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RESEARCH ARTICLE

Solving Nonlinear Boundary Value Problems with Nonlinear Integral Boundary Conditions by Local and Nonlocal Boundary Shape Functions Methods

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Abstract

The paper considers the second-order nonlinear boundary value problem (NBVP), which is equipped with nonlinear integral boundary conditions (BCs). Two novel iterative algorithms are developed to overcome the difficulty of NBVP with double nonlinearities involved. In the first iterative algorithm, two nonlocal shape functions incorporating the linear integral terms are derived, and a nonlocal boundary shape function (NBSF) is formulated to assist the solution. Let the solution be the NBSF so that the NBVP can be exactly transformed into an initial value problem. The new variable is a free function in the NBSF, and its initial values are given. For the NBVP with linear integral BCs, three unknown constants are to be determined, while for the nonlinear integral BCs, five unknown constants are to be determined. Two-point local shape functions and local boundary shape functions are derived for the second iterative algorithm, wherein the integral terms in the boundary conditions are viewed as unknown constants. By a few iterations, four unknown constants can be determined quickly. Through numerical experiments, these two iterative algorithms are found to be powerful for seeking quite accurate solutions. The second algorithm is slightly better than the first, with fewer iterations and a more accurate solution.

Keywords: Nonlinear BVP, Nonlinear integral boundary conditions, Nonlocal shape functions, Local shape functions, Iterative algorithms

1. Introduction

Nonlinear boundary value problems (NBVPs) are frequently encountered in scientific and engineering problems. In particular, second-order NBVPs have been studied extensively. Agarwal [1] investigated the existence and uniqueness of solutions of BVPs of higher order. To solve NBVPs, Kubíček and Hlaváček [2] conducted a complete survey of the development, analysis, and application of numerical techniques. Keller [3] described an elementary yet rigorous account of practical numerical methods for solving general two-point boundary value problems (BVPs).

Recently, Hajipou et al. [4] and Jajarmi and Baleanu [5] proposed a new iterative method to solve high-order nonlinear fractional boundary value problems. Then, different studied views further are applied to analyze nonlinear boundary value problems such as stability [6], nonlinear fractional-order derivatives [7] and optimal control problems [8,9]. Mahariq [10] and Mahariq et al. [11−13] applied the spectral element method to solve the application field's electromagnetic and photonic nanojet problems.

In this paper, we consider nonlinear and nonlocal boundary conditions (BCs) for second-order NBVPs. They are different from the conventional two-point

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BCs, which are specified at boundary points of a given interval. The nonlocal BCs involving certain integrals are specified for the solution at all points within the given interval. Therefore, the nonlocal NBVP is more challenging to solve than the local NBVP. As defined by Lin et al. [14], the boundary shape function (BSF) automatically satisfies the BCs, which includes the solution of BVP as a special case since the solution must exactly satisfy the specified BCs. Liu and Chang [15] extended the work of Lin et al. [14] to address the multipoint boundary conditions for NBVPs. Liu and Chang [16] applied the BSF to solve nonlinear singularly perturbed problems with Robin boundary conditions. Furthermore, Liu [17] used the BSF to analyze nonlinear composite beams subjected to nonlinear boundary moment conditions. The idea of the BSF has been adopted to solve some BVPs with conventional local BCs and then extended to solve 2D and 3D nonlinear problems [18–20]. However, the method of BSF has not yet been developed for solving the NBVP endowed with nonlinear and nonlocal BCs.

For the numerical method of nonlocal BVP, it is of utmost importance to preserve the given BCs. However, this is not an easy task when the given BCs involve the solution in the entire interval, which is itself an unknown function in the interval. In the case of an NBVP subject to nonlinear and nonlocal BCs, reducing the boundary error and then the error of the solution in the entire interval is an important issue. Hereon, we attempt to develop novel methods for providing accurate numerical solutions to nonlinear and nonlocal NBVPs. To exactly preserve the nonlinear and nonlocal BCs, we formulate numerical algorithms based on local and nonlocal boundary shape functions.

In this paper, we also cover the Duffing-type NBVP, which is a nonlinear ordinary differential equation (ODE) well-known in applied science as a powerful model to discuss practical phenomena such as nonlinear mechanical oscillators, bending models of DNA, and the prediction of diseases. Some works on the forced Duffing equation with integral boundary conditions [21–24] are effective methods for solving the NBVP with linear integral BCs [24–26]. To date, most papers have developed numerical methods for solving the Duffing-type NBVP with linear integral BCs [23,27,28]. The NBVP with nonlinear integral BCs is more difficult; hence, few papers are devoted to solving this problem.

To address nonlinear and nonlocal BCs, we will develop two novel iterative algorithms to determine the solution for the NBVP with nonlinear integral boundary conditions. For both iterative algorithms, the basic idea is to transform the NBVP to the corresponding initial value problem for the new variable, whose initial conditions are given. In the transformed ODE for the new variable, some unknown constants need to be determined. Detailed descriptions of local and nonlocal BSFs are provided in the next section. The paper is structured as follows. In Section 2, we derive two nonlocal shape functions and nonlocal boundary shape functions for a second-order NBVP. An iterative algorithm for linear integral BCs of the second-order NBVP is developed in Section 3, where three unknown constants are to be determined. Section 4 gives the first iterative algorithm for the nonlinear integral BCs of the second-order NBVP, where five unknown constants are to be determined. In Section 5, we develop a second iterative algorithm for the nonlinear integral BCs of the second-order NBVP, where four unknown constants are to be determined. In Section 6, several examples are tested. Finally, Section 7 draws conclusions.

2. Nonlocal boundary shape function

Consider a second-order nonlinear ODE:

$$u''(x) = f(x, u(x), u'(x)), \quad 0 < x < 1, \tag{1}$$

where f satisfies the Lipschitz condition, which is endowed with the integral BCs:

$$a_1 u(0) + b_1 u'(0) = \int_0^1 q_1(x, u(x))dx, \tag{2}$$

$$a_2 u(1) + b_2 u'(1) = \int_0^1 q_2(x, u(x))dx.$$

When q_1 and q_2 are constants, the Robin-type BCs are specified at two boundary points. When q_1 and q_2 are linear functions of u, the BCs are linear integral BCs; otherwise, they are nonlinear integral BCs.

To explore the new iterative method more clearly, we start from the linear integral BCs:

$$a_1 u(0) + b_1 u'(0) - \rho_1 \int_0^1 u(x)dx = p_1, \tag{3}$$

$$a_2 u(1) + b_2 u'(1) - \rho_2 \int_0^1 u(x)dx = p_2, \tag{4}$$

which are obtained from Eq. (2) by inserting $q_1 = \rho_1 u + p_1$ and $q_2 = \rho_2 u + p_2$.

Upon defining linear operators

$$L_1\{u(x)\} := a_1u(0)+b_1u'(0)-\rho_1\int_0^1 u(x)dx, \tag{5}$$

$$L_2\{u(x)\} := a_2u(0)+b_2u'(0)-\rho_2\int_0^1 u(x)dx, \tag{6}$$

Eqs. (3) and (4) can be written as

$$L_1\{u(x)\}=p_1,\ L_2\{u(x)\}=p_2. \tag{7}$$

Theorem 1. If there are nonlocal shape functions $s_1(x)$ and $s_2(x)$ satisfying

$$L_1\{s_1(x)\}=1,\ L_2\{s_1(x)\}=0, \tag{8}$$

$$L_1\{s_2(x)\}=0,\ L_2\{s_2(x)\}=1 \tag{9}$$

then for any free function $z(x)\in C^1[0,1]$,

$$\begin{aligned} u(x) = z(x) - s_1(x)\big[L_1\{z(x)\}-p_1\big] \\ - s_2(x)\big[L_2\{z(x)\}-p_2\big], \end{aligned} \tag{10}$$

which satisfies the linear integral BCs (3) and (4).
Proof. We first prove Eq. (3). Applying L_1 to Eq. (10) and using the linear property of L_1, we have

$$\begin{aligned} L_1\{u(x)\} = L_1\{z(x)\} - L_1\{s_1(x)\}\big(L_1\{z(x)\}-p_1\big) \\ - L_1\{s_2(x)\}\big(L_2\{z(x)\}-p_2\big), \end{aligned}$$

which, with the aid of the first equations in Eqs. (8) and (9) becomes

$$L_1\{u(x)\}=L_1\{z(x)\}-\big(L_1\{z(x)\}-p_1\big)=p_1.$$

Similarly, applying L_2 to Eq. (10) and using the linear property of L_2 yields

$$\begin{aligned} L_2\{u(x)\} = L_2\{z(x)\} - L_2\{s_1(x)\}\big(L_1\{z(x)\}-p_1\big) \\ - L_2\{s_2(x)\}\big(L_2\{z(x)\}-p_2\big), \end{aligned}$$

which, with the aid of the second equations in Eqs. (8) and (9) becomes

$$L_2\{u(x)\}=L_2\{z(x)\}-\big(L_2\{z(x)\}-p_2\big)=p_2.$$

We have proven Eq. (7), and thus Eqs. (3) and (4) are proven.

3. Numerical algorithm for linear integral BCs

Eqs. (8) and (9), including the integrals of $s_1(x)$ and $s_2(x)$, are called nonlocal shape functions (NSFs). With the help of Theorem 1, a feasible and efficient way to obtain $u(x)$ is transformed into a new variable $z(x)$ by

$$z(x)=u(x)+W(x), \tag{11}$$

where

$$\begin{aligned} W(x) = \big[L_1\{z(x)\}-p_1\big]s_1(x) \\ + \big[L_2\{z(x)\}-p_2\big]s_2(x). \end{aligned} \tag{12}$$

If we can determine $z(x)$, then Eqs. (3) and (4) are automatically satisfied by $u(x) = z(x) - W(x)$, as proven in Theorem 1.

3.1. Initial value problem

From Eqs. (1) and (11), $z(x)$ is governed by a new ODE:

$$\begin{aligned} z''(x) &= H(x,z(x),z'(x)), \\ &:= W(x)+f(x,z(x)-W(x),z'(x)-W'(x)), \end{aligned} \tag{13}$$

where $W(x)$ given by Eq. (12) involves three unknown constants:

$$\alpha := z(1),\ \beta := z'(1),\ \gamma := \int_0^1 z(x)dx, \tag{14}$$

as shown in Eqs. (5) and (6). Eq. (13) is an initial value problem (IVP), upon giving the initial values $z(0)$ and $z'(0)$.

In Theorem 1, $s_1(x)$ and $s_2(x)$ can be determined as follows. Suppose that $s_1(x) = a + bx + cx^2$ and insert it into Eq. (8). Then, we can obtain

$$aa_1+b_1b-\rho_1\left[a+\frac{b}{2}+\frac{c}{3}\right]=1, \tag{15}$$

$$a_2[a+b+c]+b_2[b+2c]-\rho_2\left[a+\frac{b}{2}+\frac{c}{3}\right]=0, \tag{16}$$

which are underdetermined linear systems to determine a, b, and c. There are many solutions of a, b, and c. Similarly, we can do this for $s_2(x)$. We choose the suitable values of a, b, and c such that the Runge−Kutta method RK4 can be applied to integrate the ODE (13) with the given initial values $z(0)$ and $z'(0)$.

3.2. Iterative algorithm

When RK4 is adopted to integrate ODE (13) with the given initial conditions, we can iteratively determine α, β, and γ until they are convergent as follows: (i) given $z_1(0) = a_0$, $z_2(0) = b_0$, α_0, β_0, γ_0 and ϵ; (ii) for $k = 0, 1, 2, \ldots$, integrate

$$z_1'(x)=z_2(x),$$

$$z_2'(x) = H(x, z_1(x), z_2(x); \alpha_k, \beta_k, \gamma_k), \tag{17}$$

$$z_3'(x) = z_1(x), \; z_3'(x) = z_1(x),$$

where $z_1(0) = a_0$, $z_2(0) = b_0$, and e given initial values. Take

$$\alpha_{k+1} = z_1(1), \; \beta_{k+1} = z_2(1), \; \gamma_{k+1} = z_3(1). \tag{18}$$

Until

$$r_k := \sqrt{\left\{ \begin{array}{c} (\alpha_{k+1} - \alpha_k)^2 + (\beta_{k+1} - \beta_k)^2 \\ + (\gamma_{k+1} - \gamma_k)^2 \end{array} \right\}} < \grave{o} \tag{19}$$

is satisfied. If Eq. (19) is not fulfilled, then go to (ii) for the next iteration.

4. First iterative algorithm for nonlinear integral BCs

When Eq. (1) is subjected to the nonlinear integral BCs in Eq. (2), the BVP is more challenging to solve than that with linear integral BCs (3) and (4). We suppose that q_1 and q_2 can be decomposed as

$$q_2(x, u) = \rho_2 u + p_2 + F_2(x, u), \tag{20}$$

where F_1 and F_2 are nonlinear functions of u. Let

$$\begin{aligned} c_1 &:= p_1 + \int_0^1 F_1(x, u(x))dx, \\ c_2 &:= p_2 + \int_0^1 F_2(x, u(x))dx, \end{aligned} \tag{21}$$

where unknown constants c_1 and c_2 are to be determined. Eq. (2) is rewritten as

$$\begin{aligned} a_1 u(0) + b_1 u'(0) - \rho_1 \int_0^1 u(x)dx &= c_1, \\ a_2 u(1) + b_2 u'(1) - \rho_2 \int_0^1 u(x)dx &= c_2. \end{aligned} \tag{22}$$

4.1. Transformation to IVP

Let

$$u(x) = y(x) + G(x), \tag{23}$$

$$G(x) = [L_1\{y(x)\} - c_1]s_1(x) + [L_2\{y(x)\} - c_2]s_1(x), \tag{24}$$

such that $u(x)$ satisfies Eq. (22) automatically, as proven in Theorem 1.

It follows from Eqs. (1) and (23) that

$$\begin{aligned} y''(x) &= F(x, y(x), y'(x)), \\ &:= G''(x) + f(x, f(x) - G(x), y'(x) - G'(x)), \end{aligned} \tag{25}$$

where we can give $y(0)$ and $y'(0)$ as the initial values for y, while $y(1)$, $y'(1)$, $\gamma := \int_0^1 y(x)dx$, c_1 and c_2 are five unknown values in $G(x)$ as shown by Eq. (24).

4.2. First iterative algorithm

For solving the NBVP in Eqs. (1) and (22), we list the iterative algorithm. (i) Given $y_1(0)$, $y_2(0)$, d_0, e_0, γ_0, c_1^0, c_2^0, ϵ, and N; (ii) for $k = 0, 1, 2, \ldots$, apply the RK4 to integrate

$$\begin{aligned} y_1'(x) &= y_2(x), \\ y_2'(x) &= F(x, y_1(x), y_2(x); d_k, e_k, \gamma_k, c_1^k, c_2^k), \\ y_3'(x) &= y_1(x), \\ y_4'(x) &= F_1(x, y_1(x) - G(x)) + p_1, \\ y_5'(x) &= F_2(x, y_1(x) - G(x)) + p_2, \end{aligned}$$

where $y_3(0) = y_4(0) = y_5(0) = 0$. Take $d_{k+1} = y_1(1)$, $e_{k+1} = y_2(1)$, $\gamma_{k+1} = y_3(1)$, $c_1^{k+1} = y_4(1)$, and $c_2^{k+1} = y_5(1)$.

We terminate the iterations if the residual satisfies

$$\sqrt{\left\{ \begin{array}{c} (d_{k+1} - d_k)^2 + (e_{k+1} - e_k)^2 + (\gamma_{k+1} - \gamma_k)^2 \\ + (c_1^{k+1} - c_1^k)^2 + (c_2^{k+1} - c_2^k)^2 \end{array} \right\}} < \grave{o};$$

otherwise, go to (ii) for the next iteration.

When c_1^k and c_2^k are convergent, the solution $u(x)$ is given by

$$u(x) = y(x) - s_1(x)\left[L_1\{y(x)\} - c_1^k\right] - s_2(x)\left[L_1\{y(x)\} - c_2^k\right].$$

5. Second iterative algorithm for nonlinear integral BC

In Section 4, five unknown constants d, e, γ, c_1, and c_2 are determined. Instead of the nonlocal shape functions $s_1(x)$ and $s_2(x)$, we may consider two-point local shape functions $T_1(x)$ and $T_2(x)$ as determined by

$$a_1 T_1(0) + b_1 T_1'(0) = 1, \; a_2 T_1(0) + b_2 T_1'(0) = 0, \tag{26}$$

$$a_1 T_2(0) + b_1 T_2'(0) = 0, \; a_2 T_2(0) + b_2 T_2'(0) = 1. \tag{27}$$

In Eq. (2), we let

$$c_1 := \int_0^1 q_1(x, u(x))dx, \; c_2 := \int_0^1 q_2(x, u(x))dx \tag{28}$$

be unknown constants to be determined. Then, replacing $s_1(x)$ and $s_2(x)$ in Section 4 with $T_1(x)$ and $T_2(x)$, we have the following iterative algorithm: (i) given $y_1(0)$, $y_2(0)$, d_0, e_0, c_1^0, c_2^0, ϵ, and N; (ii) for $k = 0, 1, 2, \ldots$, apply RK4 with a step size $\Delta x = 1/N$ to integrate

$$
\begin{aligned}
&Q(x) = T_1(x)\left[a_1 y(0) + b_1 y'(0) - c_1^k\right] + \\
&\qquad T_2(x)\left[a_2 d_k + b_2 e_k - c_2^k\right], \\
&y_1'(x) = y_2(x), \\
&y_2'(x) = F\left(x, y_1(x), y_2(x); d_k, e_k, c_1^k, c_2^k\right), \\
&y_3'(x) = q_1\left(x, y_1(x) - Q(x)\right), \\
&y_4'(x) = q_2\left(x, y_1(x) - Q(x)\right),
\end{aligned}
$$

where $y_3(0) = y_4(0) = 0$ and $F = Q''(x) + f(x, y(x) - Q(x), y'(x) - Q'(x))$. Take $d_{k+1} = y_1(1)$, $e_{k+1} = y_2(1)$, $c_1^{k+1} = y_3(1)$, and $c_2^{k+1} = y_4(1)$.

The iterations are terminated if the residual satisfies

$$
\sqrt{\left\{\begin{array}{l} (d_{k+1} - d_k)^2 + (e_{k+1} - e_k)^2 + \\ (c_1^{k+1} - c_1^k)^2 + (c_2^{k+1} - c_2^k)^2 \end{array}\right\}} < \text{ò};
$$

otherwise, go to the next iteration in (ii).

The first iterative algorithm, as presented in Section 4.2, is reduced to the present algorithm if $\rho_1 = \rho_2 = 0$ since $s_1(x) = T_1(x)$ and $s_2(x) = T_2(x)$, and the parameter γ in the first iterative algorithm is no longer needed. However, when $\rho_1 \neq 0$ and $\rho_2 \neq 0$, the first iterative algorithm is different from the second iterative algorithm. The latter is more efficient since only four unknown constants d, e, c_1, and c_2 are to be determined. The second iterative algorithm can be used to solve the NBVP with the nonlinear integral BCs in Eq. (2).

6. Numerical examples

6.1. Example 1

Consider an NBVP

$$u''(x) + u'(x) - u^2(x) = 3e^x + 2xe^x - x^2 e^{2x}, \tag{29}$$

$$
\begin{aligned}
&u(0) + u'(0) - \int_0^1 u(x)dx = 0, \\
&u(1) - u'(1) - 2\int_0^1 u(x)dx = e - 2,
\end{aligned} \tag{30}
$$

whose exact solution is

$$u(x) = xe^x. \tag{31}$$

In Eq. (30), $a_1 = 1$, $b_1 = 1$, $\rho_1 = 1$, $p_1 = 0$, $F_1 = 0$, $a_2 = 1$, $b_2 = -1$, $\rho_2 = 2$, $p_2 = e - 2$, and $F_2 = 0$. For this problem, $s_1(x) = -2/3 + 2x$ and $s_2(x) = -2x - 3x^2$ are derived. Utilizing $z(0) = 0$, $z'(0) = 1$, $\alpha_0 = \beta_0 = \gamma_0 = 0$, $\epsilon = 10^{-10}$, and $N = 500$, the NBSF method converges, as shown in Fig. 1(a), with 84 iterations. Upon comparing numerical and exact solutions, Fig. 1(b) displays the absolute numerical error whose maximum error (ME) is 6.61×10^{-12}, which is very accurate.

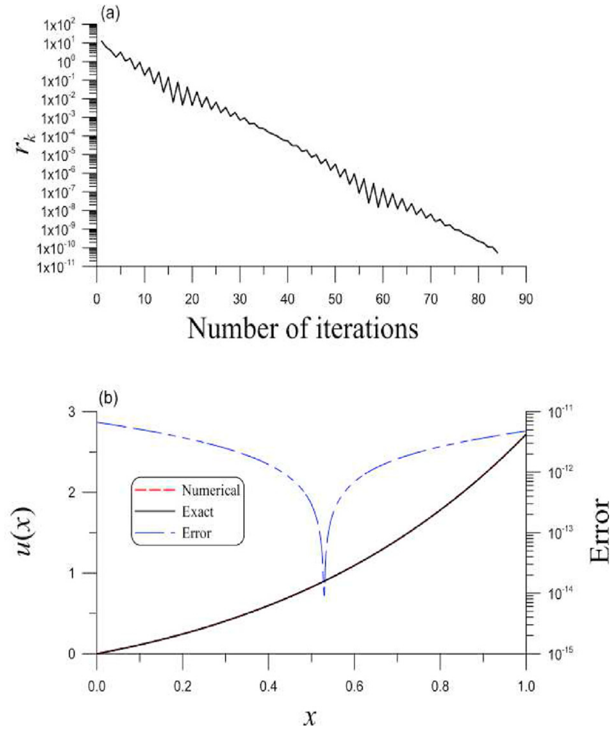

Fig. 1. For example 1 with NBSF method: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

The above results are computed by the method in Section 3.2, which is a special case of the first iterative algorithm in Section 4.2 with $F_1 = 0$ and $F_2 = 0$. Next, we apply the second iterative algorithm in Section 5 to solve this problem, of which $T_1(x) = x$ and $T_2(x) = x - 1$ are derived. In Eq. (30), $a_1 = 1$, $b_1 = 1$, $q_1 = u$, $a_2 = 1$, $b_2 = -1$, and $q_2 = 2u + e - 2$. Taking $y(0) = y'(0) = \alpha_0 = \beta_0 = c_1^0 = c_2^0 = 0$, $\epsilon = 10^{-10}$, and $N = 500$, as shown in Fig. 2(a), the second iterative algorithm is convergent within 19 iterations. Upon comparing numerical and exact solutions, Fig. 2(b) displays the absolute numerical error, whose ME $= 1.31 \times 10^{-12}$ is very accurate.

The method in Section 5 converges more quickly and more accurately than that in Section 3.2, where 84 iterations and ME $= 6.61 \times 10^{-12}$ were obtained for this problem.

6.2. Example 2

The following NBVP is to be solved:

$$u''(x) + u'(x) - 2u(x) + 2\sin^2(x) = 0, \tag{32}$$

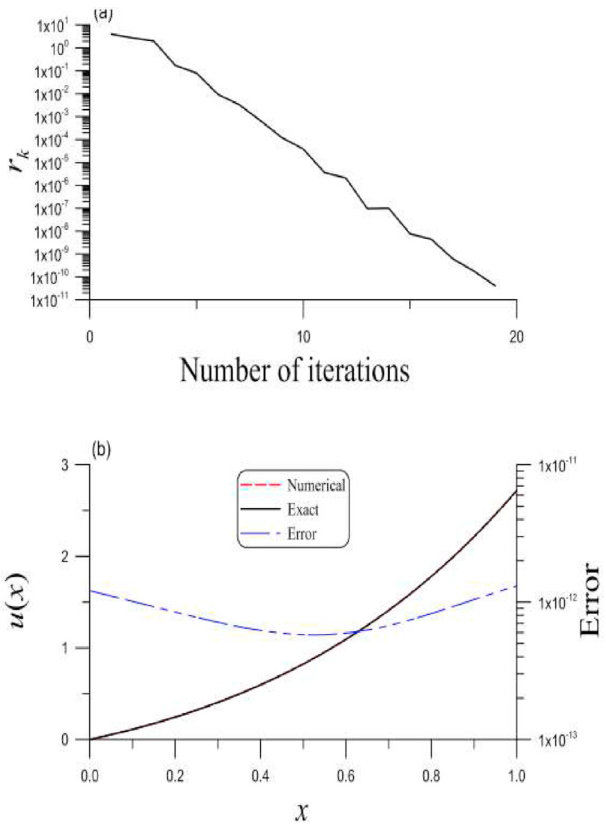

Fig. 2. Example 1 is solved by second iterative algorithm: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

$$
\begin{aligned}
&u(0)-\frac{3}{8}u'(0)-\frac{1}{4}\int_0^1 u(x)dx-\int_0^1 u^2(x)dx=-1,\\
&u(1)+\frac{1}{4}u'(1)-\frac{1}{2}\int_0^1 u(x)dx=1,
\end{aligned}
\tag{33}
$$

of which no closed-form solution is available.

In Eq. (33), $a_1=1$, $b_1=-3/8$, $\rho_1=1/4$, $p_1=-1$, $F_1=u^2$, $a_2=1$, $b_2=1/4$, $\rho_2=1/2$, $p_2=1$, and $F_2=0$. For this problem, $s_1(x)=1-x/2$ and $s_2(x)=1+23x/14-3x^2/7$ are derived. Because a nonlinear integral term occurs on the left side, only c_1^k. is unknown.

Utilizing $y(0)=y'(0)=d_0=e_0=c_1^0=\gamma_0=0$, $\epsilon=10^{-10}$ and $N=100$, as shown in Fig. 3(a), the first iterative algorithm converges with 45 iterations, and in Fig. 3(b), we plot the numerical solution. Since there exists no exact solution, we assess the numerical error by showing the absolute errors of BCs, which are 1.5×10^{-8} for the left boundary condition and 2.4×10^{-8} for the right boundary condition in Eq. (33).

The above results are computed by the first iterative algorithm in Section 4.2. Next, we apply the second iterative algorithm in Section 5 to solve this problem. In Eq. (33), $a_1=1$, $b_1=-3/8$, $q_1=u^2+u/4-1$, $a_2=1$, $b_2=1/4$, and $q_2=u/2+1$. $T_1(x)=10/13-8x/13$ and $T_2(x)=3/13+8x/13$ are derived. Taking $y(0)=0$, $y'(0)=1$, $\alpha_0=\beta_0=c_1^0=c_2^0=0$, $\epsilon=10^{-5}$, and $N=100$, the second iterative algorithm converges with 44 iterations. The absolute errors of BCs are 8.74×10^{-11} for the left boundary condition and 4.35×10^{-9} for the right boundary condition in Eq. (33). The second iterative algorithm is more accurate than the first iterative algorithm.

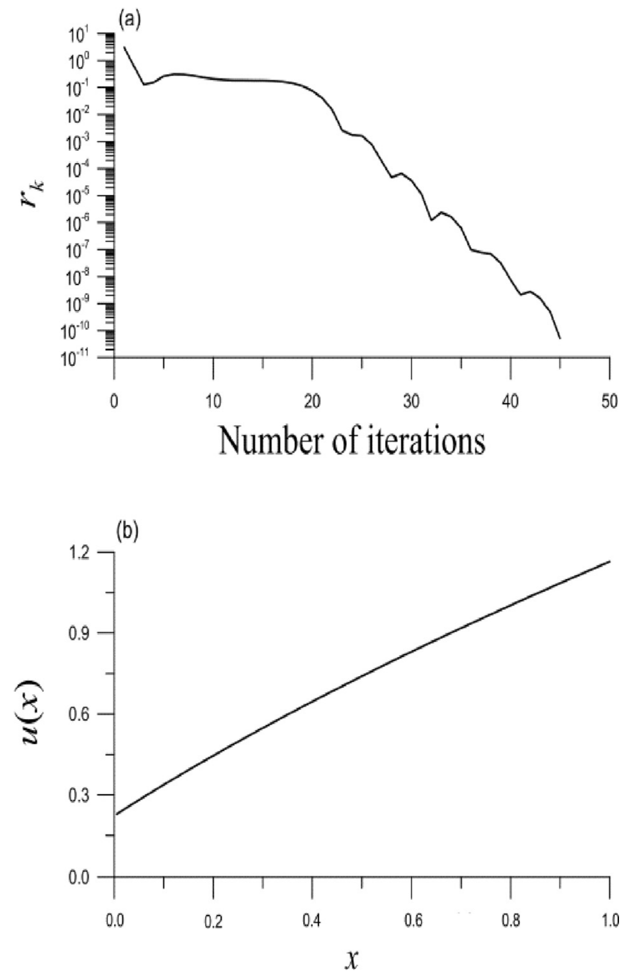

Fig. 3. For example 2: (a) convergence of absolute error of iterations, and (b) numerical solution.

6.3. Example 3

Consider [28]:

$$u''(x)+u'(x)+x(1-x)u^3(x)=F(x), \tag{34}$$

$$
\begin{aligned}
&u(0)-\frac{2}{\pi^2}u'(0)=-\int_0^1 u(x)dx,\\
&u(1)+\frac{1}{\pi^2}u'(1)=-\int_0^1 xu(x)dx,
\end{aligned}
\tag{35}
$$

$$u(x) = \sin(\pi x). \tag{36}$$

$F(x)$ can be obtained by inserting $u(x) = \sin(\pi x)$ into Eq. (34).

In Eq. (35), $a_1 = 1$, $b_1 = -2/\pi$, $q_1 = -u$, $a_2 = 1$, $b_2 = 1/\pi^2$, and $q_2 = xu^2$. For this problem, we take $T_1(x) = (\pi^2+1)/(\pi^2+3) - \pi^2 x/(\pi^2+3)$ and $T_2(x) = 2/(\pi^2+3) + \pi^2 x/(\pi^2+3)$. Utilizing $y(0) = 0$, $y'(0) = 2$, $d_0 = e_0 = c_1^0 = c_2^0 = 0$, $\epsilon = 10^{-10}$, and $N = 100$, the second iterative algorithm converges with 77 iterations, and the maximum relative error is 2.54×10^{-8}.

To compare with the reference shown in Table 1, we list the relative errors at some points and compare the results to those obtained by Geng and Cui [28]. For this problem, it can be seen that the presented accuracy obtained by the second iterative algorithm is approximately four orders of magnitude more accurate than that obtained by Geng and Cui [28].

6.4. Example 4

Let us consider

$$u''(x) = \frac{3}{2}u^2(x), \tag{37}$$

$$u(0) = 4, \ u(1) = \int_0^1 \left[u^2(x) - \frac{11}{3}\right]dx, \tag{38}$$

$$u(x) = \frac{4}{(1+x)^2}. \tag{39}$$

In Eq. (38), $a_1 = 1$, $b_1 = 0$, $q_1 = 4$, $a_2 = 1$, $b_2 = 0$, and $q_2 = u^2 - 11/3$. Because a nonlinear integral term occurs on the right side, only c_2^k is unknown. We derive $T_1(x) = 1 - 2x + x^2$ and $T_2(x) = x^2$. Using $y(0) = y'(0) = d_0 = c_2^0 = 0$, $\epsilon = 10^{-6}$, and $N = 100$, the second iterative algorithm converges with two iterations, as shown in Fig. 4(a). In Fig. 4(b), we compare the solutions, and the numerical error is ME $= 7.92 \times 10^{-9}$. This value is smaller than that obtained in [29], where the Lie-group shooting method (LGSM) was adopted to solve the problem under the usual boundary condition:

$$u(0) = 4, \ u(1) = 1.$$

The results show that the proposed algorithm avoids the need for step-by-step adjustment of the weighting value and multisolution problems by the LGSM. The proposed method is very efficient and stable in approximating the true solution.

Table 1. *For example 3: comparison of relative errors at some points.*

x	Exact $u(x)$	[28]	Present
0.01	0.314108	7.49×10^{-5}	2.48×10^{-9}
0.08	0.248690	8.17×10^{-5}	1.07×10^{-10}
0.16	0.481754	8.11×10^{-5}	5.03×10^{-11}
0.32	0.844328	8.09×10^{-5}	1.62×10^{-10}
0.48	0.998027	8.08×10^{-5}	3.12×10^{-10}
0.64	0.904827	8.03×10^{-5}	5.82×10^{-10}
0.80	0.587785	7.97×10^{-5}	1.26×10^{-9}
0.96	0.125333	7.85×10^{-5}	6.46×10^{-9}

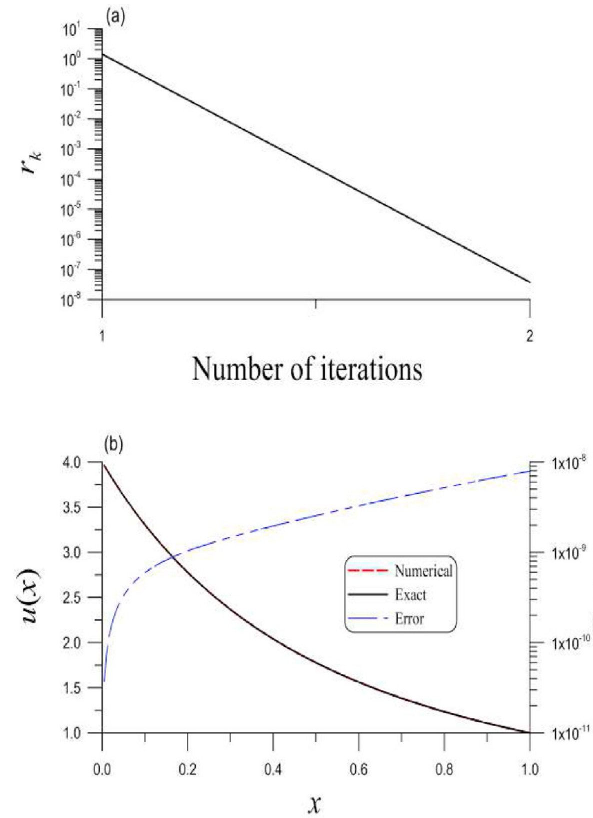

Fig. 4. For example 4: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

6.5. Example 5

Consider Eq. (29) again, which is now subjected to nonlinear integral BCs

$$u(0) - \int_0^1 \left[u^2(x) - \frac{e^2-1}{4}\right]dx = 0, \quad u(1) - \int_0^1 \left[u^2(x) - \frac{e^2-1}{4} + e\right]dx = 0, \tag{40}$$

the exact solution is still given by Eq. (31).

In Eq. (40), $a_1 = 1$, $b_1 = 0$, $q_1 = u^2 - (e^2-1)/4$, $a_2 = 1$, $b_2 = 0$, and $q_2 = u^2 - (e^2-1)/4 + e$. We derive $T_1(x) = 1 - x$ and $T_2(x) = x$. Because integral BCs occur on both sides, c_1^k and c_2^k are unknown constants. Utilizing $y(0) = 0$, $y'(0) = 1$, $d_0 = e_0 = c_1^0 = c_2^0 = 0$, $\epsilon = 10^{-10}$, and $N = 500$, the second iterative algorithm converges with two iterations, as shown in Fig. 5(a). The absolute numerical error is shown in Fig. 5(b), with ME $= 3.53 \times 10^{-11}$ being very accurate.

6.6. Example 6

Let us consider the following NBVP [2,29]:

$$u''(x)+\frac{1}{x}u'(x)=-\delta e^{u(x)}, \tag{41}$$

$$u'(x)=0,$$
$$u(1)+u'(1)=\int_0^1\left[e^{u(x)/2}-2-\frac{\pi}{2}\right]dx=0. \tag{42}$$

For $\delta = 2$, there is only one solution:

$$u(x)=\ln\frac{4}{(1+x^2)^2}. \tag{43}$$

In Eq. (42), $a_1 = 0$, $b_1 = 0$, $q_1 = 0$, $a_2 = 1$, $b_2 = 1$, and $q_2 = e^{u/2} - 2 - \pi/2$. Because the integral boundary condition occurs on the right side, only c_2^k is unknown. We obtain $T_1(x) = x - 2x^2/3$ and $T_2(x) = x^2 - 2$. Taking $y(0) = 0$, $y'(0) = 0.5$, $d_0 = e_0 = 0$, $c_2^0 = -1.5$, $\epsilon = 10^{-2}$, and $N = 500$, the second iterative algorithm converges with 89 iterations, as shown in Fig. 6(a), and the absolute numerical error is shown in Fig. 6(b) with ME $= 1.13 \times 10^{-3}$. This is more accurate than that in [29], who employed the LGSM to solve Eqs. (41) and (42).

6.7. Example 7

Let us consider the following NBVP:

$$u''(x)+u'(x)+u^3(x)=3+2x+\left(x+x^2\right)^3, \tag{44}$$

$$u(0)+\int_0^1 u(x)dx+\int_0^1 u^2(x)dx=\frac{28}{15},$$
$$u(1)-\frac{1}{2}\int_0^1(1+2x)u(x)dx=0, \tag{45}$$

$$u(x)=x+x^2. \tag{46}$$

In Eq. (45), $a_1 = 1$, $b_1 = 0$, $q_1 = (28/15) - u - u^2$, $a_2 = 1$, $b_2 = 0$, $q_2 = (1 + 2x)u$, $T_1(x) = 1 - x$, and $T_2(x) = x$ are derived. In the second iterative algorithm, we take $y(0) = 0$, $y'(0) = 1$, $d_0 = e_0 = c_1^0 = c_2^0 = 0$, $\epsilon = 10^{-6}$, and $N = 100$. This converges with

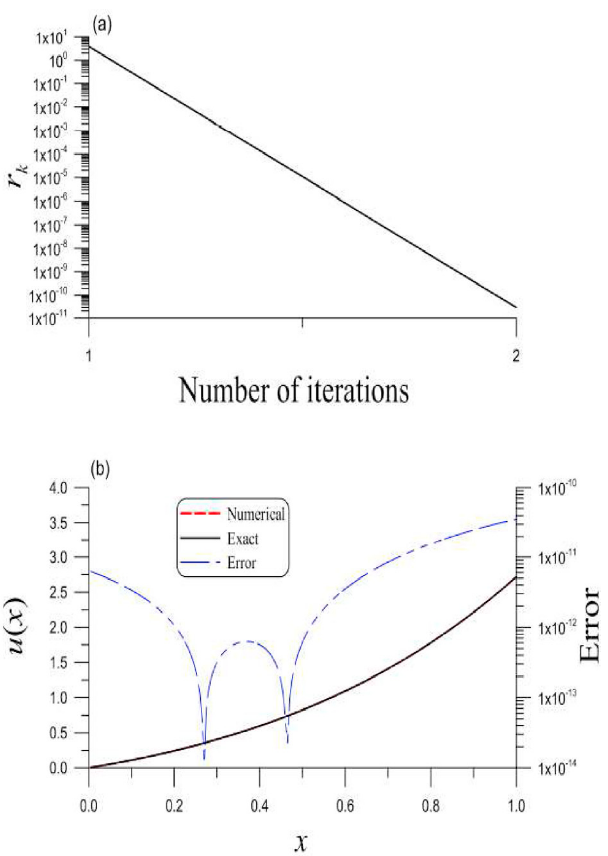

Fig. 5. For example 5: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

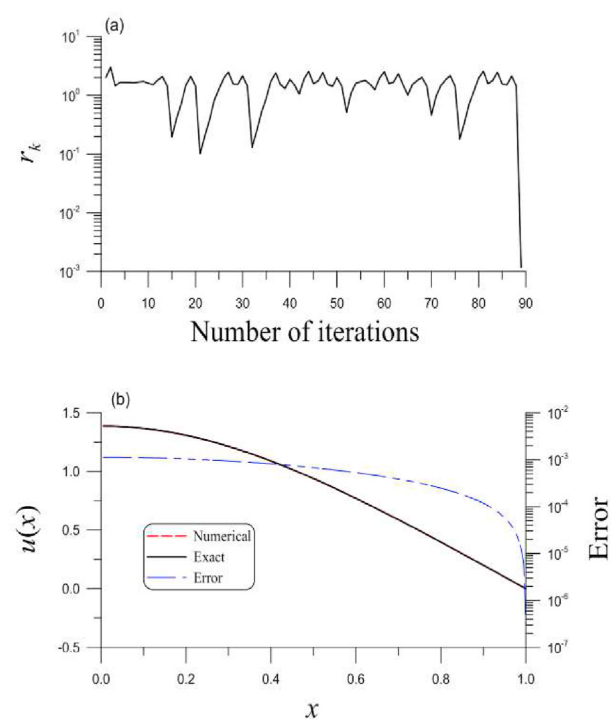

Fig. 6. For example 6: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

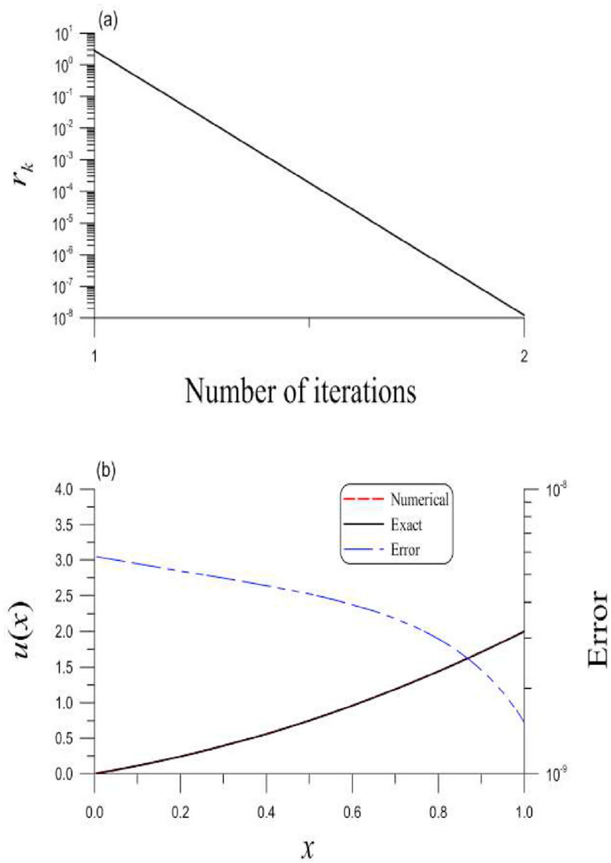

Fig. 7. For example 7: (a) convergence of absolute error and (b) numerical and exact solutions and absolute error.

two iterations, as shown in Fig. 7(a). The absolute numerical error is shown in Fig. 7(b) with $\mathrm{ME} = 5.78 \times 10^{-9}$.

7. Conclusions

In this paper, nonlocal boundary shape functions and local boundary shape functions were demonstrated. Using a new concept of these functions, we developed two novel iterative algorithms to determine the solution for the NBVP with nonlinear integral boundary conditions. For both iterative algorithms, the basic idea is to transform the NBVP to the corresponding initial value problem for the new variable, whose initial conditions are given. In the transformed ODE for the new variable, the BSF automatically satisfies a two-point solution, and unknown constants of the first and second iterative algorithm can be determined iteratively. With regard to numerical stability and computational efficiency, the proposed algorithms avoid directly solving the nonlinear governing equations and instead use the BSF indirectly to iteratively satisfy the two-point solution on the BCs. The proposed algorithms successfully avoid the need for step-by-step adjustment of the weighting value and multi-solution problems by the LGSM. Simultaneously, the BSF does not need to use high-order series to obtain approximate solutions. Numerical tests confirmed that the proposed methods are straightforward, easy to implement, and able to approximate the true solution very accurately. Moreover, the numerical solution precisely satisfied the nonlinear integral boundary conditions of the NBVP. The second iterative algorithm is slightly better than the first iterative algorithm, with fewer iterations and a more accurate solution. Future work will extend the BFS to solve optimization parameters and period orbits of chaos in terms of nonlinear problems.

Declaration of competing interest

The authors have no conflict of interest to declare.

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References

[1] Agarwal RP. Boundary value problems for high order differential equations. Singapore: World Scientific; 1986.
[2] Kubíček M, Hlaváček V. Numerical solution of nonlinear boundary value problems with applications. New York: Prentice-Hall; 1983.
[3] Keller HB. Numerical methods for two-point boundary value problems. New York: Dover; 1992.
[4] Hajipou M, Jajarmi A, Baleanu D. On the accurate discretization of a highly nonlinear boundary value problem. Numer Algorithm 2018;79:679−95.
[5] Jajarmi A, Baleanu D. A new iterative method for the numerical solution of high-order non-linear fractional boundary value problems. Front Physiol 2020;8(220):1−8.
[6] Ali A, Mahariq I, Shah K, Abdeljawad T, Al-Sheikh B. Stability analysis of initial value problem of pantograph-type implicit fractional differential equations with impulsive conditions. Adv Differ Equ 2021;2021(55):1−14.
[7] Ali A, Shah K, Abdeljawad T, Mahariq I, Rashdan M. Mathematical analysis of nonlinear integral boundary value problem of proportional delay implicit fractional differential equations with impulsive conditions. Bound Value Probl 2021;2021(7):1−27.
[8] Baleanu D, Sajjadi SS, Jajarmi A, Ö Defterli. On a nonlinear dynamical system with both chaotic and non-chaotic behaviours: a new fractional analysis and control. Adv Differ Equ 2021;2021(234):1−17.
[9] Jajarmi A, Baleanu D, Zarghami Vahid K, Mobayen S. A general fractional formulation and tracking control for immunogenic tumor dynamics. Math Methods Appl Sci 2022;45(2):667−80.
[10] Mahariq I. On the application of spectral element method in electromagnetic problems involving domain decomposition. Turk J Electr Eng Comput Sci 2017;25:1059−69.

[11] Mahariq I, Kuzuoğlu M, Tarman IH, Kurt H. Photonic nanojet analysis by spectral element method. IEEE Photon J 2014;6(5):6802714.
[12] Mahariq I, Abdeljawad T, Karar AS, Alboon SA, Kurt H, Maslov AV. Photonic nanojets and whispering gallery modes in smooth and corrugated micro-cylinders under point-source illumination. Photonics 2020;7(3):50.
[13] Mahariq I, Kuzuoğlu M, Tarman IH. On the attenuation of the perfectly matched layer in electromagnetic scattering problems with the spectral element method. ACES J 2014; 29(9):701–10.
[14] Lin J, Zhang Y, Liu CS. Solving nonlinear third-order three-point boundary value problems by boundary shape functions methods. Adv Differ Equ 2021;2021(1):1–23.
[15] Liu CS, Chang CW. Boundary shape function method for nonlinear BVP, automatically satisfying prescribed multipoint boundary conditions. Bound Value Probl 2020;2020(1):1–22.
[16] Liu CS, Chang JR. Boundary shape functions methods for solving the nonlinear singularly perturbed problems with Robin boundary conditions. Int J Nonlinear Sci Numer Stimul 2020;21(7–8):797–806.
[17] Liu CS. A boundary shape function method for analyzing nonlinear composite beams, subjecting to nonlinear boundary moment conditions. Compos Struct 2021;264:113636.
[18] Chang CW. A new meshless method for solving 3D inverse conductivity issues of highly nonlinear elliptic equations. Symmetry 2022;14(5):1044.
[19] Liu CS, Liu DJ. A homogenization boundary function method for determining inaccessible boundary of a rigid inclusion for the Poisson equation. Eng Anal Bound Elem 2018;86:56–63.
[20] Lin J. Simulation of 2D and 3D inverse source problems of nonlinear time-fractional wave equation by the meshless homogenization function method. Eng Comput 2021. https://doi.org/10.1007/s00366-021-01489-2.
[21] Boucherif A. Second-order boundary value problems with integral boundary condition. Nonlinear Anal Theor Methods Appl 2009;70(1):364–71.
[22] Dehghan M. Numerical solution of a non-local boundary value problem with Neumann's boundary conditions. Commun Numer Methods Eng 2003;19(1):1–12.
[23] Chen Z, Jiang W, Du H. A new reproducing kernel method for Duffing equations. Int J Comput Math 2021;98(11):2341–54.
[24] Ahmad B, Alsaedi A, Alghamdi BS. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. Nonlinear Anal R World Appl 2008; 9(4):1727–40.
[25] Goodrich CS. On a nonlocal BVP with nonlinear boundary conditions. Results Math 2013;63(3):1351–64.
[26] Webb JRL, Infante G. Positive solutions of nonlocal boundary value problems involving integral conditions. NoDEA-Nonlinear Differ. Equ. Appl. 2008;15(1):45–67.
[27] Najafi R, Saray BN. Numerical solution of the forced Duffing equations using Legendre multiwavelets. Comput. Methods Differ. Equ. 2017;5(1):43–55.
[28] Geng F, Cui M. New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions. J Comput Appl Math 2009;233(2): 165–72.
[29] Liu CS. The Lie-group shooting method for nonlinear two-point boundary value problems exhibiting multiple solutions. CMES-Comp. Model. Eng. Sci. 2006;13(2):149–63.