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Solving Nonlinear Boundary Value Problems with Nonlinear Integral Boundary Conditions by Local and Nonlocal Boundary Shape Functions Methods

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Abstract

The paper considers the second-order nonlinear boundary value problem (NBVP), which is equipped with nonlinear integral boundary conditions (BCs). Two novel iterative algorithms are developed to overcome the difficulty of NBVP with double nonlinearities involved. In the first iterative algorithm, two nonlocal shape functions incorporating the linear integral terms are derived, and a nonlocal boundary shape function (NBSF) is formulated to assist the solution. Let the solution be the NBSF so that the NBVP can be exactly transformed into an initial value problem. The new variable is a free function in the NBSF, and its initial values are given. For the NBVP with linear integral BCs, three unknown constants are to be determined, while for the nonlinear integral BCs, five unknown constants are to be determined. Two-point local shape functions and local boundary shape functions are derived for the second iterative algorithm, wherein the integral terms in the boundary conditions are viewed as unknown constants. By a few iterations, four unknown constants can be determined quickly. Through numerical experiments, these two iterative algorithms are found to be powerful for seeking quite accurate solutions. The second algorithm is slightly better than the first, with fewer iterations and a more accurate solution.

Keywords: Nonlinear BVP, Nonlinear integral boundary conditions, Nonlocal shape functions, Local shape functions, Iterative algorithms

1. Introduction

Nonlinear boundary value problems (NBVPs) are frequently encountered in scientific and engineering problems. In particular, second-order NBVPs have been studied extensively. Agarwal [1] investigated the existence and uniqueness of solutions of BVPs of higher order. To solve NBVPs, Kubiček and Hlaváček [2] conducted a complete survey of the development, analysis, and application of numerical techniques. Keller [3] described an elementary yet rigorous account of practical numerical methods for solving general two-point boundary value problems (BVPs).

Recently, Hajipou et al. [4] and Jajarmi and Baleanu [5] proposed a new iterative method to solve high-order nonlinear fractional boundary value problems. Then, different studied views further are applied to analyze nonlinear boundary value problems such as stability [6], nonlinear fractional-order derivatives [7] and optimal control problems [8,9]. Mahariq [10] and Mahariq et al. [11–13] applied the spectral element method to solve the application field’s electromagnetic and photonic nanojet problems.

In this paper, we consider nonlinear and nonlocal boundary conditions (BCs) for second-order NBVPs. They are different from the conventional two-point...
BCs, which are specified at boundary points of a given interval. The nonlocal BCs involving certain integrals are specified for the solution at all points within the given interval. Therefore, the nonlocal NBVP is more challenging to solve than the local NBVP. As defined by Lin et al. [14], the boundary shape function (BSF) automatically satisfies the BCs, which includes the solution of BV as a special case since the solution must exactly satisfy the specified BCs. Liu and Chang [15] extended the work of Lin et al. [14] to address the multipoint boundary conditions for NBVPs. Liu and Chang [16] applied the BSF to solve nonlinear singularly perturbed problems with Robin boundary conditions. Furthermore, Liu [17] used the BSF to analyze nonlinear composite beams subjected to nonlinear boundary moment conditions. The idea of the BSF has been adopted to solve some BV with conventional local BCs and then extended to solve 2D and 3D nonlinear problems [18–20]. However, the method of BSF has not yet been developed for solving the NBVP endowed with nonlinear and nonlocal BCs.

For the numerical method of nonlocal BV, it is of utmost importance to preserve the given BCs. However, this is not an easy task when the given BCs involve the solution in the entire interval, which is itself an unknown function in the interval. In the case of an NBVP subject to nonlinear and nonlocal BCs, reducing the boundary error and then the error of the solution in the entire interval is an important issue. Hereon, we attempt to develop novel methods for providing accurate numerical solutions to nonlinear and nonlocal NBVPs. To exactly preserve the nonlinear and nonlocal BCs, we formulate numerical algorithms based on local and nonlocal boundary shape functions.

In this paper, we also cover the Duffing-type NBVP, which is a nonlinear ordinary differential equation (ODE) well-known in applied science as a powerful model to discuss practical phenomena such as nonlinear mechanical oscillators, bending models of DNA, and the prediction of diseases. Some works on the forced Duffing equation with integral boundary conditions [21–24] are effective methods for solving the NBVP with linear integral BCs [24–26]. To date, most papers have developed numerical methods for solving the Duffing-type NBVP with linear integral BCs [23,27,28]. The NBVP with nonlinear integral BCs is more difficult; hence, few papers are devoted to solving this problem.

To address nonlinear and nonlocal BCs, we will develop two novel iterative algorithms to determine the solution for the NBVP with nonlinear integral boundary conditions. For both iterative algorithms, the basic idea is to transform the NBVP to the corresponding initial value problem for the new variable, whose initial conditions are given. In the transformed ODE for the new variable, some unknown constants need to be determined. Detailed descriptions of local and nonlocal BSFs are provided in the next section. The paper is structured as follows. In Section 2, we derive two nonlocal shape functions and nonlocal boundary shape functions for a second-order NBVP. An iterative algorithm for linear integral BCs of the second-order NBVP is developed in Section 3, where five unknown constants are to be determined. Section 4 gives the first iterative algorithm for the nonlinear integral BCs of the second-order NBVP, where four unknown constants are to be determined. In Section 5, we develop a second iterative algorithm for the nonlinear integral BCs of the second-order NBVP, where four unknown constants are to be determined. In Section 6, several examples are tested. Finally, Section 7 draws conclusions.

2. Nonlocal boundary shape function

Consider a second-order nonlinear ODE:

$$u''(x) = f(x, u(x), u'(x)), \quad 0 < x < 1,$$

where \( f \) satisfies the Lipschitz condition, which is endowed with the integral BCs:

$$a_1u(0) + b_1u'(0) = \int_0^1 q_1(x, u(x))\,dx,$$

$$a_2u(1) + b_2u'(1) = \int_0^1 q_2(x, u(x))\,dx.$$ 

When \( q_1 \) and \( q_2 \) are constants, the Robin-type BCs are specified at two boundary points. When \( q_1 \) and \( q_2 \) are linear functions of \( u \), the BCs are linear integral BCs; otherwise, they are nonlinear integral BCs.

To explore the new iterative method more clearly, we start from the linear integral BCs:

$$a_1u(0) + b_1u'(0) - \rho_1 \int_0^1 u(x)\,dx = p_1,$$

$$a_2u(1) + b_2u'(1) - \rho_2 \int_0^1 u(x)\,dx = p_2,$$

which are obtained from Eq. (2) by inserting \( q_1 = \rho_1 u + p_1 \) and \( q_2 = \rho_2 u + p_2 \).

Upon defining linear operators
3. Numerical algorithm for linear integral BCs

Eqs. (3) and (4), including the integrals of \( s_1(x) \) and \( s_2(x) \), are called nonlocal shape functions (NSFs). With the help of Theorem 1, a feasible and efficient way to obtain \( u(x) \) is transformed into a new variable \( z(x) \) by

\[
z(x) = u(x) + W(x),
\]

where

\[
W(x) = [L_1\{z(x)\} - p_1]s_1(x) + [L_2\{z(x)\} - p_2]s_2(x).
\]

If we can determine \( z(x) \), then Eqs. (3) and (4) are automatically satisfied by \( u(x) = z(x) - W(x) \), as proven in Theorem 1.

3.1. Initial value problem

From Eqs. (1) and (11), \( z(x) \) is governed by a new ODE:

\[
z''(x) = H(x, z(x), z'(x)),
\]

\[
: = W(x) + f(x, z(x) - W(x), z'(x) - W'(x)),
\]

where \( W(x) \) given by Eq. (12) involves three unknown constants:

\[
\alpha := z(1), \quad \beta := z'(1), \quad \gamma := \int_0^1 z(x)\,dx,
\]

as shown in Eqs. (5) and (6). Eq. (13) is an initial value problem (IVP), upon giving the initial values \( z(0) \) and \( z'(0) \).

In Theorem 1, \( s_1(x) \) and \( s_2(x) \) can be determined as follows. Suppose that \( s_1(x) = a + bx + cx^2 \) and insert it into Eq. (8). Then, we can obtain

\[
(aa_1 + b_1b - p_1\left[a + \frac{b + c}{3}\right] = 1,
\]

\[
a_2(a + b + c) + b_2[2b + 2c] - p_2\left[a + \frac{b + c}{3}\right] = 0,
\]

which are underdetermined linear systems to determine \( a, b, \) and \( c \). There are many solutions of \( a, b, \) and \( c \). Similarly, we can do this for \( s_2(x) \). We choose the suitable values of \( a, b, \) and \( c \) such that the Runge–Kutta method RK4 can be applied to integrate the ODE (13) with the given initial values \( z(0) \) and \( z'(0) \).

3.2. Iterative algorithm

When RK4 is adopted to integrate ODE (13) with the given initial conditions, we can iteratively determine \( \alpha, \beta, \) and \( \gamma \) until they are convergent as follows: (i) given \( z_1(0) = a_0, z_2(0) = b_0, \) \( a_0, b_0, \beta_0, \gamma_0 \) and \( \epsilon \); (ii) for \( k = 0, 1, 2, \ldots \), integrate

\[
z_k'(x) = z_k(x),
\]

and use the linear property of

\[
W(x) = [L_1\{z(x)\} - p_1]s_1(x) + [L_2\{z(x)\} - p_2]s_2(x).
\]
where \( z_1(0) = a_0, \) \( z_2(0) = b_0, \) and \( e \) given initial values. Take
\[
\alpha_k = z_1(1), \quad \beta_k = z_2(1), \quad \gamma_k = z_3(1).
\]
Until
\[
r_k := \sqrt{\left( \frac{(\alpha_k - \alpha_k)^2 + (\beta_k - \beta_k)^2}{\gamma_k - \gamma_k} \right)} < \delta
\]
is satisfied. If Eq. (19) is not fulfilled, then go to (ii) for the next iteration.

4. First iterative algorithm for nonlinear integral BCs

When Eq. (1) is subjected to the nonlinear integral BCs in Eq. (2), the BVP is more challenging to solve than that with linear integral BCs (3) and (4). We suppose that \( q_1 \) and \( q_2 \) can be decomposed as
\[
q_2(x, u) = \rho_1 u + p_2 + F_2(x, u),
\]
where \( F_1 \) and \( F_2 \) are nonlinear functions of \( u \). Let
\[
c_1 := p_1 + \int_0^1 F_1(x, u(x)) \, dx,
\]
\[
c_2 := p_2 + \int_0^1 F_2(x, u(x)) \, dx,
\]
where unknown constants \( c_1 \) and \( c_2 \) are to be determined. Eq. (2) is rewritten as
\[
\begin{align*}
a_1 u(0) + b_1 u'(0) - \rho_1 \int_0^1 u(x) \, dx &= c_1, \\
a_2 u(1) + b_2 u'(1) - \rho_2 \int_0^1 u(x) \, dx &= c_2.
\end{align*}
\]

4.1. Transformation to IVP

Let
\[
u(x) = y(x) + G(x),
\]
\[
G(x) = [L_1 \{y(x)\} - c_1 s_1(x)] + [L_2 \{y(x)\} - c_2 s_1(x)],
\]
such that \( u(x) \) satisfies Eq. (22) automatically, as proven in Theorem 1.

It follows from Eqs. (1) and (23) that
\[
y''(x) = F(x, y(x), y'(x)) = G''(x) + f(x, f(x) - G(x), y'(x) - G'(x)),
\]
where we can give \( y(0) \) and \( y'(0) \) as the initial values for \( y \), while \( y(1), y'(1), \gamma := \int_0^1 y(x) \, dx \), \( c_1 \) and \( c_2 \) are five unknown values in \( G(x) \) as shown by Eq. (24).

4.2. First iterative algorithm

For solving the NBVP in Eqs. (1) and (22), we list the iterative algorithm. (i) Given \( y_1(0), y_2(0), a_0, e_0, \gamma_0, c_0, c_1, c_2, \) and \( N \); (ii) for \( k = 0, 1, 2, ... \), apply the RK4 to integrate
\[
y_f(x) = y_2(x), \quad y_2(x) = F(x, y_1(x), y_2(x); d_1, c_1, c_2), \quad y_3(x) = y_1(x), \quad y_4(x) = F_1(x, y_1(x) - G(x)) + p_1,
\]
\[
y_5(x) = F_2(x, y_1(x) - G(x)) + p_2,
\]
where \( y_1(0) = y_2(0) = y_3(0) = 0. \) Take \( d_{k+1} = y_1(1), e_{k+1} = y_2(1), \gamma_{k+1} = y_3(1), c_{k+1} = y_4(1), \) and \( c_{k+1} = y_5(1). \)

We terminate the iterations if the residual satisfies
\[
\sqrt{\left( (d_{k+1} - d_k)^2 + (e_{k+1} - e_k)^2 + (\gamma_{k+1} - \gamma_k)^2 + (c_{k+1} - c_k)^2 + (c_{k+1} - c_k)^2 \right)} < \delta;
\]
otherwise, go to (ii) for the next iteration.

When \( c_1 \) and \( c_2 \) are convergent, the solution \( u(x) \) is given by
\[
u(x) = y(x) - s_1(x) [L_1 \{y(x)\} - c_1] - s_2(x) [L_1 \{y(x)\} - c_2].
\]

5. Second iterative algorithm for nonlinear integral BC

In Section 4, five unknown constants \( d, e, \gamma, c_1, \) and \( c_2 \) are determined. Instead of the nonlocal shape functions \( s_1(x) \) and \( s_2(x) \), we may consider two-point local shape functions \( T_1(x) \) and \( T_2(x) \) as determined by
\[
a_1 T_1(0) + b_1 T_1'(0) = 1, \quad a_2 T_1(0) + b_2 T_1'(0) = 0,
\]
\[
a_1 T_2(0) + b_1 T_2'(0) = 0, \quad a_2 T_2(0) + b_2 T_2'(0) = 1.
\]

In Eq. (2), we let
\[
c_1 := \int_0^1 q_1(x, u(x)) \, dx, \quad c_2 := \int_0^1 q_2(x, u(x)) \, dx
\]
be unknown constants to be determined. Then, replacing \( s_1(x) \) and \( s_2(x) \) in Section 4 with \( T_1(x) \) and \( T_2(x) \), we have the following iterative algorithm: (i) given \( y_1(0), y_2(0), a_0, e_0, c_0, c_1, c_2, \) and \( N \); (ii) for \( k = 0, 1, 2, ... \), apply RK4 with a step size \( \Delta x = 1/N \) to integrate
6. Numerical examples

6.1. Example 1

Consider an NBVP

\[ u''(x) + u'(x) - u^2(x) = 3e^x + 2xe^x - x^2e^x, \]  
\[ u(0) + u'(0) - \int_0^1 u(x) dx = 0, \]  
\[ u(1) - u'(1) - 2 \int_0^1 u(x) dx = e - 2, \]  
whose exact solution is

\[ u(x) = xe^x. \]  

In Eq. (30), \( a_1 = 1, b_1 = 1, \rho_1 = 1, p_1 = 0, F_1 = 0, a_2 = 1, b_2 = -1, \rho_2 = 2, p_2 = e - 2, \) and \( F_2 = 0. \) For this problem, \( s_1(x) = -2/3 + 2x \) and \( s_2(x) = -2x - 3x^2 \) are derived. Utilizing \( z(0) = 0, z'(0) = 1, \alpha_0 = \beta_0 = \gamma_0 = 0, \epsilon = 10^{-10}, \) and \( N = 500, \) the NBSF method converges, as shown in Fig. 1(a), with 84 iterations. Upon comparing numerical and exact solutions, Fig. 1(b) displays the absolute numerical error whose maximum error (ME) is \( 6.61 \times 10^{-12}, \) which is very accurate.

6.2. Example 2

The following NBVP is to be solved:

\[ u''(x) + u'(x) - 2u(x) + 2\sin^2(x) = 0, \] (32)
of which no closed-form solution is available.

In Eq. (33), \( a_1 = 1, b_1 = -3/8, \rho_1 = 1/4, p_1 = -1, F_1 = u^2, a_2 = 1, b_2 = 1/4, \rho_2 = 1/2, p_2 = 1, \) and \( F_2 = 0. \) For this problem, \( s_1(x) = 1 - x/2 \) and \( s_2(x) = 1 + 23x/14 - 3x^2/7 \) are derived. Because a nonlinear integral term occurs on the left side, only \( c_1^0 \) is unknown.

Using \( y(0) = y'(0) = d_0 = e_0 = c_1^0 = \gamma_0 = 0, \) \( \epsilon = 10^{-10}, \) and \( N = 100, \) as shown in Fig. 3(a), the first iterative algorithm converges with 45 iterations, and in Fig. 3(b), we plot the numerical solution. Since there exists no exact solution, we assess the numerical error by showing the absolute errors of BCs, which are \( 1.5 \times 10^{-8} \) for the left boundary condition and \( 2.4 \times 10^{-8} \) for the right boundary condition in Eq. (33).

The above results are computed by the first iterative algorithm in Section 4.2. Next, we apply the second iterative algorithm in Section 5 to solve this problem. In Eq. (33), \( a_1 = 1, b_1 = -3/8, q_1 = u^2 + u/4 - 1, a_2 = 1, b_2 = 1/4, \) and \( q_2 = u/2 + 1. \) \( T_1(x) = 10/13 - 8x/13 \) and \( T_2(x) = 3/13 + 8x/13 \) are derived. Taking \( y(0) = 0, y'(0) = 1, \) \( a_0 = \beta_0 = c_2^0 = c_2^0 = 0, \) \( \epsilon = 10^{-5}, \) and \( N = 100, \) the second iterative algorithm converges with 44 iterations. The absolute errors of BCs are \( 8.74 \times 10^{-11} \) for the left boundary condition and \( 4.35 \times 10^{-9} \) for the right boundary condition in Eq. (33). The second iterative algorithm is more accurate than the first iterative algorithm.

6.3. Example 3

Consider [28]:

\[
\frac{d^2}{n} x + u'(x) + x(1-x)u^2(x) = F(x), \tag{34}
\]

\[
u(0) - \frac{1}{2} u'(0) = - \int_0^1 u(x) dx, \tag{35}
\]

\[
u(0) + \frac{1}{2} u'(1) = - \int_0^1 x u(x) dx.
\]


\[ u(x) = \sin(\pi x). \quad \text{(36)} \]

F(x) can be obtained by inserting \( u(x) = \sin(\pi x) \) into Eq. (34).

In Eq. (35), \( a_1 = 1, b_1 = -2/\pi, q_1 = -u, a_2 = 1, b_2 = 1/\pi^2 \), and \( q_2 = x^2 \). For this problem, we take \( T_1(x) = (\pi^2 + 1)/(\pi^2 + 3) - \pi^2 x/(\pi^2 + 3) \) and \( T_2(x) = 2/(\pi^2 + 3) + \pi^2 x/(\pi^2 + 3) \). Utilizing \( y(0) = 0, y'(0) = 2, d_0 = c_0 = c_1^0 = c_2^0 = 0, \epsilon = 10^{-10} \), and \( N = 100 \), the second iterative algorithm converges with 77 iterations, and the maximum relative error is \( 2.54 \times 10^{-8} \).

To compare with the reference shown in Table 1, we list the relative errors at some points and compare the results to those obtained by Geng and Cui [28]. For this problem, it can be seen that the presented accuracy obtained by the second iterative algorithm is approximately four orders of magnitude more accurate than that obtained by Geng and Cui [28].

6.4. Example 4

Let us consider

\[ u^n(x) = \frac{3}{2} u^2(x), \quad \text{(37)} \]

\[ u(0) = 4, \quad u(1) = \int_0^1 \left[ u^2(x) - \frac{11}{3} \right] \, dx, \quad \text{(38)} \]

\[ u(x) = \frac{4}{(1 + x)^2}. \quad \text{(39)} \]

In Eq. (38), \( a_1 = 1, b_1 = 0, q_1 = 4, a_2 = 1, b_2 = 0 \), and \( q_2 = u^2 - 11/3 \). Because a nonlinear integral term occurs on the right side, only \( c_2^1 \) is unknown. We derive \( T_1(x) = 1 - 2x + x^2 \) and \( T_2(x) = x^2 \). Using \( y(0) = y'(0) = d_0 = c_0 = 0, \epsilon = 10^{-10} \), and \( N = 100 \), the second iterative algorithm converges with two iterations, as shown in Fig. 4(a). In Fig. 4(b), we compare the solutions, and the numerical error is ME = 7.92 \times 10^{-9}. This value is smaller than that obtained in [29], where the Lie-group shooting method (LGSM) was adopted to solve the problem under the usual boundary condition:

\[ u(0) = 4, \quad u(1) = 1. \]

The results show that the proposed algorithm avoids the need for step-by-step adjustment of the weighting value and multisolution problems by the LGSM. The proposed method is very efficient and stable in approximating the true solution.

6.5. Example 5

Consider Eq. (29) again, which is now subjected to nonlinear integral BCs

\[ u(0) - \int_0^1 \left[ u^2(x) - \frac{e^2 - 1}{4} \right] \, dx = 0, \quad \text{(40)} \]

\[ u(1) - \int_0^1 \left[ u^2(x) - \frac{e^2 - 1}{4} + \epsilon \right] \, dx = 0, \]

the exact solution is still given by Eq. (31).

In Eq. (40), \( a_1 = 1, b_1 = 0, q_1 = u^2 - (e^2 - 1)/4, a_2 = 1, b_2 = 0, \) and \( q_2 = u^2 - (e^2 - 1)/4 + \epsilon \). We derive \( T_1(x) = 1 - x \) and \( T_2(x) = x \). Because integral BCs occur on both sides, \( c_1^1 \) and \( c_2^1 \) are unknown

<table>
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<td>0.08</td>
<td>0.248690</td>
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constants. Utilizing $y(0) = 0$, $y'(0) = 1$, $d_0 = e_0 = c_0^1 = c_0^2 = 0$, $\epsilon = 10^{-10}$, and $N = 500$, the second iterative algorithm converges with two iterations, as shown in Fig. 5(a). The absolute numerical error is shown in Fig. 5(b), with $ME = 3.53 \times 10^{-11}$ being very accurate.

6.6. Example 6

Let us consider the following NBVP [2,29]:

$$u^{''}(x) + \frac{1}{x}u'(x) = -\delta e^{u(x)}, \quad (41)$$

$$u'(x) = 0,$$

$$u(1) + u'(1) = \int_0^1 \left[ e^{u(x)/2} - 2 - \frac{\pi}{2} \right] dx = 0. \quad (42)$$

For $\delta = 2$, there is only one solution:

$$u(x) = \ln \frac{4}{(1 + x^2)^2}. \quad (43)$$

In Eq. (42), $a_1 = 0$, $b_1 = 0$, $q_1 = 0$, $a_2 = 1$, $b_2 = 1$, and $q_2 = e^{u/2} - 2 - \pi/2$. Because the integral boundary condition occurs on the right side, only $c^2_0$ is unknown. We obtain $T_1(x) = x - 2x^2/3$ and $T_2(x) = x^2 - 2$. Taking $y(0) = 0$, $y'(0) = 0.5$, $d_0 = e_0 = 0$, $c_0^1 = -1.5$, $\epsilon = 10^{-2}$, and $N = 500$, the second iterative algorithm converges with 89 iterations, as shown in Fig. 6(a), and the absolute numerical error is shown in Fig. 6(b) with $ME = 1.13 \times 10^{-3}$. This is more accurate than that in [29], who employed the LGSM to solve Eqs. (41) and (42).

6.7. Example 7

Let us consider the following NBVP:

$$u^{''}(x) + u'(x) + u^3(x) = 3 + 2x + (x + x^2)^3, \quad (44)$$

$$u(0) + \int_0^1 u(x) dx + \int_0^1 u^2(x) dx = \frac{28}{15}, \quad (45)$$

$$u(1) - \frac{1}{2} \int_0^1 (1 + 2x)u(x) dx = 0,$$

$$u(x) = x + x^2. \quad (46)$$

In Eq. (45), $a_1 = 1$, $b_1 = 0$, $q_1 = (28/15) - u - u^2$, $a_2 = 1$, $b_2 = 0$, $q_2 = (1 + 2x)u$, $T_1(x) = 1 - x$, and $T_2(x) = x$ are derived. In the second iterative algorithm, we take $y(0) = 0$, $y'(0) = 1$, $d_0 = e_0 = c_0^1 = 0$, $c_0^2 = 0$, $\epsilon = 10^{-6}$, and $N = 100$. This converges with

![Fig. 5. For example 5: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.](image1)

![Fig. 6. For example 6: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.](image2)
two iterations, as shown in Fig. 7(a). The absolute numerical error is shown in Fig. 7(b) with ME = 5.78 × 10⁻⁹.

7. Conclusions

In this paper, nonlocal boundary shape functions and local boundary shape functions were demonstrated. Using a new concept of these functions, we developed two novel iterative algorithms to determine the solution for the NBVP with nonlinear integral boundary conditions. For both iterative algorithms, the basic idea is to transform the NBVP to the corresponding initial value problem for the new variable, whose initial conditions are given. In the transformed ODE for the new variable, the BSF automatically satisfies a two-point solution, and unknown constants of the first and second iterative algorithm can be determined iteratively. With regard to numerical stability and computational efficiency, the proposed algorithms avoid directly solving the nonlinear governing equations and instead use the BSF indirectly to iteratively satisfy the two-point solution on the BCs. The proposed algorithms successfully avoid the need for step-by-step adjustment of the weighting value and multisolution problems by the LGSM. Simultaneously, the BSF does not need to use high-order series to obtain approximate solutions. Numerical tests confirmed that the proposed methods are straightforward, easy to implement, and able to approximate the true solution very accurately. Moreover, the numerical solution precisely satisfied the nonlinear integral boundary conditions of the NBVP. The second iterative algorithm is slightly better than the first iterative algorithm, with fewer iterations and a more accurate solution. Future work will extend the BFS to solve optimization parameters and period orbits of chaos in terms of nonlinear problems.

Declaration of competing interest

The authors have no conflict of interest to declare.

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References


