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Chein-Shan Liu
Center of Excellence for Ocean Engineering, Center of Excellence for the Oceans, National Taiwan Ocean University, Keelung 20231, Taiwan

Jian-Hung Shen
Department of Marine Engineering, National Taiwan Ocean University, Keelung 202301, Taiwan, 20666006@email.ntou.edu.tw

Yung-Wei Chen
Department of Marine Engineering, National Taiwan Ocean University, Keelung 202301, Taiwan, cyw0710@mail.ntou.edu.tw

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RESEARCH ARTICLE

Numerical and Approximate Analytic Solutions of Second-order Nonlinear Boundary Value Problems

Chein-Shan Liu [a], Jian-Hung Shen [b], Yung-Wei Chen [b,*]

[a] Center of Excellence for Ocean Engineering, Center of Excellence for the Oceans, National Taiwan Ocean University, Keelung, 20231, Taiwan
[b] Department of Marine Engineering, National Taiwan Ocean University, Keelung, 202301, Taiwan

Abstract

The shooting method consists of guessing unknown initial values, transforming a second-order nonlinear boundary value problem (BVP) to an initial value problem and integrating it to obtain the values at the right end to match the specified boundary condition, which acts as a target equation. In the shooting method, the key issue is accurately solving the target equation to obtain highly precise initial values. Due to the implicit and nonlinear property, we develop a generalized derivative-free Newton method (GDFNM) to solve the target equation, which offers very accurate initial values. Numerical examples are examined to show that the shooting method together with the GDFNM can generate a very accurate solution. Additionally, the GDFNM can successfully solve the three-point nonlinear BVPs with high accuracy. A new splitting-linearizing method is developed to express the approximate analytic solutions of nonlinear BVPs in terms of elementary functions, which adopts the Lyapunov technique by inserting a dummy parameter into the governing equation and the power series solution. Then, linearized differential equations are sequentially solved to derive the analytic solution.

Keywords: Nonlinear boundary value problems, Bratu problem, Shooting method, Generalized derivative-free Newton method, Splitting-linearizing method, Lyapunov technique

1. Introduction

For numerical solution of a boundary value problem (BVP), it is considered to be precise when it satisfies the boundary conditions precisely. Many computational methods have been developed to solve BVPs [1–7]. The singularly perturbed problem always exhibits a boundary layer, which is a narrow region where the solution varies rapidly, and the numerical methods to overcome this difficulty can be found in [8–15]. The present paper develops a powerful numerical solver with a generalized derivative-free Newton method to solve the target equation, even when a singularity appears in the boundary layer.

For ordinary differential equations (ODEs), the group-preserving scheme (GPS) was developed by Liu [16] for the solutions of initial value problems (IVPs). Recently, Liu [17] developed a more powerful GPS to solve IVPs. Liu [8,18,19] extended and modified the GPS for ODEs to obtain a Lie-group shooting method (LGSM) for solving the second-order BVPs based on the proper orthochronous Lorentz group. According to the LGSM, the two-point solution of nonlinear dynamic systems can be derived and applied to determine the initial value and heat source in the nonlinear backwards-in-time partial differential equations (PDEs) [20–22]. However, solving the one-dimensional nonlinear ODEs is more challenging than solving the PDEs when considering only two- or three-point boundary conditions. If the boundary conditions (BCs) are insufficient, the solver will have to deal with a multi-solution situation.

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* Corresponding author. Department of Marine Engineering, National Taiwan Ocean University, Keelung, Taiwan.
E-mail address: cyw0710@mail.ntou.edu.tw (Y.-W. Chen).

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For the numerical solutions of BVPs, Liu [18] introduced a one-step GPS by utilizing the closure property of the Lie-group. It is called the Lie-group shooting method (LGSM). Next, Liu [23] solved an inverse Sturm-Liouville problem by using the LGSM, and Liu [24,25] solved the Sturm-Liouville problem and the generalized Sturm-Liouville problem by using the LGSM to determine the eigenvalues and eigenfunctions. Then Liu [13] developed the LGSM for solving nonlinear singularly perturbed boundary value problems. Recently, Hajiketabi and Abbasbandy [26] developed a simple, efficient and accurate LGSM for solving nonlinear boundary value problems. In addition, Liu et al. [27] developed three novel fifth-order iterative schemes for solving nonlinear equations. Then, Lin et al. [28] used boundary shape function methods (BSFM) to solve nonlinear third-order three-point BVPs. Next, Liu and Chang [29] modified the LGSM and combined it with the BSFM to solve nonlinear BVPs with Robin boundary conditions.

Liu [30] developed an $SL(2, R)$ shooting method to solve the generalized Sturm-Liouville problem. Moreover, Liu [31] developed an $SL(3, R)$ shooting method to solve the Falkner-Skan boundary layer equation. Both the LGSM and $SL(2, R)$ shooting method possess a great advantage in that they determine the missing initial values through the determination of a weight factor in a small and definite range of $r\in[0,1]$. However, the Lie-group shooting method was only applicable to the nonlinear BVP with simple Dirichlet or Neumann boundary conditions but not to the BVP equipped with the Robin boundary conditions. The $SL(2, R)$ shooting method needs to iteratively determine the missing initial values at each r and seek the best r by solving a target equation. As an extension, we develop a more powerful and simpler shooting method directly based on the ODEs themselves, instead of the Lie-groups $SL(n, R)$ and $SO_o(n, 1)$ [32] for solving the nonlinear BVPs. Our shooting method, which resorts to a generalized derivative-free Newton iterative method, is simpler than the previous works of Liu [13,24,25].

As stated in [33], Lyapunov developed a dummy parameter technique to investigate the conditions of stability of the Hill equation:

$$\ddot{y}(t)+p(t)y(t)=0,\ y(0)=1,\ \dot{y}(0)=0, \tag{1}$$

where $p(t+T)=p(t)$ for some $T>0$. Lyapunov recast Eq. (1) as

$$\ddot{y}(t)=\mu p(t)y(t), \tag{2}$$

where $\mu\in R$ is a dummy parameter. When $\mu=-1$, Eq. (2) recovers to Eq. (1). The solution of Eq. (1) can be determined as the sum of a convergent power series of the parameter μ:

$$y(t)=\sum_{k=0}^{\infty}\mu^k\varphi_k(t). \tag{3}$$

Substituting Eq. (3) into Eq. (1) and equating equal powers of μ yields

$$\ddot{\varphi}_0(t)=0,\ \ddot{\varphi}_k(t)=p(t)\varphi_{k-1}(t),\ k=1,2,..., \tag{4}$$

which is a recurrent formula to sequentially determine $\varphi_k(t)$ from the previous step solution $\varphi_{k-1}(t)$, by starting from $\varphi_0(t)=1$ and subject to $\varphi_k(0)=\dot{\varphi}_k(0)=0$. Lyapunov proved that

$$|\varphi_k(t)|\le\frac{M^k t^{2k}}{2k!},\ k=1,2,..., \tag{5}$$

where M is an upper bound of $p(t)$, and obtained the convergent solution of Eq. (1):

$$y(t)=\sum_{k=0}^{\infty}(-1)^k\varphi_k(t). \tag{6}$$

In the present paper, we will call the above method the Lyapunov technique.

There exists no study using the shooting method together with the generalized derivative-free Newton iterative method, which will be developed here, to solve nonlinear BVP. The remaining portions of the paper are arranged as follows. In Section 2, we introduce a target equation for the second-order nonlinear BVP. Motivated by the Newton method, a generalized derivative-free Newton iterative method (GDFNM) is developed in Section 3, and we assess its convergence by using the computed order of convergence (COC). Six numerical examples are tested in Section 4. To seek an approximate analytic solution of the second-order nonlinear BVP, we develop a splitting-linearizing method (SLM) in Section 5, where the Lyapunov technique is adopted and two examples are given. Section 6 presents the conclusions.

2. Target equation

We consider a second-order boundary value problem (BVP):

$$u''(x)=F(x,u(x),u'(x)),\ x\in(0,1), \tag{7}$$

$$a_1u(0)+b_1u'(0)=c_1, \tag{8}$$

$$a_2u(1)+b_2u'(1)=c_2, \tag{9}$$

where c_1 and c_2 are given constants, while a_1 and b_1 are not both zeros and a_2 and b_2 are not both zeros. The Lie-group shooting method developed by Liu [18] is not available for solving Eqs. (7)–(9).

The basic idea of the conventional shooting method is to transform Eqs. (7)–(9) into an initial value problem and solve the target equation Eq. (9). If $b_1 \neq 0$, we assume that

$$u(0)=A,\ u'(0)=\frac{c_1-a_1A}{b_1}, \tag{10}$$

where A is a constant to be determined. If $b_1 = 0$, $u(0) = c_1/a_1$ and $u'(0) = B$ constitute the initial conditions where B is to be determined.

Eq. (7) together with Eq. (10) is an initial value problem endowed with an unknown value of A to be determined. For each A, we can integrate Eq. (7) to obtain

$$f(A)=a_2u(1,\ A)+b_2u'(1,\ A)-c_2=0, \tag{11}$$

which is a target equation to be solved for A. The function of $f(A)$ with respect to A is a target curve. The integration of Eq. (7) will be carried out by the fourth-order Runge–Kutta method, whose accuracy is $(\Delta x)^4$ depending on the step size $\Delta x = 1/N$, where N is the number of integrating points.

3. A generalized derivative-free Newton iterative method

Eq. (11) is indeed an implicit and highly nonlinear equation of A. To reduce the computational burden, a generalized derivative-free Newton method (GDFNM) for solving a scalar equation $f(x) = 0$ is motivated by the Newton method:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\ n=0,\ 1,\ldots, \tag{12}$$

and we modify it below.

To eliminate the derivative term $f'(x_n)$ in Eq. (12), we consider

$$f'(x_n)=f'(x^*)+f''(x^*)(x_n-x^*)+\frac{1}{2}f'''(x^*)(x_n-x^*)^2+\cdots. \tag{13}$$

Neglecting the higher-order terms and inserting it into Eq. (12), we have

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x^*)+f''(x^*)(x_n-x^*)}. \tag{14}$$

However, we have

$$f(x_n)=f'(x^*)(x_n-x^*)+\frac{1}{2}f''(x^*)(x_n-x^*)^2+\cdots. \tag{15}$$

Neglecting the higher-order terms and replacing $x_n - x^*$ in Eq. (14) by $f(x^n)/f'(x^*)$, we can derive a derivative-free Newton method (DFNM):

$$x_{n+1}=x_n-\frac{f(x_n)}{a+bf(x_n)}, \tag{16}$$

where

$$a=f'(x^*),\ b=\frac{f''(x^*)}{f'(x^*)}. \tag{17}$$

Theorem 1. The iterative scheme (16) with the parameters a and b given by Eq. (17) for solving f (x) = 0 has second-order convergence. Furthermore, by taking

$$a=f'(x^*),\ b=\frac{f''(x^*)}{2f'(x^*)}, \tag{18}$$

the iterative scheme (16) has the third-order convergence.

Proof. For the proof of convergence, we let x^* be a simple solution of $f(x)= 0$, i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$. Thus, let

$$e_n=x_n-x^*, \tag{19}$$

be a small solution error. It follows that

$$e_{n+1}=e_n+x_{n+1}-x_n, \tag{20}$$

$$f(x_n)=f'(x^*)\left[e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots\right], \tag{21}$$

where

$$c_k:=\frac{f^{(k)}(x^*)}{k!f'(x^*)},\ k=2,\ 3,\ldots, \tag{22}$$

Inserting Eq. (21) into Eq. (16) yields

$$\begin{aligned}\frac{f(x_n)}{a+bf(x_n)}&=\frac{e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots}{1+be_n+bc_2e_n^2+bc_3e_n^3+\cdots}\\&=e_n+D_2e_n^2+D_3e_n^3+D_4e_n^4+\cdots,\end{aligned} \tag{23}$$

where we have used the first one in Eq. (17), and D_2, D_3 and D_4 are given by

$$\begin{aligned}&D_2=c_2-b,\ \ D_3=c_3-2c_2b+b^2,\\&D_4=2b^2c_2-bc_3+c_2\left(b^2-bc_2\right)-c_3b+c_4-b^3.\end{aligned} \tag{24}$$

Inserting Eq. (23) into Eq. (16) and using Eq. (20) yields

$$\begin{aligned} e_{n+1} &= e_n - e_n - D_2 e_n^2 - D_3 e_n^3 - D_4 e_n^4 - \cdots \\ &= -D_2 e_n^2 - D_3 e_n^3 - D_4 e_n^4 - \cdots \\ &= (b - c_2) e_n^2 + O(e_n^3). \end{aligned} \tag{25}$$

If we take $b = c_2$, i.e., b given by Eq. (18), e_{n+1} reduces to $O(e_n^3)$, which ends the proof of this theorem.

Then, we turn our attention to the determination of a and b in Eq. (17), whose values will influence the convergence speed. Similar to the half-interval method, the first step is choosing two initial guesses x_0 and x_2 such that $f(x_0) f(x_2) < 0$ to render $x^* \in (x_0, x_2)$. Then, we take $x_1 = (x_0 + x_2)/2$. As the approximations of a and b in Eq. (17) with generalization by a constant factor β, we can evaluate them by finite differences:

$$a = \frac{f(x_2) - f(x_0)}{x_2 - x_0},$$

$$\begin{aligned} b &= \frac{\beta}{a} \frac{f(x_2) - 2f(x_1) + f(x_0)}{(x_1 - x_0)^2} \\ &= \frac{4\beta f(x_2) - 8\beta f(x_1) + 4\beta f(x_0)}{(x_2 - x_0)\left[f(x_2) - f(x_0)\right]}. \end{aligned} \tag{26}$$

The resulting iterative algorithm is termed the generalized derivative-free Newton method (GDFNM).

The iterative algorithm with the GDFNM for solving $u(x)$ in Eqs. (7)–(9) are summarized as follows: (i) Given β, the initial guesses A_0 and A_2 are made to render $[a_2 u(1, A_0) + b_2 u'(1, A_0) - c_2][a_2 u(1, A_2) + b_2 u'(1, A_2) - c_2] < 0$ by inspecting the target curve, and give ϵ, and $\Delta x = 1/N$. (ii) Compute $A_1 = (A_0 + A_2)/2$, $u(1, A_1)$, and a and b by

$$a = \frac{u(1, A_2) - u(1, A_0)}{A_2 - A_0},$$

$$b = \frac{\beta}{a} \frac{u(1, A_2) - 2u(1, A_1) + u(1, A_0)}{(A_1 - A_0)^2}$$

(iii) Let $A^0 = A_0$ and for k = 0, 1, …, doing

$$A^{k+1} = A^k - \frac{a_2 u(1, A^k) + b_2 u'(1, A^k) - c_2}{a + b\left[a_2 u(1, A^k) + b_2 u'(1, A^k) - c_2\right]}$$

until $r_k = \left|a_2 u(1, A^k) + b_2 u'(1, A^k) - c_2\right| < \epsilon$,

where r_k is the residual to match the right boundary condition. Unless specified otherwise, we will take $\beta = 1$ for all computations.

In each iteration, an integration of Eq. (7) is required subject to the initial conditions:

$$u(0) = A^k, \; u'(0) = \frac{c_1 - a_1 A^k}{b_1},$$

to obtain the end values $u(1, A^k)$ and $u'(1, A^k)$, which is time saving if the number of iterations is small.

For the case with $b_1 = 0$, Eq. (8) is a Dirichlet boundary condition, and we can take $u'(0) = B$ and repeat the same process to determine B. To solve a scalar equation $f(x) = 0$, the numerically computed order of convergence (COC) is approximated by [28].

$$\text{COC} := \frac{\ln|(x_{n+1} - r)/(x_n - r)|}{\ln|(x_n - r)/(x_{n-1} - r)|}, \tag{27}$$

where r is a solution of $f(x) = 0$ and the sequence x_n is generated from an iterative scheme. In the computation of COC, we store the values of A^n where $n \le k_0 - 1$ and take $r = A^{k_0}$, where k_0 is the number of iterations for convergence.

4. Numerical examples

4.1. Example 1

$$u'' = \frac{3}{2} u^2, \tag{28}$$

$$2u(0) + u'(0) = 0, \; u(1) - 2u'(1) = 3. \tag{29}$$

An exact solution is

$$u(x) = \frac{4}{(1 + x)^2}. \tag{30}$$

By

$$u(0) = A, \; u'(0) = -2A, \tag{31}$$

where A is an unknown constant to match the target equation $u(1) - 2u'(1) - 3 = 0$. We apply the shooting method together with GDFNM to solve this problem with $N = 5000$ and $\epsilon = 10^{-14}$. The target curve shown in Fig. 1(a) possesses two intersection points to the zero line. For the first solution, we take $(A_0, A_2) = (3.9, 4.1)$, which is convergent with 9 iterations, and the numerical solution coincides with the exact solution in Eq. (30) with the maximum error (ME) $= 6.21 \times 10^{-15}$, as shown in Fig. 1(b). To test the stability and accuracy of the present algorithm, we consider the same setting and add random noise with a maximum level of 2.247×10^{-2} at the right condition in Eq. (29). The results show that ME $= 8.381 \times 10^{-3}$ and u

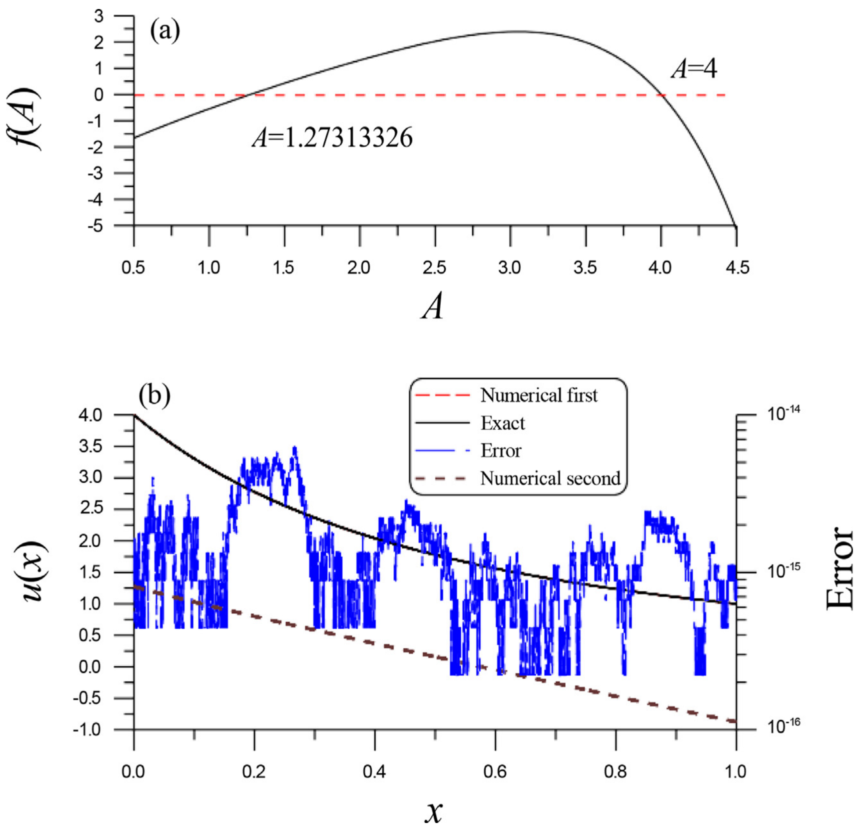

Fig. 1. For example, 1, (a) the target curve with two intersection points to the zero line, (b) comparing the first and second solutions obtained by the shooting method with DFNM and showing the error for the first solution.

$(0) = A = 3.9964917777$, and our algorithm has very good accuracy and stability even under random noise.

As shown in Table 1, the COC reveals that the GDFNM converges fast for $\beta = 0.5$, 1, 1.2; however, NI is not sensitive to β. When we take $(A_0, A_2) = (1.26, 1.3)$, we obtain the second solution as shown in Fig. 1(b) by the dashed line.

which is convergent with 5 iterations and obtains $A = 1.273133257721444$.

With the aid of the target curve, as shown in Fig. 1(a), it is easy to select A_0 and A_2 such that $A \in (A_0, A_2)$ is the solution of the target equation. Without inspecting the target curve, we may choose A_0 and A_2, which do not satisfy $[u\,(1, A_0) - 2u'\,(1, A_0) - 3]\,[u\,(1, A_2) - 2u'\,(1, A_2) - 3] < 0$; however, we find that the GDFNM is still applicable but with slower convergence. For example, when we take $(A_0, A_2) = (4.3, 4.4)$, it is convergent with 42 iterations to obtain the first solution with $ME = 6.22 \times 10^{-15}$. When we take $(A_0, A_2) = (2.1, 2.2)$, it is convergent with 27 iterations to obtain the second solution with $u\,(0) = A = 1.273133257721443$, which is very close to the above solution with an error of 10^{-15}. Table 2 with $(A_0, A_2) = (4.3, 4.4)$ lists the NI and COC.

4.2. Example 2

A reaction problem was studied by Finlayson [36], where an isothermal situation with an n-th order irreversible reaction leads to

$$u'' = Pe(u' + Ru^n), \tag{32}$$

$$Peu(0) - u'(0) = Pe,\ u'(1) = 0, \tag{33}$$

where $Pe = 1$, $R = 2$ and $n = 2$.

Table 1. For example, 1 with different values of β lists the number of iterations (NI) and COC.

β	0	0.2	0.5	1.0	1.2	1.3
NI	10	9	8	9	11	10
COC	0.27575	0.83985	0.99899	1.13599	1.20898	0.36327

Table 2. For example, 1 with different values of β and (A0, A2) = (4.3, 4.4), listing the NI and COC.

β	0	0.2	0.5	1.0	1.2	1.3
NI	41	41	41	42	42	42
COC	1.1537	1.1364	1.4682	1.0402	1.1537	1.2676

For $N = 5000$ and $\epsilon = 10^{-15}$ and $(A_0, A_2) = (0.63, 0.65)$, we obtain the first solution as shown in Fig. 2 by a solid line, which is convergent with 7 iterations with the error of the right boundary condition being 4.69×10^{-16}. COC = 0.96497 is computed. With $(A_0, A_2) = (-1, 0)$, we obtain the second solution as shown in Fig. 2 by the dashed line, which is convergent with 10 iterations with the error of the right boundary condition being 7.83×10^{-16}.

4.3. Example 3

We solve a nonlinear singular perturbation problem [31]:

$$\epsilon u'' + 2u' + e^u = 0, \tag{34}$$

$$u(0) = 0,\ u(1) = 0. \tag{35}$$

For the purpose of comparison, we write a uniform approximation provided by Bender and Orszag [37]:

$$u(x) = \ln\frac{2}{1+x} - e^{-2x/\epsilon}\ln 2. \tag{36}$$

However, we let the above $u(x)$ be an exact solution of the following BVP:

$$\epsilon u'' + 2u' + e^u = \frac{\epsilon}{(1+x)^2} + \frac{2^{\left[1-e^{-2x/\epsilon}\right]} - 2}{1+x}, \tag{37}$$

which is subjected to the Robin boundary conditions:

$$u(0) - u'(0) = 1 - \frac{2\ln 2}{\epsilon},\ 2u(1) + \epsilon u'(1) = -\frac{\epsilon}{2}. \tag{38}$$

In [38], using the boundary shape function method with 221 iterations, the authors find the numerical solution with ME = 1.993×10^{-4} for $\epsilon = 0.02$. Here, we take $\epsilon = 0.001$ for a highly singular case, and with the parameters $N = 5000$ and $\epsilon = 10^{-15}$ and $(A_0, A_2) = (-0.1, 0.1)$, we obtain the solution as shown in Fig. 3 by a solid line that is convergent with 7 iterations, as shown in Fig. 3(a). COC = 1.2721 reveals that the GDFNM converges fast. The numerical solution coincides with the exact solution in Eq. (36), and the numerical error is shown in Fig. 3(b) with ME = 1.35×10^{-4}. In Table 3, we tabulate the absolute errors at different x. 2.1E-4 means that 2.1×10^{-4}. Because we imposed the Robin boundary condition at the right end, the error is on the order of 10^{-4} at the singular point, which is within a strongly singular boundary layer, and after that, the error quickly tends to the orders of 10^{-14} and 10^{-15}.

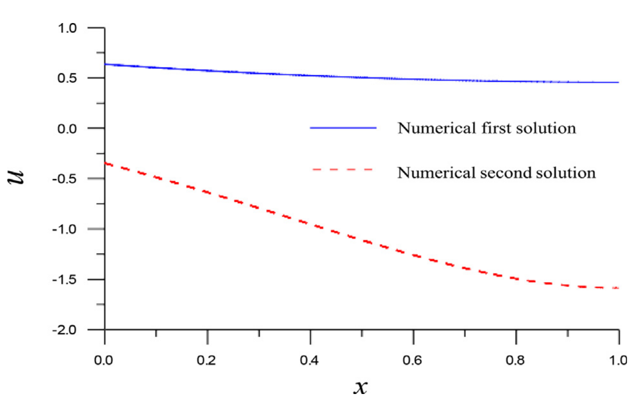

Fig. 2. For example, 2, compare the first and second solutions obtained by the shooting method with DFNM.

4.4. Example 4

Let us calculate the Bratu equation [39]:

$$u''(x) + \lambda e^{u(x)} = 0, \tag{39}$$

$$u(0) = 0,\ u(1) = 0, \tag{40}$$

which has an exact solution:

$$u(x) = -2\ln\left[\frac{\cosh\left(x - \frac{1}{2}\right)\frac{\theta}{2}}{\cosh\frac{\theta}{4}}\right], \tag{41}$$

where θ satisfies

$$\sqrt{2\lambda}\cosh\frac{\theta}{4} = \theta. \tag{42}$$

The Bratu problem has zero, one and two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$, respectively, where $\lambda_c = 3.513830719$.

In the shooting method, we assume that $u'(0) = A$ as an initial slope to be determined. We take $\lambda = 2$, $N = 1 \times 10^4$, $\epsilon = 10^{-15}$ and $(A_0, A_2) = (8.1, 8.3)$, obtaining $A = 8.268763180545193$, which is very close to the exact value with an error of 3.55×10^{-15}. With 9 iterations, as shown in Fig. 4(a), for convergence, we obtain the solution shown in Fig. 4(b) by a solid line. The numerical solution coincides with the exact solution in Eq. (41), and the numerical error is shown in Fig. 4(b) with ME = 7.55×10^{-15}. COC = 0.9445 is obtained. In Table 4, we tabulate the absolute errors and compare them to those obtained in [39–41].

A smaller solution with the initial slope $A = 1.248217517758$ is plotted in Fig. 4(b) by a dashed line. In the shooting method with GDFNM, we take $(A_0, A_2) = (1.2, 1.3)$, and with 6 iterations for

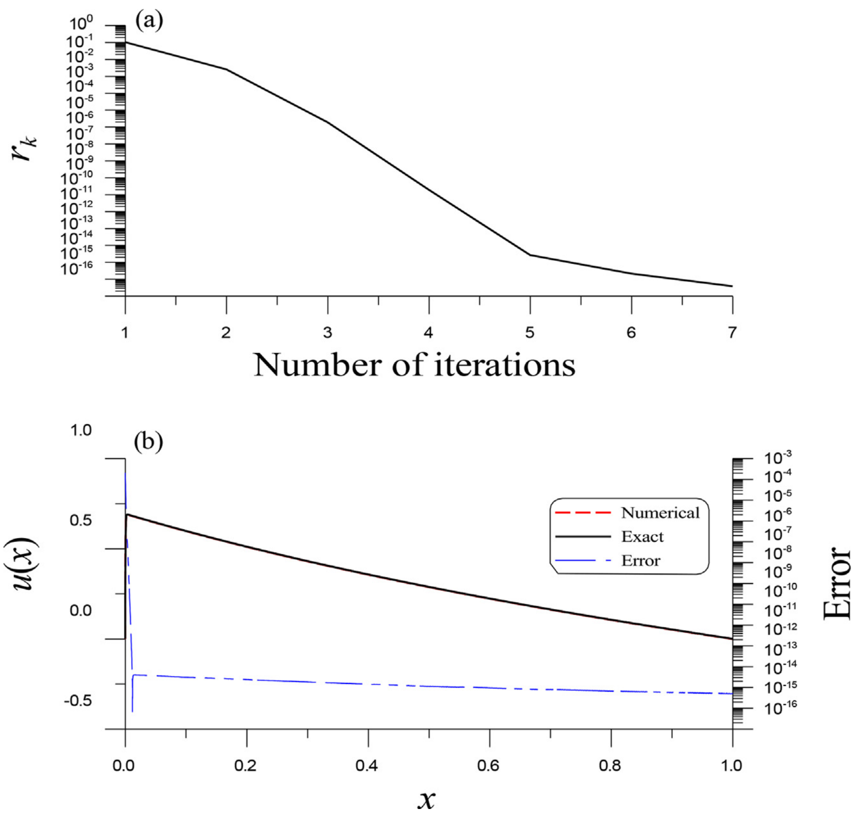

Fig. 3. For example, 3 of a highly singular perturbed problem, (a) convergence rate and (b) showing the solution obtained by the shooting method with DFNM and errors.

the convergence, we obtain a very accurate second solution with ME $= 1.06 \times 10^{-15}$.

4.5. Example 5

Let us calculate a three-point boundary value problem and $1 \le x \le 3$:

$$u''(x) - \frac{1}{8}\left[32 + 2x^3 - u(x)u'(x)\right] = 0, \quad (43)$$

$$u(1) = 17,\ u(2) + u(3) = \frac{79}{3}, \quad (44)$$

which has an exact solution:

$$u(x) = x^2 + \frac{16}{x}. \quad (45)$$

Previously, Liu [42] employed a two-stage Lie-group-shooting method to solve this problem, whose procedures are quite complicated. We suppose that $u'(1) = A$ is an unknown constant and use the GDFNM to solve the target equation $u(2) + u(3) - 79/3 = 0$. We take $N = 1 \times 10^4$, $\epsilon = 10^{-15}$ and $(A_0, A_2) = (-14.1, -13.8)$, obtaining $A = -13.9999994188434$, which is very close to the exact one $A = -14$ with an error of 5.8×10^{-7}. With 6 iterations for convergence, the numerical solution coincides with the exact solution in Eq. (45) with ME $= 3.45 \times 10^{-7}$. COC $= 1.05156$ is obtained. The accuracy is limited by the target equation, whose value is already zero at the sixth iteration; hence, we cannot further raise the accuracy by solving $u(2) + u(3) - 79/3 = 0$.

Table 3. For example, 3 with $\epsilon = 0.001$ list errors at different x.

x	0	0.1	0.2	0.3	0.5	0.7	0.9	1
Error	2.1*E*-4	3.0*E*-14	2.4*E*-14	1.8*E*-14	1.1*E*-14	7.9 *E*-15	5.7 *E*-15	4.9*E*-15

4.6. Example 6

Another three-point boundary value problem is [42,43]:

$$u''(x) + \frac{u^2(x)}{1 + u(x)} = 0,\ u(0) - u'(0) = 0,\ u(1) - \frac{1}{3}u(0.5) = 1, \quad (46)$$

We suppose that $u(0) = u'(0) = A$ is an unknown constant and use the GDFNM to solve the target

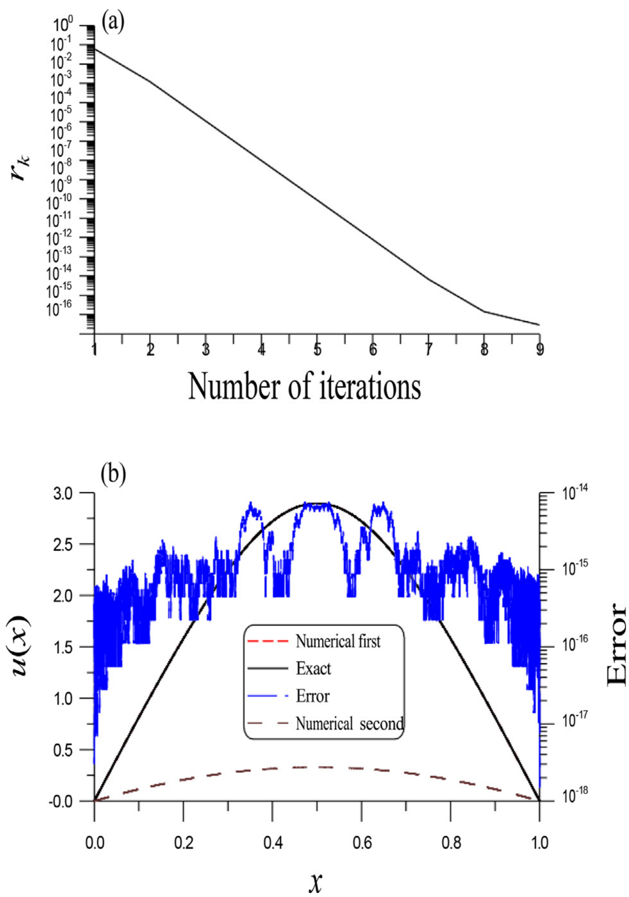

Fig. 4. For example, 4, (a) convergence rate and (b) showing two solutions obtained by the shooting method with DFNM and errors for the first solution.

equation $u(1) - u(0.5)/3 - 1 = 0$. We take $N = 1000$, $\epsilon = 10^{-15}$ and $(A_0, A_2) = (0.8, 0.85)$, obtaining $A = 0.841091466$, as shown in Fig. 5(a), with one intersection point. With 8 iterations for convergence, the numerical solution shown in Fig. 5(b) can match the target equation with an error of 2.22×10^{-16}.

5. Splitting-linearizing method and examples

The splitting-linearizing method was adopted in [34,44] to solve a nonlinear equation, which is quite promising. Later, this method was employed by Liu et al. [45] to solve nonlinear elliptic equations and by Liu et al. [46] to solve nonlinear BVPs. In this section, we employ the splitting-linearizing method together with the Lyapunov technique to determine the approximate analytic solutions of nonlinear BVPs.

Table 4. For example, 4 with $\lambda = 2$, comparing errors at different x.

x	Present	[39]	[40]	[41]
0.1	3.33 E−16	4.03 E−6	1.52 E−2	2.13 E−3
0.2	8.88 E−16	5.70 E−6	1.47 E−2	4.21 E−3
0.3	1.78 E−15	5.22 E−6	5.89 E−3	6.19 E−3
0.4	2.67 E−15	3.08 E−6	3.25 E−3	8.00 E−3
0.5	7.10 E−15	1.46 E−6	6.98 E−3	9.60 E−3
0.6	3.11 E−15	3.05 E−6	3.25 E−3	1.09 E−3
0.7	2.22 E−15	5.20 E−6	5.89 E−3	1.19 E−2
0.8	1.55 E−15	5.68 E−6	1.47 E−2	1.24 E−2
0.9	3.33 E−15	4.01 E−6	1.52 E−2	1.09 E−2

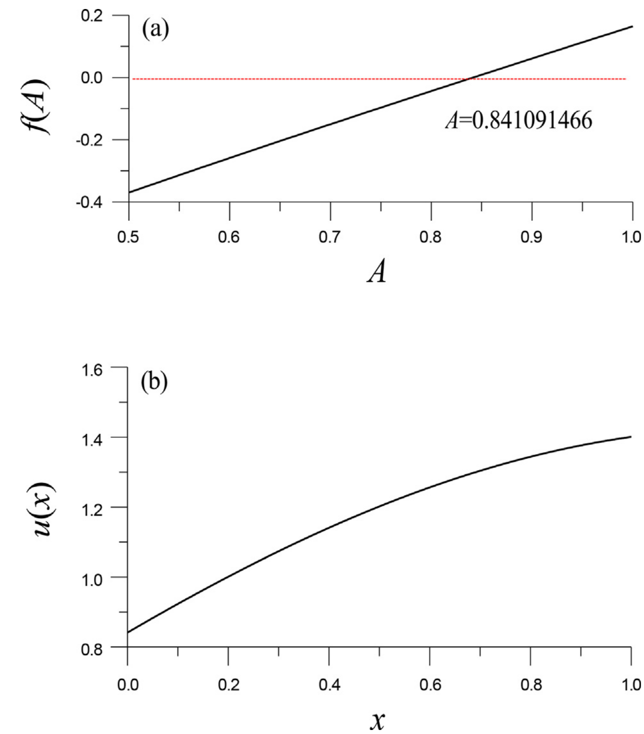

Fig. 5. For example, 6, (a) the target curve with one intersection point to the zero line, (b) displaying the solution obtained by the shooting method with DFNM.

5.1. Example 7

We employ the following example to demonstrate the splitting-linearizing method (SLM) [47]:

$$u''(x) + 3u(x)u'(x) + u^3(x) = 0,\ u(0) = 1,\ u(1) = 1, \tag{47}$$

whose exact solution is

$$u(x) = \frac{2x+1}{x^2+x+1}. \tag{48}$$

Let

$$u'(x) = y(x) \Rightarrow u(x) = 1 + \int_0^x y(s)ds, \tag{49}$$

where the left condition $u(0) = 1$ is considered. We suppose that $u'(0) = A$ is unknown, such that

$$y(0)=A, \tag{50}$$

where A is to be determined by the GDFNM in Section 3. Eq. (47) can be written as

$$y'(x)+3\left[1+\int_0^x y(s)ds\right]y(x)=-\left[1+\int_0^x y(s)ds\right]^3. \tag{51}$$

We suppose that

$$y_0(x)=(A+b)e^{-\lambda x}-be^{-2\lambda x}, \tag{52}$$

which satisfies $y_0(0)=A$, where b and λ are parameters. Then, we have

$$\begin{aligned}&1+\int_0^x y_0(s)ds\\ &=\frac{b+2\lambda+2A}{2\lambda}-\frac{A+b}{\lambda}e^{-\lambda x}+\frac{b}{2\lambda}e^{-2\lambda x},\\ &=a_1e^{-\lambda x}+a_2e^{-2\lambda x}\end{aligned} \tag{53}$$

$$b=-2\lambda-2A,\ a_1:=-\frac{A+b}{\lambda},\ a_2:=\frac{b}{2\lambda}, \tag{54}$$

where b is selected such that the constant term in Eq. (53) is zero.

Now, we recast Eq. (51) to

$$\begin{aligned}&y'(x)+3\mu q_0(a_1e^{-\lambda x}+a_2e^{-2\lambda x})y(x)\\ &=3\mu(q_0-1)(a_1e^{-\lambda x}+a_2e^{-2\lambda x})y_0(x)\\ &\quad-\mu(a_1e^{-\lambda x}+a_2e^{-2\lambda x})^3,\end{aligned} \tag{55}$$

where μ is a dummy parameter. Then, the analytic solution is determined by

$$\begin{aligned}y(x)&=y_0(x)+\sum_{k=1}^m(-\mu)^k y_k(x)\\ &=y_0(x)-\mu y_1(x)+\mu^2y_2(x)+\cdots,\end{aligned} \tag{56}$$

where $y_k(x)$, $k=1, 2, \ldots, m$ are to be determined.

Inserting Eq. (56) into Eq. (55) and equating the coefficients preceding μ^k, $k=1, 2, \ldots, m$, we can derive

$$\begin{aligned}&y_1'(x)=3(a_1e^{-\lambda x}+a_2e^{-2\lambda x})y_0(x)+(a_1e^{-\lambda x}+a_2e^{-2\lambda x})^3,\\ &y_1(0)=0,\end{aligned} \tag{57}$$

$$\begin{aligned}&y_k'(x)=(3q_0a_1e^{-\lambda x}+3q_0a_2e^{-2\lambda x})y_{k-1}(x),\\ &y_k(0)=0, k=2,\ldots,m.\end{aligned} \tag{58}$$

For the first-order solution, inserting Eq. (52) into Eq. (57), we have

$$y_1'(x)=a_{12}e^{-2\lambda x}+a_{13}e^{-3\lambda x}+a_{14}e^{-4\lambda x}+a_{15}e^{-5\lambda x}+a_{16}e^{-6\lambda x}, \tag{59}$$

where

$$\begin{aligned}&a_{12}=3a_1(1+b), a_{13}=3a_2(1+b)+a_1^3-3a_1b\\ &a_{14}=3a_1^2a_2-3a_2b, a_{15}=3a_1a_2^2, a_{16}=a_2^2.\end{aligned} \tag{60}$$

Let us define

$$\begin{aligned}E_k(x)&=\int_0^x e^{-k\lambda s}ds=\frac{1}{k\lambda}(1-e^{-k\lambda x}),\\ F_k(x)&=\int_0^x E_k(s)ds=\frac{x}{k\lambda}[x-E_k(x)].\end{aligned} \tag{61}$$

It follows from Eqs. (49), (56) and (59) with $m=1$ and $\mu=-1$ that

$$\begin{aligned}&u(x)=a_1e^{-\lambda_1x}+a_2e^{-2\lambda_1x}+a_{12}F_2(x)\\ &+a_{13}F_3(x)+a_{14}F_4(x)+a_{15}F_5(x)+a_{16}F_6(x),\end{aligned} \tag{62}$$

where we have replaced λ in the first two terms by λ_1 to control the rising part of the curve.

We take $\lambda=1.05$, $\lambda_1=1.5$, $A_0=0.9$, and $A_2=2$, and $A=0.78556551$ is obtained by using the GDFNM through 18 iterations under $\epsilon=10^{-15}$. The first-order approximate analytic solution is quite close to the exact one in Eq. (48), as shown in Fig. 6, with $ME=4.42\times10^{-3}$. We consider the same setting and add random noise with a maximum level of 6.95×10^{-3} on the right condition at $x=1$. The results show that $ME=6.95\times10^{-3}$ and $A=0.7783669605352804$, the absolute error depends on the boundary conditions. Thus the present

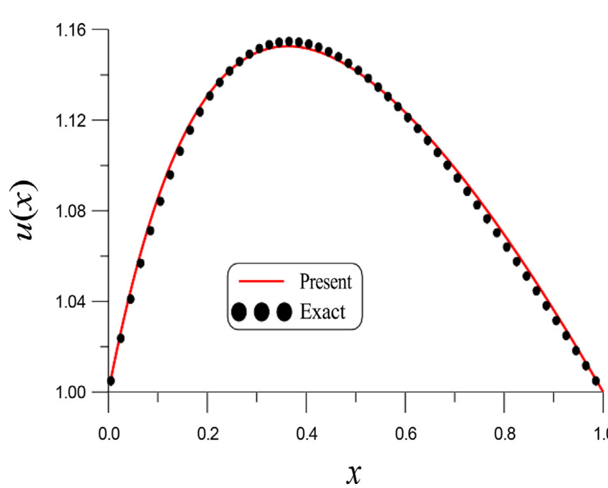

Fig. 6. For example, 7 compares the first-order approximate analytic solution obtained by the SLM to the exact solution.

algorithm provides very good numerical stability, even when considering random noise.

5.2. Example 8

We consider a second-order BVP [35]:

$$u''(x)=\frac{3}{2}u^2(x),\ u(0)=4,\ u(1)=1, \tag{63}$$

whose exact solution is given by Eq. (30).

For this BVP, the SLM presented in Section 5.1 is not applicable, since Eq. (63) does not include the term $u'(x)$. We give the zeroth order solution with

$$u_0(x)=4-3x, \tag{64}$$

satisfying $u_0(0) = 4$ and $u_0(1) = 1$ and directly considering the linearization of Eq. (63) by

$$u''(x)+\frac{3}{2}q_0u_0u(x)=\frac{3}{2}(q_0+1)u_0^2(x), \tag{65}$$

Inserting

$$u(x)=u_0(x)+\sum_{k=1}^{m}p^ku_k(x), \tag{66}$$

where p is a dummy parameter, into

$$u''(x)+\frac{3}{2}pq_0u_0u(x)=\frac{3}{2}p(q_0+1)u_0^2(x). \tag{67}$$

and equating the coefficients preceding p^k, $k = 1, 2, \ldots, m$, we can derive

$$u_1''(x)=\frac{3}{2}u_0^2(x),\ u_1(0)=0,\ u_1(1)=0, \tag{68}$$

$$u_k''(x)=-\frac{3}{2}q_0u_0(x)u_{k-1}(x),\ u_k(0)=0,\ u_k(1)=0,\ k=2,\ldots,m. \tag{69}$$

We can sequentially solve the above linear BVPs to derive $u_k(x)$ and insert them into Eq. (66) with $p = 1$ to obtain an analytic solution of m-order.

We take $m = 2$ and $q_0 = -1$, and the second-order approximate analytic solution is a polynomial with seventh order:

$$F_1(x)=\frac{3}{2}\int\int^x u_0^2(s)ds,\ D=F_1(1),\ u_1(x)=D-2Dx+F_1(x), \tag{70}$$

$$F_2(x)=-\frac{3q_0}{2}\int\int^x u_0(s)u_1(s)ds,\ E=F_2(1),\ u_2(x)=E-2Ex+F_2(x), \tag{71}$$

$$u(x)=u_0(x)+u_1(x)+u_2(x), \tag{72}$$

which is quite close to the exact one in Eq. (30), as shown in Fig. 7, with ME = 2.51×10^{-2}.

When the considered boundary conditions are given by

$$2u(0)+u'(0)=0,\ u(1)=1. \tag{73}$$

the procedure is more complicated, where the boundary conditions become $2u_0(0) + u'_0(0) = 0$, $u_0(1) = 1$ and $2u_k(0) + u'_k(0) = 0$, $u_k(1) = 0$, $k = 1, 2$. We start from

$$u_0(x)=A-2Ax+(1+A)x^2, \tag{74}$$

where A is to be determined such that the approximate analytic solution in Eq. (72) is close to the exact one, in which we insert Eq. (74) for $u_0(s)$. We take $m = 2$ and $q_0 = -320$, and with $A_0 = 0.35$ and $A_2 = 0.4$, the GDFNM to determine A is obtained as $A = 0.3779179154$ through 9 iterations under $\epsilon = 10^{-14}$. The second-order approximate analytic solution is a polynomial with tenth order, which is quite close to the exact solution in Eq. (30), as shown

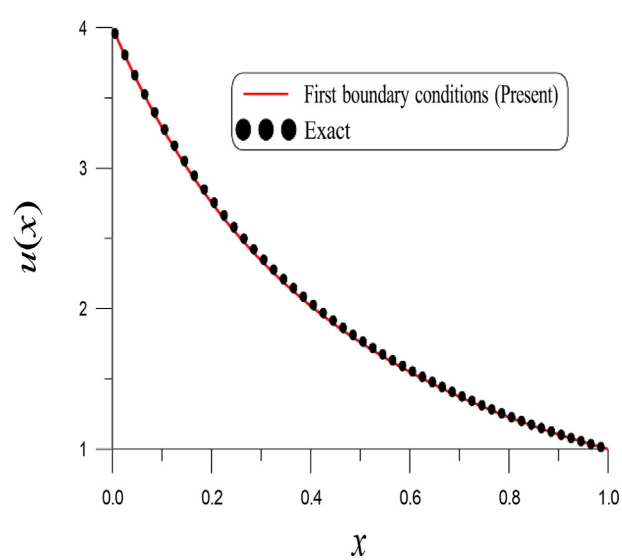

Fig. 7. For example, 8 under the first boundary conditions, the second-order approximate analytic solution obtained by the SLM is compared to the exact solution.

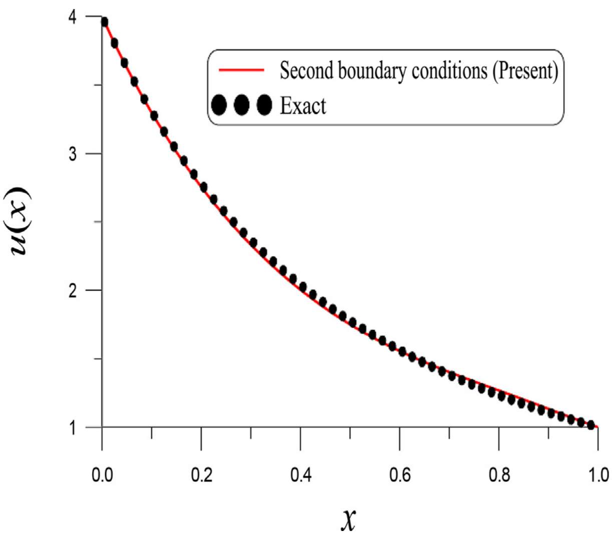

Fig. 8. For example, 8 under second boundary conditions, comparing the second-order approximate analytic solution obtained by the SLM to the exact one.

in Fig. 8, with ME $= 3.734 \times 10^{-2}$. Here, we consider the same setting and add random noise with a maximum level of 3.18×10^{-2} on the right condition at $x = 1$. The results show that ME $= 5.28 \times 10^{-2}$ and $A = 0.376301981031081$. Hence, the present algorithm is very robust and stable even when considering random noise.

6. Conclusions

Based on the shooting method, a novel and effective solver with a generalized derivative-free Newton method (GDFNM) was developed in this paper to solve the second-order nonlinear BVPs. The convergence analysis resulted in a second-order and third-order convergence of the iterative scheme GDFNM for $\beta = 1$ and $\beta = 0.5$. The involved a and b in the proposed iterative scheme $x_{n+1} = x_n - f(x_n)/[a + bf(x_n)]$ were approximated by the finite difference technique on the data at three points. As a demonstration of the use of the shooting method and GDFNM, we have investigated the numerical solutions of the Bratu problem et al., whose missing initial slope is obtained quickly and accurately. The initial guessed value of A can be obtained quickly by inspecting the intersection points of the target curve with the zero line. Furthermore, we can easily find multiple solutions of the considered problems. The results clearly showed that this method provides excellent approximations to the true solution of the nonlinear BVP with high accuracy, which is of the order of magnitude $(\Delta x)^4$, by using the fourth-order Runge−Kutta method to integrate the ODE with "almost exact" initial values obtained by the presented method. We have derived an approximate analytic solution for the BVPs involving the first-order differential term with exponential functions and for those without having the first-order differential term with polynomials as the approximate elements. With first-order or second-order approximations, the analytic results are good enough even for the Robin-type boundary condition.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] Agarwal RP. Boundary value problems for high order differential equations. Singapore: World Scientific; 1986.
[2] Kubíček M, Hlaváček V. Numerical solution of nonlinear boundary value problems with applications. New York: Prentice-Hall; 1983.
[3] Keller HB. Numerical methods for two-point boundary value problems. New York: Dover; 1992.
[4] Cash JR. On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections, Part 1: a survey and comparison of some one-step formulae. Comput Math Appl 1986;12(10):1029−48.
[5] Cash JR. On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections, Part 2: the development and analysis of highly stable deferred correction formulae. SIAM J Numer Anal 1988;25(4):862−82.
[6] Cash JR, Wright RW. Continuous extensions of deferred correction schemes for the numerical solution of nonlinear two-point boundary value problems. Appl Numer Math 1998;28(2−4):227−44.
[7] Ascher UM, Mattheij RMM, Russell RD. Numerical solution of boundary value problems for ordinary differential equations. Philadelphia: SIAM; 1995.
[8] Liu CS. The Lie-group shooting method for singularly perturbed two-point boundary value problems, Comput. Model Eng Sci 2006;15(3):179−96.
[9] Awoke A, Reddy YN. An exponentially fitted special second-order finite difference method for solving singular perturbation problems. Appl Math Comput 2007;190(2):1767−82.
[10] Patidar KC. High order parameter uniform numerical method for singular perturbation problems. Appl Math Comput 2007;188(1):720−33.
[11] Vigo-Aguiar J, Natesan S. An efficient numerical method for singular perturbation problems. J Comput Appl Math 2006; 192(1):132−41.
[12] Lin TC, Schultza DH, Zhang W. Numerical solutions of linear and nonlinear singular perturbation problems. Comput Math Appl 2008;55(11):2574−92.
[13] Liu CS. The Lie-group shooting method for solving nonlinear singularly perturbed boundary value problems. Commun Nonlinear Sci Numer Simulat 2012;17(4):1506−21.

[14] Khuri SA, Sayfy A. Self-adjoint singularly perturbed boundary value problems: an adaptive variational approach. Math Methods Appl Sci 2013;36(9):1070–9.
[15] Dogan N, Erturk VS, Akin O. Numerical treatment of singularly perturbed two-point boundary value problems by using differential transformation method. Discrete Dynam Nat Soc 2012;2012:1–10.
[16] Liu CS. Cone of non-linear dynamical system and group preserving schemes. Int J Non Lin Mech 2001;36(7):1047–68.
[17] Liu CS. A novel Lie-group theory and complexity of nonlinear dynamical systems. Commun Nonlinear Sci Numer Simulat 2015;20(1):39–58.
[18] Liu CS. The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. Comput Model Eng Sci 2006;13(2):149–63.
[19] Liu CS. Efficient shooting methods for the second order ordinary differential equations. Comput Model Eng Sci 2006; 15(2):69–86.
[20] Liu CS. The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. Comput Model Eng Sci 2006;13(2):149–63.
[21] Chen YW. Simultaneous determination of the heat source and the initial data by using an explicit Lie-group shooting method, Numer. Heat Transf B: Fundam 2019;75(4):239–64.
[22] Chang YS, Chen YW, Liu CS, Chang JR. A non-iteration solution for solving the backward-in-time two-dimensional Burgers' equation with a large Reynolds number. J Mar Sci Technol-Taiwan 2022;30(1):75–85.
[23] Liu CS. Solving an inverse Sturm-Liouville problem by a Lie-group method. Bound Value Probl 2008;2008(1):1–18.
[24] Liu CS. A Lie-group shooting method for computing eigenvalues and eigenfunctions of Sturm-Liouville problems. Comput Model Eng Sci 2008;26(3):157–68.
[25] Liu CS. The Lie-group shooting method for computing the generalized Sturm-Liouville problems. Comput Model Eng Sci 2010;56(1):85–112.
[26] Hajiketabi M, Abbasbandy S. A simple, efficient and accurate new Lie-group shooting method for solving nonlinear boundary value problems. Int J Nonlinear Anal Appl 2021; 12(1):761–81.
[27] Liu CS, El-Zahar ER, Chang CW. Three novel fifth-order iterative schemes for solving nonlinear equations. Math Comput Simulat 2021;187:282–93.
[28] Lin J, Zhang YH, Liu CS. Solving nonlinear third-order three-point boundary value problems by boundary shape functions methods. Adv Differ Equ 2021;146:1–23.
[29] Liu CS, Chang CW. Lie-group shooting/boundary shape function methods for solving nonlinear boundary value problems. Symmetry 2022;14(4):778.
[30] Liu CS. Computing the eigenvalues of the generalized Sturm-Liouville problems based on the Lie group SL(2, R). J Comput Appl Math 2012;236(17):4547–60.
[31] Liu CS. An SL(3, R) shooting method for solving the Falkner-Skan boundary layer equation. Int J Non Lin Mech 2013;49: 145–51.
[32] Liu CS. The Lie-group shooting method for solving multi-dimensional nonlinear boundary value problems. J Optim Theor Appl 2012;152(2):468–95.
[33] Farkas M. Periodic motions. New York: Springer-Verlag; 1994.
[34] Liu CS, Hong HK, Lee TL. A splitting method to solve a single nonlinear equation with derivative-free iterative schemes. Math Comput Simulat 2021;190:837–47.
[35] Ha SN, Lee CR. Numerical study for two-point boundary value problems using Green's functions. Comput Math Appl 2002;44(12):1599–608.
[36] Finlayson B. The method of weighted residuals and variational principles. New York: Academic; 1972.
[37] Bender CM, Orszag SA. Advanced mathematical methods for scientists and engineers. New York: McGraw-Hill; 1978.
[38] Liu CS, Chang JR. Boundary shape functions methods for solving the nonlinear singularly perturbed problems with Robin boundary conditions. Int J Nonlinear Sci Numer Stimul 2020;21(7–8):797–806.
[39] Abbasbandy S, Hashemi MS, Liu CS. The Lie-group shooting method for solving the Bratu equation. Commun Nonlinear Sci Numer Simulat 2011;16(11):4238–49.
[40] Deeba E, Khuri SA, Xie S. An algorithm for solving boundary value problems. J Comput Phys 2000;159(2):125–38.
[41] Khuri SA. A new approach to Bratu's problem. Appl Math Comput 2004;147(1):131–6.
[42] Liu CS. A two-stage LGSM for three-point BVPs of second-order ODEs. Bound Value Probl 2008;2008:1–22.
[43] Kwong MK, Wong JSW. The shooting method and nonhomogeneous multipoint BVPs of second order ODE. Bound Value Probl 2007;2007:1–16.
[44] Liu CS. A new splitting technique for solving nonlinear equations by an iterative scheme. J Math Res 2020;12(4):1–40.
[45] Liu CS, El-Zahar ER, Chen YW. Solving nonlinear elliptic equations in arbitrary plane domains by using a new splitting and linearization technique. Eng Anal Bound Elem 2021; 125:124–34.
[46] Liu CS, El-Zahar ER, Chang CW. Solving nonlinear boundary value problems by a boundary shape function method and a splitting and linearizing method. Int J Nonlinear Sci Numer 2022:1–18.
[47] Sachdev PL. Nonlinear ordinary differential equations and their applications. New York: Marcel Dekker; 1991.