marine Science and Technology*: Vol. 31: Iss. 2, Article 9.
DOI: 10.51400/2709-6998.2695
Available at: https://jmstt.ntou.edu.tw/journal/vol31/iss2/9

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RESEARCH ARTICLE

Algorithms for the Generalized Inverse Solution and Direct Solution: Using an Algebra Computer-based System to Obtain Meridian Arc Length

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Abstract

In this study, the idea employed in the article of [1] to treat the problem of the arc-length of meridian is applied to the arc-length determined by the third flattening (*n*) instead of the eccentricity (*e*) of an Earth-reference ellipsoid. With WGS 84 ellipsoid datum used in modern navigation, the meridian arc-length is a crucial factor for map projections and geodetic and distance calculations such as Mercator charts, the universal transverse Mercator, great circle (ellipse), rhumb line, and geodesic. Because navigation software lacks officially standardized calculation methods, these "black box solutions" used in navigational systems are unknown. Therefore members of the general public must be provided with logical and simpler formulas. Because the previous general formulas of meridian arc-length are unnecessarily complicated and difficult to understand, this paper deduces a new general formula of meridian arc-length by using the binomial theorem and general terms of sinusoidal even power. The general formula presented herein is suitable for computer algorithm programming and other common navigational uses.

Keywords: Meridian, Rhumb line, Great circle (ellipse), Geodesic, Rectifying latitude

1. Introduction

IMO [2] ANNEX 24, RESOLUTION MSC.232 (82) *"ADOPTION OF THE REVISED PERFORMANCE STANDARDS FOR ELECTRONIC CHART DISPLAY AND INFORMATION SYSTEMS (ECDIS)"* article 12.3 stipulates that the system should be capable of performing and presenting the results of at least the following calculations:

1. True distance and azimuth between two geographical positions;
2. Geographic position from a known position and distance/azimuth; and
3. Geodetic calculations such as spheroidal distance, rhumb line, and great circle.

The geodetic calculations of the inverse problem and the direct problem of the meridian arc-length are crucial considerations for voyage plans, such as Mercator charts, great circle (ellipse), rhumb line, normal section, and geodesic. Because geodetic calculations encounter problems of advanced mathematics, in general, navigational education curricula always omit this topic and replace it with that related to a simplified perfect spherical Earth model. With improvement of computing power and user-friendly environments, basic geodetic calculation should not

Received 16 January 2023; revised 15 April 2023; accepted 6 June 2023.
Available online 30 June 2023

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https://doi.org/10.51400/2709-6998.2695
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be ignored. This paper mainly introduces the inverse generalized series of a meridian arc-length by third flattening (n), and then uses a computer algebra system (CAS) to calculate two series of the direct problem and inverse problems. It is hoped that this paper provides straightforward method for such calculations and that it can gives readers an insight into mathematics to help them understand geodetic and navigational algorithms.

The meridian (green) is the special curve of the intersection (section, orange) of a plane (Fig. 1) and an Earth-reference ellipsoid. The plane passes through the two poles of the Earth (Fig. 2) and is perpendicular to the equator. Both meridian and section are ellipses on an ellipsoid [3]. Longitude can be determined according to the angle from the Greenwich meridian and the local meridian. Analyzing the intersection of a plane and ellipsoid is common in interdisciplinary research. These intersections are crucial equations for navigation, geodesy, satellite orbits in space, a wide range of elliptical motions (e.g., planetary motions), and the curvature of surfaces, and those concerning eye-related radiotherapy (e.g., such as the anterior surface of the cornea, which is usually represented in the mathematical form of an ellipsoid) [4].

We developed an algorithm for the general formula of meridian arc-length and used a CAS to obtain the inverse solution of the meridian arc-length using the algorithm of symbolic computation for reversion formula.

In other aspects [5], research focuses on improving the Sumner method's issues in ex-meridian and meridian sights using adaptive boundaries technique. On the other hand [6], research presents an algorithm for extracting meteorological information for ship routing, which could help enhance safety and efficiency in maritime transportation. These achievements provide significant contributions to the safety of ships sailing on the sea.

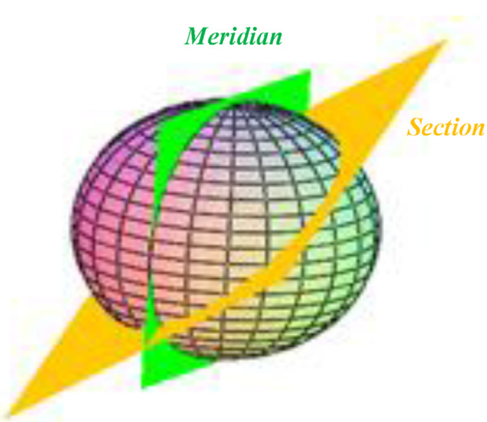

Fig. 1. The meridian (green) is the special intersection (section, orange) of a plane.

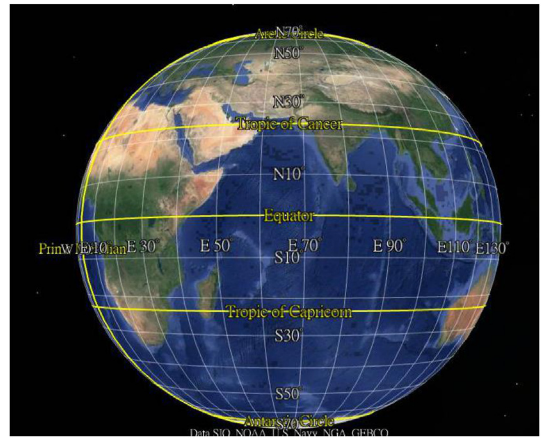

*Fig. 2. Longitude determined by the angle from the Greenwich meridian to the local meridian (Snapshot from Google **Earth**).*

2. Geodetic calculation of meridian

In this section, the meridian arc-length formulas provided by others are briefly summarized. When the radius of curvature in the prime vertical section is known, the Cartesian coordinates of an ellipsoid with regard to geographic (geodetic) latitude can be determined as follows cited from [7]:

$$P(\phi,\lambda)=N(\phi)\left[\cos\phi\cos\lambda,\cos\phi\sin\lambda,\left(1-e^2\right)\sin\phi\right] \tag{1}$$

where the radius of curvature in the prime vertical is $N(\phi) = a/(1-e^2\sin^2\phi)^{1/2}$.

From Fig. 3, Cayley's parametric equations, the Cartesian coordinates of an ellipsoid can be easily deciphered with regard to reduced (or parametric) latitude, using the following formula:

$$P(\beta,\lambda)=[a\cos\beta\cos\lambda,a\cos\beta\sin\lambda,b\sin\beta] \tag{2}$$

where ϕ is geodetic latitude, β is reduced latitude, a is major radius, b is minor radius, and e is eccentricity (Fig. 3).

Assuming that the Earth is an ellipsoid, the well-known formula of meridian arc-length is derived from the integral of meridian curvature from the equator to reduced latitude and expressed using the following formula (Tseng et al., 2014, Rapp 1993):

$$s(\beta)==\int_0^x\sqrt{|\partial P(\beta)/\partial\beta|}d\beta=\int_0^x\sqrt{a^2\sin^2\beta+b^2\cos^2\beta}d\beta \tag{3}$$

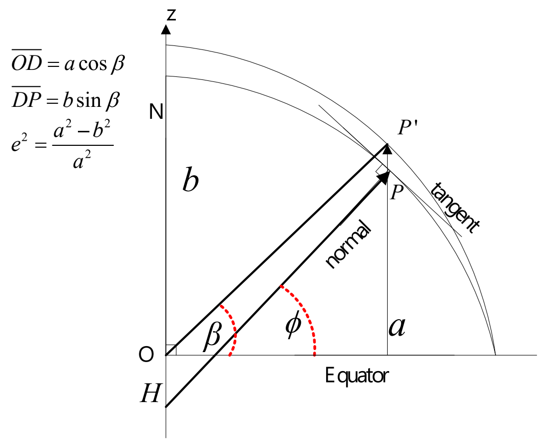

Fig. 3. Reduced (β) and geodetic (ϕ) latitudes of point P.

The integral in Eq. (3) is an incomplete elliptic integral of the second kind and it cannot be expressed in a closed form in terms of basic calculus. This formula is also the most straightforward possible form for the meridian arc-length and the section of an ellipsoid and a plane.

Because geodetic latitude is the most commonly used latitude in practical applications, considering the relationship between geodetic latitude (ϕ) and reduced (β) latitude [Eq. (4)] [8] provides the arc-length integral with respect to geodetic latitude (ϕ) from the equator as shown in Eq. (5) [9].

$$\beta = \tan^{-1}\left(\left(1-e^2\right)^{1/2}\tan\varphi\right), \tag{4}$$

$$S(\varphi) = \int_0^{\varphi} \frac{a(1-e^2)}{\left(1-e^2\sin^2\varphi\right)^{3/2}} d\varphi. \tag{5}$$

As Eq. (5) originates from an elliptic integral, it also cannot be evaluated in a closed form. The calculation can be performed by using the binomial expansion of the denominator, which yields Eq. (6),

$$S(\varphi) = a\left(1-e^2\right)\int_0^{\varphi}\left(1+\frac{3}{2}e^2\sin^2\varphi - \frac{15}{8}e^4\sin^4\varphi + \frac{35}{16}e^6\sin^6\varphi \ldots\right)d\varphi. \tag{6}$$

Then [10], integrates Eq. (6) by rearrangement with trigonometric identities to provide the infinite series of the meridian arc-length in Eq. (7), which is the distance from the equator to an arbitrary geodetic latitude (ϕ). Because the powers of e are small, Eq. (7) is a rapidly converging series depending on the accuracy required. Truncating the first few terms can maintain appropriate accuracy for practical use [11].

$$S(\phi) = a\left(1-e^2\right)\begin{bmatrix} \left(1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6\right)\phi + \\ \left(-\frac{3}{8}e^2-\frac{15}{32}e^4-\frac{525}{1024}e^6\right)\sin(2\phi) + \\ \left(\frac{15}{256}e^4+\frac{105}{1024}e^6\right)\sin(4\phi) + O\left(e^8\right) \end{bmatrix} \tag{7}$$

Eq. (7) is presented in numerous textbooks and published papers [8,12–15].

2.1. BESSLEL's formula

[16] provided a meridian arc-length formula by introducing a new quantity n as the third flattening of the Earth-reference ellipsoid. Thus Eq. (5) can be rewritten as Eq. (8) [17]:

$$S(\varphi) = \int_0^{\varphi} \frac{a(1-n)^2(1+n)}{\left(1+2n\cos 2\varphi + n^2\right)^{3/2}} d\varphi \tag{8}$$

Where $n = \frac{1-(1-e^2)^{1/2}}{1+(1-e^2)^{1/2}} \cong \frac{e^2}{4}$ and $e^2 = \frac{4n}{(1+n)^2}$.

Further integrating Eq. (9) provides the following formula:

$$S(\phi) = a(1-n)^2(1+n)\begin{bmatrix} \left(1+\frac{9}{4}n^2+\frac{255}{64}n^4+\cdots\right)\phi \\ -\frac{3}{2}\left(n+\frac{15}{8}n^3+\frac{175}{64}n^5\right)\sin(2\phi) \\ +\frac{15}{16}\left(n^2+\frac{7}{4}n^4\right)\sin(4\phi) \\ -\frac{35}{48}\left(n^3+\frac{27}{16}n^5\right)\sin(6\phi)+O\left(e^8\right) \end{bmatrix} \tag{9}$$

Because the value of n is approximately one quarter of the square of e, Eq. (9) has better convergent quality than Eq. (7) and reduces the number of terms with almost equal accuracy.

2.2. HELMERTS's formula

[18] derived a meridian arc-length formula by using the equivalent value of $(1-n)^2(1+n) = (1-n)(1-n^2)$ Bessel's formula, which can be summarized as

$$S(\varphi)=\frac{a}{1+n}\cdot\left[\left(1+\frac{1}{4}n^2+\frac{1}{64}n^4\right)\varphi-\frac{3}{2}\left(n-\frac{1}{8}n^3\right)\sin2\varphi+\frac{15}{16}\left(n^2-\frac{1}{4}n^4\right)\sin4\varphi-\frac{35}{48}n^3\sin6\varphi+\frac{315}{512}n^4\sin8\varphi\ldots\right] \tag{10}$$

This formula derived by Helmert is straightforward and concise and it can be considered an alternative to Eq. (7).

3. General formulas by KAZUSHIGE KAWASE

The aforementioned works present formulas of meridian arc-length with nongeneral forms, these formulas cause the formula patterns to not be truly represented and can be deemed too complicated for those who unfamiliar. Users other than those specialized in a given field have a high likelihood of making mistakes when inputting the terms in computer algorithms, and when such terms are input incorrectly, it can result in substantial errors in navigational systems. The aforementioned reasons indicates that the general formula for meridian arc-length must be derived.

[13] presented the general formulas to calculate the meridian arc-length from the equator to an arbitrary geodetic latitude. The formulas are derived by the generalized Helmert's formula [18] and are shown as Eqs. (11) and (12):

$$S(\varphi)=\frac{a}{1+n}\sum_{j=1}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{3n}{2k}-n\right)\right\}^2\left[\varphi+\sum_{l=1}^{2j}\left(\frac{1}{l}-4l\right)\sin2l\varphi\prod_{m=1}^{l}\left\{\frac{3n}{2j+2(-1)^m\lfloor m/2\rfloor}-n\right\}^{(-1)^m}\right] \tag{11}$$

or

$$S(\varphi)=a(1-n)^2(1+n)\sum_{j=1}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{-n}{2k}-n\right)\right\}^2\left[\varphi+\sum_{l=1}^{2j}\frac{\sin2l\varphi}{l}\prod_{m=1}^{l}\left\{\frac{-n}{2j+2(-1)^m\lfloor m/2\rfloor}-n\right\}^{(-1)^m}\right] \tag{12}$$

where $[x]$ is the floor function.

Understanding the valuable study of Kawase requires insightful familiarity with differential geometry. This paper sets forth an alternative algorithm instead of Kawase's derivations and formulas. The method provided herein is easier to understand than that in previous work such as that of Kawase.

The formulas provided by [13] are sophisticated, because they involve floor functions, Π functions, and n definitions from Bessel's formula. Those unfamiliar with such formulas may require more effort to comprehend such formulas and apply them for practical purposes.

In the following sections, this paper derives a new general formula that is straightforward, direct, and highly suited for computer programming. The novel general formula has greater accuracy and precision than those of Kawase and Delambre, and can be used directly in navigational and geodetic calculations.

4. New generalized reverse solution

Eq. (5) can be rewritten as follows [13,16]:

$$dS=a(1-n)^2(1+n)\left(1+2n\cos(2\phi)+n^2\right)^{-3/2}d\phi. \tag{13}$$

Substituting Euler's identities into Eq. (13) gives

$$dS=a(1-n)^2(1+n)\left(1+n\cdot e^{2\phi\cdot i}\right)^{\frac{-3}{2}}\left(1+n\cdot e^{-2\phi\cdot i}\right)^{\frac{-3}{2}}d\phi, \tag{14}$$

where,

$$2\cos(2\phi)=e^{2\phi i}+e^{-2\phi i}, \tag{15}$$

and

$$\left(1+n\cdot e^{\pm2\phi\cdot i}\right)^{\frac{-3}{2}}=\sum_{j=0}^{\infty}C_j^{-3/2}n^j e^{\pm2j\phi i}, \tag{16}$$

The product of the above two series in Eq. (16) will be the series in terms of $e^{2j\phi\cdot i}+e^{-2j\cdot\phi\cdot i}=2\cos(2j\cdot\phi)$. Using the product of Eq. (14), simplification and integration provide the generalized series

$$S(\phi)=a(1-n)^2(1+n)\left[\sum_{k=0}^{\infty}\left(C_k^{-3/2}n^k\right)^2\phi+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}j^{-1}C_k^{-3/2}C_{k+j}^{-3/2}n^{2k+j}\sin(2j\phi)\right]. \tag{17}$$

Finally, using the coefficients of geodetic latitude and its even multiple sine functions gives the new general formula of the meridian arc-length from the equator to arbitrary geodetic latitude:

$$S(\phi) = a(1-n)^2(1+n)\left[\mathbf{M}_0 \cdot \phi + \sum_{j=1}^{\infty} \mathbf{M}_j \sin(2j\phi)\right], \tag{18}$$

where $\mathbf{M}_0 = \sum_{k=0}^{\infty}\left(C_k^{-3/2} n^k\right)^2$ and $\mathbf{M}_j = \sum_{k=0}^{\infty} j^{-1} C_k^{-3/2} C_{k+j}^{-3/2} n^{2k+j}$.

*in the above equations ($\mathbf{M}_0$ and $\mathbf{M}_j$), the binomial series for half-integer multiples is defined by: $\begin{pmatrix} -\frac{3}{2} \\ k \end{pmatrix} = \frac{\frac{-3}{2}\cdot\frac{-5}{2}\cdots\left(\frac{-3}{2}-k+1\right)}{k\cdot(k-1)\cdot\cdots\cdot 1}$, where k is the truncated term.

To represent the calculation results in matrices, the expansion of eq. (18) yields

$$S(\phi) = a\left(1-e^2\right)(1+n)(\mathbf{M}_0\phi + \mathbf{M}_1 \sin(2\phi) + \cdots + \mathbf{M}_5 \sin(10\phi)) + O\left(n^{12}\right) = (\mathbf{S}_1 \cdot \mathbf{M} \cdot \mathbf{N}) + O\left(n^{12}\right), \tag{19}$$

where $\mathbf{S}_1 = [\phi \quad \sin(2\phi) \quad \sin(4\phi) \quad \sin(6\phi) \quad \sin(8\phi)\sin(10\phi)]$ and $\mathbf{N} = \begin{bmatrix} 1 & n^2 & n^4 & n^6 & n^8 & n^{10} \end{bmatrix}$.

The coefficients of Eq. (19) truncated at order n^{10} and $\mathbf{M}_5$ can be computed using symbolic computation system (such as Maple). The results of this derivation are listed in Eq. (20). The results up to terms of $\mathbf{M}_5$ are more accurate than those given by others; this confirms the new general formula to be more accurate and reliable. If higher accuracy is required, such as for precise geodetic, mapping and navigational calculations, the amount of coefficients required is truncated. The more truncated terms are used in Eq. (19), the more accurate the meridian arc-length is

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{9}{4} & \frac{225}{64} & \frac{1225}{256} & \frac{99225}{16384} & \frac{480249}{65536} \\ 0 & -\frac{3}{2} & -\frac{45}{16} & -\frac{525}{128} & -\frac{11025}{2048} & -\frac{218295}{32768} \\ 0 & \frac{15}{16} & \frac{105}{64} & \frac{4725}{2048} & \frac{24255}{8192} & \frac{945945}{262144} \\ 0 & 0 & -\frac{35}{48} & -\frac{315}{256} & -\frac{3456}{2048} & -\frac{35035}{16384} \\ 0 & 0 & \frac{315}{512} & \frac{2079}{2048} & \frac{45045}{32768} & \frac{255255}{131072} \\ 0 & 0 & 0 & -\frac{693}{1280} & -\frac{9009}{10240} & -\frac{19305}{16384} \end{bmatrix}. \tag{20}$$

5. Geodetic latitude from inverse series of the new general formula

Substituting the latitude $\pi/2$ into Eq. (18) gives the quadrant distance from the equator to the Pole as

$$S(\pi/2) = a(1-n)^2(1+n)\mathbf{M}_0 \cdot \pi \big/ 2. \tag{21}$$

The spherical radius of the equivalent circumference is

$$R = \frac{S(\pi/2)}{\pi/2} = a(1-n)^2(1+n)\mathbf{M}_0. \tag{22}$$

The rectifying latitude [9,15], which provides a sphere with a correct distance along the meridian, can be given by:

$$\mu = S(\phi)/R \tag{23}$$

An expression of the rectifying latitude μ in terms of geodetic latitude can be obtained from dividing Eq. (18) by Eq. (21) as:

$$\mu = \phi + \sum_{i=1}^{\infty} \frac{\mathbf{M}_i}{\mathbf{M}_0} \sin(2i\phi). \tag{24}$$

Transforming Eq. (23) gives the following form:

$$\phi = \mu - f(\mu). \tag{25}$$

Lagrange reversion theorem provided series or formal power series expansions of certain implicitly defined functions [9]. The Lagrange reversion the-

Table 1. Algorithm of symbolic computation for the reversion formula of geodetic latitude in terms of meridian arc-length.

Initial setting	*f1 = Series(f(μ),n,U),* Lat = μ+f1,ff = f1,i = 1
Do while	*i ≤ **U**; truncated N terms for Eq. (26)* *ff = ff*f1,* *ff = Series(ff,n,* **U**) *df = diff(ff,μ,i)* *Lat = Lat + df/(i+1)!*
Loop	*Lat = Combine(Lat,trig)*

Note: For general CASs, the functions listed in this table have the following definitions (cited and modified from Maple Help):
* The ***series(f,u = a,* U)** function computes a truncated series expansion of *f*, with respect to the variable ***u***, around the point ***a***, up to order **U**. If *N* is infinity then an asymptotic expansion is given.
* The ***diff(f,μ,i)*** function computes the ***i***-order derivative of *f* with respect to ***μ***.
* The ***combine(f,trig)*** function applies a repeated application of the transformations which ***f's*** products and powers of ***trigonometric*** terms involving sine and cosine are combined into a sum of the trigonometric term.

Table 2. Errors between latitude and reverted latitude, and errors between arc-tween arc-length and reverted arc-length.

Lat	Length	Recovered Lat	Recovered Len.	Error (μm)	Error:Deg
0	0	0.00000000000000	0	0.00	0.00E+00
5	552885.4511	4.99999999998665	552885.4511	1.48	1.34E-11
10	1105854.833	9.99999999998695	1105854.833	1.44	1.31E-11
15	1658989.589	15.00000000000380	1658989.589	−0.42	−3.80E-12
20	2212366.254	20.00000000002650	2212366.254	−2.93	−2.65E-11
25	2766054.169	25.00000000003780	2766054.169	−4.18	−3.78E-11
30	3320113.398	30.00000000002650	3320113.398	−2.94	−2.65E-11
35	3874592.902	34.99999999999620	3874592.902	0.42	3.77E-12
40	4429529.03	39.99999999996410	4429529.03	3.99	3.59E-11
45	4984944.378	44.99999999994970	4984944.378	5.59	5.03E-11
50	5540847.042	49.99999999996210	5540847.042	4.21	3.79E-11
55	6097230.313	54.99999999999370	6097230.313	0.70	6.25E-12
60	6654072.819	60.00000000002500	6654072.819	−2.79	−2.50E-11
65	7211339.117	65.00000000003790	7211339.117	−4.23	−3.79E-11
70	7768980.728	70.00000000002750	7768980.728	−3.07	−2.75E-11
75	8326937.587	75.00000000000470	8326937.587	−0.52	−4.69E-12
80	8885139.872	79.99999999998710	8885139.872	1.44	1.29E-11
85	9443510.141	84.99999999998640	9443510.141	1.51	1.36E-11
90	10001965.73	90.00000000000000	10001965.73	0.00	0.00E+00

orem can be used to expand any function in Eq. (24) into the following form:

$$\phi = \mu + \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\left(\frac{\partial}{\partial\mu}\right)^k f(\mu)^{k+1}, \tag{26}$$

where

$$f(\mu) = -\sum_{i=1}^{\infty} \frac{\mathbf{M}_i}{\mathbf{M}_0}\sin(2i\mu), \tag{27}$$

Using special arrangements with algebra computer systems expands the symbolic expression of Eq. (25) which can avoid high-order terms and the requirement for substantial memory in computational procedures. The pseudocode is shown in Table 1:

Implementing the algorithm in Table 1 and arranging the result can provide the formula of geodetic latitude from the equator to arbitrary arc-length (rectifying latitude) in the following form:

$$\phi = \mu + \sum_{k=1}^{\infty} \mathbf{U}_k \sin(2k\mu). \tag{28}$$

Expanding Eq. (28) yields Eq. (29):

$$\begin{aligned}\phi &= \mu + \mathbf{U}_1\sin(2\mu) + \cdots + \mathbf{U}_{10}\sin(20\mu) + O(n^{12}) \\ &= \mathbf{L}\cdot\mathbf{U}\cdot\mathbf{N} + O(n^{12})\end{aligned} \tag{29}$$

where $\mathbf{L} = [\sin(2\mu) \quad \sin(4\mu) \quad \ldots \quad \sin(18\mu) \quad \sin(20\mu)]$ and $\mathbf{N} = [n \quad n^2 \quad \ldots \quad n^9 \quad n^{10}]$.

With the CAS, the coefficients for Eq. (28) truncated at order n^{10} and $\mathbf{U}_{10}$ ($\mathbf{U} = 10$) can be computed as Eq. (30) in matrix form. The results show that the expansions of up to $\mathbf{U}_8$ terms are more accurate than those given by others, and higher terms can also be obtained according to user's requirements. The formula given in this paper is more accurate and reliable than others.

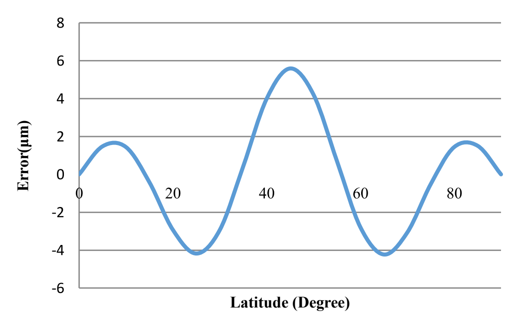

Fig. 4. Distance error deviations with regard to Latitude.

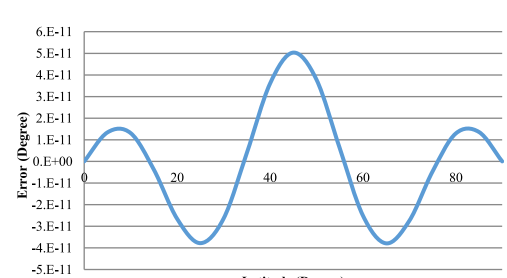

Fig. 5. Latitude error deviations with regard to rectifying latitude.

$$\mathbf{U}_{10\times10}=\begin{bmatrix}\frac{3}{2} & 0 & -\frac{27}{32} & 0 & \frac{269}{512} & 0 & -\frac{6607}{24576} & 0 & \frac{40941}{327680} & 0\\ 0 & \frac{21}{16} & 0 & -\frac{55}{32} & 0 & \frac{6759}{4096} & 0 & -\frac{155113}{122880} & 0 & \frac{39591143}{47185920}\\ 0 & 0 & \frac{151}{96} & 0 & -\frac{417}{128} & 0 & \frac{87963}{20480} & 0 & -\frac{572057}{131072} & 0\\ 0 & 0 & 0 & \frac{1097}{512} & 0 & -\frac{15543}{2560} & 0 & \frac{2514467}{245760} & 0 & -\frac{33432797}{2580480}\\ 0 & 0 & 0 & 0 & \frac{8011}{2560} & 0 & -\frac{69119}{6144} & 0 & \frac{1515771}{65536} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{293393}{61440} & 0 & -\frac{5962461}{286720} & 0 & \frac{463409979}{9175040}\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6459601}{860160} & 0 & -\frac{1258281}{32768} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{332287993}{27525120} & 0 & -\frac{8778422179}{123863040}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{116391263}{5898240} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32385167569}{990904320}\end{bmatrix}. \quad (30)$$

6. Error analysis

In this section, validations of Eq. (18) and Eq. (27) are conducted. By calculating the distances of 5° intervals from the equator with up to n^7, sin (14ϕ) of Eq. (18), and then by recovering the latitude with Eq. (27) up to n^7, sin (14μ) (shown in Table 2), the lost accuracies can be revealed (shown in the second last column). The length discrepancies are calculated by considering in the differences of the original arc-length obtained by using Eq. (18) and the lengths from the recovered latitudes (the latitude errors are also shown in the last column). Because the errors are negligible, the new general formula and reverse formula are of practical value.

The maximum distance error occurs at an approximately latitude of 45° with a value of 5.59 μm, whereas the maximum degree error occurs approximately at 45° latitude with a value of 5.03E-11°. Be-cause the characteristics of the trigonometric series are varied periodically with the geodetic and rectifying latitudes, the non-truncated terms also affect these alterations. As shown in Fig. 4 and Fig. 5, the errors varied periodically.

7. Conclusion

In this paper, a novel general formula for meridian arc-length and an algorithm for calculating latitude formulas for a given arc-length were presented. Compared with previous general formulas, the novel formulas presented herein is much simpler and more direct. Such formulas are applicable for creating computer programs for navigation systems and other fields. The more straightforward expressions and new general formulas provided herein can reduce input errors in practical applications and provide a series with high accuracy. However, the new general formulas proposed in this paper are only applicable to specific ellipsoid models, such as WGS84, GRS80, etc. If different ellipsoid models are used, corresponding formulas need to be used for the calculation. Nevertheless, the algorithms presented in this paper still represent an important advancement. The formulas can also provide alternative methods for determining whether the calculations in a navigational black box are in conformance with the standards of accuracy required for its use.

Conflict of interest

This research does not have any conflict of interest.

Acknowledgments

This work was supported in part by the Ministry of Science and Technology of Taiwan, Republic of

China under grant MOST 106-2410-H-019-017. This manuscript was edited with the assistance of Wallace Academic Editing.

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