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RESEARCH ARTICLE

A Hybrid Finite Difference Method for Singularly Perturbed Delay Partial Differential Equations with Discontinuous Convection Coefficient and Source Term

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Abstract

The article presents a hybrid finite difference scheme to solve a singularly perturbed parabolic functional differential equation with discontinuous coefficient and source. The simultaneous presence of deviating argument with a discontinuous source and coefficient makes the problem stiff. The solution of the problem exhibits turning point behaviour across discontinuity as ε tends to zero. The hybrid scheme presented is a composition of a central difference scheme and a midpoint upwind scheme on a specially generated mesh. At the same time, an implicit finite difference method is used to discretize the time variable. Consistency, stability, and convergence of the presented numerical approach have been investigated. The presented method converges uniformly independent of the perturbation parameter. Numerical results have been presented for two test examples that verify the effectiveness of the scheme.

Keywords: Singular perturbation, Functional differential equation, Finite difference method, Interior layer

1. Introduction

Differential equations model a wide range of phenomena in almost every branch of sciences and engineering. Often mathematical models assume specific behaviour or phenomenon which depends on the present and the past states of a system [17,42]. In other words, previous events have a direct impact on future outcomes. Modelling such systems leads to functional differential equations that are more realistic and frequently appear in a wide range of applications. Classical examples cover oscillatory, excitable as well as chaotic behavior [1] in physiological control systems [29], nonlinear optics [19], population dynamics [16] and neuroscience [31]. Particularly in nonlinear optics, the finite-time communication delays are typically much larger than the device's internal time-scales [39], and therefore, give rise to rich dynamical phenomena [27]. Specific examples of delay systems with (multiple) large delays include semiconductor lasers with two optical feedback loops of different lengths [30], ring-cavity lasers with optical feedback [12], and others [10,37].

When we associate a mathematical model with physical phenomena, we often capture the essentials and ignore the minor components involving small parameters. A mathematical model with small parameters is called a perturbed model, while an unperturbed or reduced model is a degenerate simplified model. If a small parameter is multiplied with the highest-order derivative term, then the problem is called singularly perturbed from a mathematical perspective [4]. In that case, the

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corresponding degenerate problem cannot satisfy the prescribed boundary/initial conditions. Moreover, the stiffness, attributed to the simultaneous presence of a deviating argument and discontinuous coefficient, give rise to multiple time scales as ε approaches the limiting value zero. There exist narrow regions across the turning points where the solution varies exponentially and approaches a discontinuous limit [28,36]. In these regions, the solution reveals sharp interior layers, and it is not easy to find an asymptotic expansion in terms of ε or to find a consistent numerical approximation.

The numerical methods on a uniform mesh fail to approximate singular perturbation functional differential equations. They require an unacceptably large number of mesh points to sustain the approximation because the mesh width depends on the perturbation parameter. This limitation of the conventional numerical methods has encouraged researchers to develop robust or adaptive numerical techniques that perform well enough independent of the perturbation parameter. Many researchers have made efforts to develop parameter uniform methods for partial functional differential equations; for an overview, see [5,20] and references therein. In [2], the authors studied a Dirichlet boundary value problem for the partial functional differential equation. In this paper, a numerical method comprising a standard finite difference operator (centered in space, implicit in time) is presented. The method presented is robust in the sense that its numerical solution converges to the exact solution uniformly. The higher-order methods are of particular interest and reasonably understood for problems without a deviating argument [8,23]. Nevertheless, attempts of having a higher order of uniform convergence for parabolic functional differential equations are lacking. In [25], a higher-order uniformly convergent hybrid method is proposed for parabolic problems with a delay. A scheme of HODIE type [24] is used in spatial direction over a generalized Shishkin mesh, whereas the implicit Euler scheme discretizes the time variable. Richardson extrapolation improves the order of convergence in time-variable. It is proved that the computed solution is uniformly convergent of $\mathscr{O}((L/N)^4 + (\Delta t)^2)$. In [22], a stable finite difference scheme is presented for singularly perturbed differential-difference equations with delay and advanced term. In [13], a standard finite difference method on an equidistributed layer adapted mesh is presented. The method approximates a singularly perturbed parabolic functional differential equations of convection-diffusion type using an upwind finite difference method on a piecewise uniform Shishkin mesh. In [38], a singularly perturbed differential−difference equation with small shifts is considered. The terms containing the delay and advance parameters are approximated by using the Taylor's series expansion. A quadratic B-spline collocation method is used in the space direction on exponentially graded mesh while the Crank−Nicolson finite difference method is used in the time direction on uniform mesh.

For a time-delayed convection-diffusion problem, a hybrid scheme is reported in [9]. The method presented is a composition of an upwind scheme and a central difference scheme and has almost second-order accuracy in space which is optimal compared to [14]. While a hybrid difference scheme for parabolic differential equation with delay and advance terms is studied in [15,34]. Besides, uniformly convergent methods based on fitted mesh and fitted operator approach are presented in [6]. The fitted operator methods used have coefficients of exponential type and are quite capable of reproducing the multiscale character of the exact solution of the problem [36]. On the other hand, the fitted mesh method [32,41] adopted concentrates on the appropriate choice of the grid points in narrow layer regions. Moreover, a Robin type boundary value problem for parabolic functional differential equation is brought to attention in [3]. A numerical method comprising a standard finite difference scheme on a rectangular piecewise uniform fitted mesh is presented. It is shown that the errors are bounded in the maximum norm by $C(N_x^{-2}\ln^2 N_x + N_t^{-1})$. In [21], the authors considered a class of singularly perturbed parabolic problem with delay and discontinuous data. An implicit numerical scheme is used to find the numerical solution. The method is uniformly convergent of $\mathscr{O}(N^{-1}\ln^2 N + \Delta t)$. However, the analysis of higher-order methods for singularly perturbed parabolic partial functional differential equations with discontinuous data and degenerating convective terms has seen little development and lacks due attention. Consequently, this paper aims to extend the idea of parameter uniform difference schemes to solve a singularly perturbed parabolic functional differential equation with discontinuity in coefficients with turning points and presents a higher-order accurate approximation.

2. The continuous problem

Consider the following initial-boundary-value problem on the domain $S^- \cup S^+ = (0,1) \times (0,T] \cup (1,2) \times (0,T]$:

$$\left.\begin{aligned} L_\varepsilon u &= \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) - b(x)u(x,t) - u_t(x,t)\\ &= f(x,t) + c(x)u(x-1,t) \text{ in } S^- \cup S^+,\\ u(x,t) &= p_0(x) \text{ on } [0,2]\times\{t=0\},\\ u(x,t) &= p_1(x,t) \text{ in } [-1,0]\times[0,T],\\ u(x,t) &= p_2(t) \text{ on } \{x=2\}\times[0,T], \end{aligned}\right\} \tag{2.1}$$

where $\varepsilon \ll 1$ is a small positive parameter, b and c are sufficiently smooth functions such that $c(x)>0$, $b(x)>0$ for all $x\in[0,2]$. Moreover, we assume that

$$\left.\begin{aligned} &a(x) = \begin{cases} a_1(x) \text{ if } 0\le x\le 1\\ a_2(x) \text{ if } 1<x\le 2,\end{cases} f(x,t) = \begin{cases} f_1(x,t) \text{ if } (x,t)\in\overline{S}^-\\ f_2(x,t) \text{ if } (x,t)\in\overline{S}^+,\end{cases}\\ &-\gamma_1^* < a_1(x) < -\gamma_1 < 0,\ \ \gamma_2^* > a_2(x) > \gamma_2 > 0,\\ &|[a]| \le C,\ \ |[f]| \le C, \end{aligned}\right\} \tag{2.2}$$

where $\gamma = \min\{\gamma_1,\ \gamma_2\}$ and $\gamma^* = \max\{\gamma_1^*,\ \gamma_2^*\}$. The solution of (2.1) satisfies $[u]=0$ and $[u_x]=0$ at $x=1$. Here, $[u]$ denotes the jump of u defined at the point of discontinuity $x=1$ as $[u](1,\ t) = u(1^+,\ t) - u(1^-,\ t)$ where $u(1^\pm,t) = \lim_{x\to 1\pm 0} u(x,t)$.

The functions p_0, p_1 and p_2 are Hölder continuous and the compatibility conditions hold at the corners $(0,0)$, $(2,0)$ and at the transition point $(1,0)$. On the domain S^-, the delay term $u(x-1,t) = p_1(x-1,t)$. Under these assumptions, the solution of (2.1) exists and unique $u\in C^{1+\lambda}(D=(0,2)\times(0,T])\cap C^{2+\lambda}(S^-\cup S^+)$ [26,35].

The presence of a discontinuity and a delay makes the problem challenging. The solution $u(x,t)$ of (2.1) displays a strong interior layer in the neighbourhood of the point $x=1$. Moreover, it is easy to follow that the differential operator L_ε satisfies the following maximum principle on $\overline{D}$.

Lemma 2.1 Suppose $P\in C^0(\overline{D})\cap C^2(S^-\cup S^+)$ satisfies $P(x,t)\le 0$ for all $(x,t)\in\Gamma := \overline{D}\setminus D$ and $L_\varepsilon P(x,t)\ge 0$ for all $(x,\ t)\in S^-\cup S^+$. Then $P(x,t)\le 0$ for all $(x,t)\in\overline{D}$.

Proof. Choose $(x^k,t^k)\in D$ such that $P(x^k,\ t^k) = \max_{(x,t)\in\overline{D}} P(x,t)$. Consequently,

$$P_x(x^k,t^k)=0,\quad P_t(x^k,t^k)=0 \quad \text{and} \quad P_{xx}(x^k,t^k)<0.$$

Suppose $P(x^k,t^k)>0$ and it follows that $(x^k,t^k)\notin\Gamma$. If $(x^k,\ t^k)\in S^-\cup S^+$, note that $L_\varepsilon P(x^k,\ t^k)<0$. A contradiction to the assumption and consequently the required result follows.

Establishing the boundedness of solution is an important application of the maximum principle. As an immediate application, we obtain

Lemma 2.2 Let u be the solution of (2.1). Then

$$\|u\|_{\infty,\overline{D}} \le \|u\|_{\infty,\Gamma} + \frac{1}{\gamma}\|f\|_{\infty,\overline{D}},\quad \gamma = \min\{\gamma_1,\gamma_2\}. \tag{2.3}$$

Proof. Consider

$$\psi_\pm = \begin{cases} -\|u\|_{\infty,\Gamma} - \dfrac{x}{\gamma}\|f\|_{\infty,\overline{D}} \pm u & \text{if } x\le 1\\ -\|u\|_{\infty,\Gamma} - \dfrac{(2-x)\|f\|_{\infty,\overline{D}}}{\gamma} \pm u & \text{if } x\ge 1.\end{cases}$$

For $(x,t)\in S^-$, it follows that

$$L_\varepsilon\psi_\pm(x,t) = \pm L_\varepsilon u - a_1(x)\frac{\|f\|}{\gamma} + b(x)\|u\| + b(x)\frac{x\|f\|}{\gamma} \ge 0$$

since $a_1(x)\le 0$ and $b(x)>0$. Similarly, for $(x,t)\in S^+$, it is easy to verify that $L_\varepsilon\psi_\pm(x,t)\ge 0$. The required result (2.3) now follows from Lemma 2.1.

In general, one can assume homogeneous boundary conditions $p_0=p_1=p_2=0$ by subtracting from u some suitable smooth function that satisfies the original boundary conditions [36]. To find sharper bounds on the solution and its derivatives, we decompose the solution u into smooth and singular components. We write $u=v+w$. The smooth component v is solution of

$$\left.\begin{aligned} &L_\varepsilon v(x,t) = f(x,t) + c(x)v(x-1,t) \text{ in } S^-\cup S^+,\\ &v(x,t) = u(x,t) \text{ in } [-1,0]\times[0,T],\\ &v(1^-,t) = j_1(t),\ v(1^+,t) = j_2(t),\ t\in(0,T],\\ &v(x,t) = u(x,t) \text{ on } \{x=2\}\times(0,T],\\ &v(x,t) = u(x,t) \text{ on } [0,2]\times\{t=0\}, \end{aligned}\right\} \tag{2.4}$$

where the functions $j_1(t)$ and $j_2(t)$ will be computed using the Theorem 2.4. The singular component w is solution of

$$\left.\begin{aligned} &L_\varepsilon w(x,t) = c(x)w(x-1,t) \text{ in } S^-\cup S^+,\\ &w(x,t) = 0 \text{ in } [-1,0]\times[0,T],\\ &w(x,t) = 0 \text{ on } \{x=2\}\times(0,T],\\ &w(x,t) = 0 \text{ on } [0,2]\times\{t=0\},\\ &[w](1,t) = -[v](1,t),\\ &\left[\frac{\partial w}{\partial x}\right](1,t) = -\left[\frac{\partial v}{\partial x}\right](1,t),\ t\in(0,T] \end{aligned}\right\} \tag{2.5}$$

Following are the direct results from [18] that will be used in Theorem 2.4.

Theorem 2.3 Let $a,\ b,\ c\in C^2[0,\ 2]$, $f\in C^{2+\lambda}(\overline{D})$ and $a(x)\geq\gamma>0$ for $x\in[0,2]$. Also suppose p_0, p_1 and p_2 are identically zero so that the compatibility conditions hold in $\Gamma_c=(-1,0)\cup(0,0)\cup(2,0)\cup(1,0)$ and

$$\frac{\partial^{k+m}F}{\partial x^k\partial t^m}=0,\quad 0\leq k+2m\leq 2,$$

where $F(x,\ t)=f(x,\ t)+c(x)u(x-1,\ t)$. Then $u\in C^{4+\lambda}(\overline{D})$ and

$$\left\|\frac{\partial^{k+m}u}{\partial x^k\partial t^m}\right\|_{\infty,D}\leq C\varepsilon^{-k},\quad 0\leq k+2m\leq 4. \tag{2.6}$$

Theorem 2.4 There exist smooth functions $j_1(t)$, $j_2(t)$ such that the smooth and singular component defined in (2.4) and (2.5), respectively, satisfy the following bounds for $0\leq k\leq 3$, $m\geq 0$ and $0\leq k+2m\leq 4$

$$\left\|\frac{\partial^{k+m}v}{\partial x^k\partial t^m}\right\|_{\infty,S^-\cup S^+}\leq C,\quad \left\|\frac{\partial^4 v}{\partial x^4}\right\|_{\infty,S^-\cup S^+}\leq C\varepsilon^{-1}$$

and

$$\left|\frac{\partial^{k+m}w}{\partial x^k\partial t^m}(x,t)\right|\leq\begin{cases}C\left(\varepsilon^{-k}\exp(-(1-x)\gamma_1/\varepsilon)\right) & \text{for } (x,t)\in S^-\\ C\left(\varepsilon^{-k}\exp(-(x-1)\gamma_2/\varepsilon)\right) & \text{for } (x,t)\in S^+,\end{cases}$$

$$\left|\frac{\partial^4 w}{\partial x^4}(x,t)\right|\leq\begin{cases}C\left(\varepsilon^{-4}\exp(-(1-x)\gamma_1/\varepsilon)\right) & \text{for } (x,t)\in S^-\\ C\left(\varepsilon^{-4}\exp(-(x-1)\gamma_2/\varepsilon)\right) & \text{for } (x,t)\in S^+.\end{cases}$$

Proof. We conduct our analysis separately in the subregions S^- and S^+ to obtain stronger bounds on the solution and its derivatives. We start with the subregion S^+ and define $\overline{D}^*$ such that $\overline{S}^+\subset\overline{D}^*$. Let $D^*=\Omega^*\times(0,T]$, $\Omega^*=[-2,2]$ and define a function v^* on D^* as

$$v^*=v_0^*+\varepsilon v_1^*+\varepsilon^2 v_2^*+\varepsilon^3 v_3^*,$$

where the functions v_0^*, v_1^* and v_2^* are solutions of

$$\left.\begin{aligned}&a^*\frac{\partial v_0^*}{\partial x}-b^*v_0^*-\frac{\partial v_0^*}{\partial t}=f^*+c^*v_0^*(x-1,t)\quad\text{in}\quad D^*,\\&v_0^*(x,t)=p_1^*(x,t)\quad\text{in}\quad[-3,0]\times[0,T],\\&v_0^*(x,t)=p_0^*(x)\quad\text{on}\quad[-2,2]\times\{t=0\},\\&a^*\frac{\partial v_i^*}{\partial x}-b^*v_i^*-\frac{\partial v_i^*}{\partial t}=-\frac{\partial^2 v_{i-1}^*}{\partial x^2}+c^*v_i^*(x-1,t)\quad\text{in}\quad D^*,\\&v_i^*(x,t)=0\quad\text{in}\quad[-3,0]\times(0,T],v_i^*(x,0)=0\quad\text{on}\quad[-2,2]\times\{t=0\},\quad i=1,2\end{aligned}\right\}\tag{2.7}$$

and v_3^* is solution of

$$\left.\begin{aligned}&\varepsilon\frac{\partial^2 v_3^*}{\partial x^2}+a^*\frac{\partial v_3^*}{\partial x}-b^*v_3^*-\frac{\partial v_3^*}{\partial t}=c^*v_3^*(x-1,t)-\frac{\partial^2 v_2^*}{\partial x^2}\ \text{in}\ D^*,\\&v_3^*(x,t)=0\ \text{in}\ [-3,0]\times[0,T],\\&v_3^*(x,0)=0\ \text{on}\ [-2,2]\times\{t=0\},\\&v_3^*(x,t)=0\ \text{on}\ \{x=2\}\times(0,T]\end{aligned}\right\}\tag{2.8}$$

Here, the coefficients a^*, b^* and c^* as well as the condition p_0^* are respective smooth extensions of a, b, c and p_0 from the domain $[1,2]$ to the domain $[-2,2]$. The functions f^* and p_1^* are the smooth extensions of f and p_1 from the domain $\overline{S}^+$ to the domain $\overline{D}^*$. In a neighbourhood of the point $(-2,0)$, the functions p_0^*, p_1^* and f^* are built such that $p_0^*=p_1^*=f^*=0$. Assuming that a^*, b^*, c^* and f^* are sufficiently smooth on $\overline{D}^*$. We set all the initial-boundary data associated with (2.7) equal to zero. Define $F^*=f^*+c^*v^*(x-1,t)$ and impose the following compatibility condition on the set Γ_c:

$$\frac{\partial^{k+m}F^*}{\partial x^k\partial t^m}=0\quad\text{for}\quad 0\leq k+m\leq 7. \tag{2.9}$$

Follows from the results of [7] for the first order differential equations defined in (2.7), we have $v_i^*\in C^{9-2i+\lambda}(\overline{D}^*)\cap C^{8-2i}(\overline{D}^*)$, $i=0,1,2$ which implies $\frac{\partial^2 v_2^*}{\partial x^2}\in C^{2+\lambda}(\overline{D}^*)$ and therefore $v_3^*\in C^{4+\lambda}(\overline{D}^*)$. Then it follows that

$$\left\|\frac{\partial^{k+m}v_i^*}{\partial x^k\partial t^m}\right\|_{\infty,D^*}\leq C,\quad i=0,1,2,$$

$$\left\|\frac{\partial^{k+m}v_3^*}{\partial x^k\partial t^m}\right\|_{\infty,D^*}\leq C\varepsilon^{-k}\quad\text{for}\quad 0\leq k+2m\leq 4.$$

The smooth component v is now defined as a restriction of v^* on the domain $\overline{S}^+$. Define $v^*(1,t) = j_2(t) = v(1,t)$. Thus v satisfies

$$L_\varepsilon v = f + c(x)v(x-1,t) \quad \text{in} \quad S^+,$$

$$v(x,t) = v^*(x,t) \quad \text{on} \quad \{x=1\} \times (0,T],$$

$$v(x,t) = u(x,t) \quad \text{on} \quad [1,2] \times \{t=0\}.$$

Since $v^* = v_0^* + \varepsilon v_1^* + \varepsilon^2 v_2^* + \varepsilon^3 v_3^*$, thus

$$\left\|\frac{\partial^{k+m} v^*}{\partial x^k \partial t^m}\right\|_{\infty,S^+} \le \left\|\frac{\partial^{k+m} v_0^*}{\partial x^k \partial t^m}\right\|_{\infty,S^+} + \varepsilon\left\|\frac{\partial^{k+m} v_1^*}{\partial x^k \partial t^m}\right\|_{\infty,S^+} + \varepsilon^2\left\|\frac{\partial^{k+m} v_2^*}{\partial x^k \partial t^m}\right\|_{\infty,S^+} + \varepsilon^3\left\|\frac{\partial^{k+m} v_3^*}{\partial x^k \partial t^m}\right\|_{\infty,S^+} \le C\left(1+\varepsilon^{3-k}\right) \quad \text{for} \quad 0 \le k+2m \le 4.$$

As v is a restriction of v^* on the domain $\overline{S}^+$, we have

$$\left\|\frac{\partial^{k+m} v}{\partial x^k \partial t^m}\right\|_{\infty,S^+} \le C\left(1+\varepsilon^{3-k}\right) \quad \text{for} \quad 0 \le k+2m \le 4.$$

Thus for $0 \le k \le 3$ and $0 \le k+2m \le 4$, the smooth component v satisfies

$$\left\|\frac{\partial^{k+m} v}{\partial x^k \partial t^m}\right\|_{\infty,S^+} \le C \quad \text{and} \quad \left\|\frac{\partial^4 v}{\partial x^4}\right\|_{\infty,S^+} \le C\varepsilon^{-1}.$$

Similarly, the desired bounds for smooth component v in the subregion S^- can be obtained. Define the barrier functions on $S^- \cup S^+$

$$\Phi^\pm(x,t) = \begin{cases} (\pm v - C)\exp\left(\dfrac{-(1-x)\gamma_1}{\epsilon}\right) \pm w, & (x,t) \in S^- \\ (\pm v - C)\exp\left(\dfrac{-(x-1)\gamma_2}{\epsilon}\right) \pm w, & (x,t) \in S^+, \end{cases} \tag{2.10}$$

where C is a constant. For $(x,\ t) \in S^-$, using assumption $(a+\gamma_1) < 0$ to obtain

$$L_\varepsilon \phi^\pm(x,t) = L_\varepsilon\left((\pm v - C)\exp\left(\frac{-(1-x)\gamma_1}{\varepsilon}\right)\right) \pm L_\varepsilon w(x,t) \ge 0. \tag{2.11}$$

Similarly, for $(x,t) \in S^+$ we obtain

$$L_\varepsilon \phi^\pm(x,t) = \frac{\gamma_2}{\varepsilon}\exp\left(\frac{-(x-1)\gamma_2}{\varepsilon}\right)\left((\pm v - C)(-a+\gamma_2) - 2\varepsilon v_x + \frac{Cb\varepsilon}{\gamma_2}\right) \pm \frac{\gamma_2}{\varepsilon}\exp\left(\frac{-(x-1)\gamma_2}{\varepsilon}\right)\left(\frac{f + cv(x-1,t)}{\gamma_2}\right) \ge 0. \tag{2.12}$$

Moreover, $\phi^\pm \in C^0(\overline{D})$ and $\phi^\pm(x,t) \le 0$ for $(x,t) \in \Gamma$. Consequently, the required bounds on w follows from Lemma 2.1. The required estimates for the derivatives of w follow from [33].

3. Difference scheme

The solution of the problem exhibits strong interior layer at $x = 1$. Therefore, we descretize the domain by constructing a rectangular mesh $\overline{D}^{N,M} = \overline{S}_x^N \times T_t^M$ in such a way that it will condense around the point $x = 1$. We write

$$[0,2] = [0,1-\sigma] \cup [1-\sigma,1] \cup [1,1+\sigma] \cup [1+\sigma,2],$$

where $\sigma = \min\left\{\frac{1}{2}, \sigma_0 \varepsilon \ln N\right\}$ and σ_0 is a constant that will be chosen later on. We place $\frac{N}{4}$ mesh points in each of the subintervals. Consequently, we obtain

$$S_x^N = \left\{x_i : i = 1,2\ldots,\frac{N}{2}-1\right\} \cup \left\{x_i : i = \frac{N}{2}+1,\ldots,N-1\right\}$$

and

$$h_i = x_i - x_{i-1}, \quad i = 1,2,\ldots,N,$$
$$\widehat{h}_i = h_i + h_{i+1}, \; i = 1,2,\ldots,N-1,$$

We define the uniform mesh for the domain $[0,T]$, as follows

$$T_t^M = \{t_k = k\Delta t, k = 0, \ \ldots \ , M, \Delta t = T/M\}.$$

$$h_i = \begin{cases} H = \dfrac{4(1-\sigma)}{N}, & i = 1, \ldots, N/4, 3N/4+1, \ldots, N \\ h = \dfrac{4\sigma}{N}, & i = N/4+1, \ldots, N/2, N/2+1, \ldots, 3N/4. \end{cases}$$

To discretize the differential operator in (2.1), we first define the finite difference operators on the mesh $\overline{D}^{N,M}$ as

$$D_x^+ v_i^k = \frac{v_{i+1}^k - v_i^k}{h_{i+1}}, \quad D_x^- v_i^k = \frac{v_i^k - v_{i-1}^k}{h_i},$$
$$D_x^0 v_i^k = \frac{v_{i+1}^k - v_{i-1}^k}{h_{i+1} + h_i}, \; D_t^- v_i^k = \frac{v_i^k - v_i^{k-1}}{\Delta t},$$

and $\delta_x^2 v_i^k = \frac{2(D_x^+ v_i^k - D_x^- v_i^k)}{h_{i+1}+h_i}$. Also define $v_{\frac{i\mp1}{2}}^k = \frac{(v_{i\mp1}^k + v_i^k)}{2}$, $v_{\frac{i\mp1}{2}} = \frac{(v_{i\mp1} + v_i)}{2}$.

We use the classical central difference scheme in the intervals $(1-\sigma, 1), (1, 1+\sigma)$ and the midpoint upwind scheme in the rest of the intervals. At the point of discontinuity, second-order one-sided difference approximations are used to keep the continuity of the spatial derivative. We use the backward-Euler method for discretizing the time derivative. The discrete problem thus reads:

$$\begin{cases} U_{i,0} = p_0(x_i) & \text{for } i = 0, \ldots, N \\ L_{mu}^{N,M,(L)} U_{i,k+1} = f_{\frac{i-1}{2},k+1} + c_{\frac{i-1}{2}} p_1 & \text{for } i = 1, \ldots, \dfrac{N}{4} \\ L_{cen}^{N,M} U_{i,k+1} = f_{i,k+1} + c_i p_1 & \text{for } i = \dfrac{N}{4}+1, \ldots, \dfrac{N}{2}-1 \\ L_{cen}^{N,M} U_{i,k+1} = f_{i,k+1} + c_i U_{i-N/2,k+1} & \text{for } i = \dfrac{N}{2}+1, \ldots, \dfrac{3N}{4}-1 \\ L_{mu}^{N,M,(R)} U_{i,k+1} = f_{\frac{i+1}{2},k+1} + c_{\frac{i+1}{2}} U_{i-N/2,k+1} & \text{for } i = \dfrac{3N}{4}, \ldots, N-1 \\ D_x^F U_{i,k+1} - D_x^B U_{i,k+1} = 0 & \text{for } i = \dfrac{N}{2} \\ \text{for } k = 0, \ldots, M-1, & \end{cases} \tag{3.1}$$

where

$$\begin{cases} \L_{mu}^{N,M,(L)}U_{i,k+1} & = \epsilon\delta_x^2 U_{i,k+1} + a_{\frac{i-1}{2}} D_x^- U_{i,k+1} - b_{\frac{i-1}{2}} U_{\frac{i-1}{2},k+1} - D_t^- U_{\frac{i-1}{2},k+1} \\ L_{cen}^{N,M}U_{i,k+1} & = \epsilon\delta_x^2 U_{i,k+1} + a_i D_x^0 U_{i,k+1} - b_i U_{i,k+1} - D_t^- U_{i,k+1} \\ \L_{mu}^{N,M,(R)}U_{i,k+1} & = \epsilon\delta_x^2 U_{i,k+1} + a_{\frac{i+1}{2}} D_x^+ U_{i,k+1} - b_{\frac{i+1}{2}} U_{\frac{i+1}{2},k+1} - D_t^- U_{\frac{i+1}{2},k+1} \end{cases} \tag{3.2}$$

and

$$\begin{cases} D_x^F U_{\frac{N}{2},k} = \left(- U_{\frac{N}{2}+2,k+1} + 4U_{\frac{N}{2}+1,k+1} - 3U_{\frac{N}{2},k+1} \right) \Big/ 2h \\ D_x^B U_{\frac{N}{2},k} = \left(U_{\frac{N}{2}-2,k+1} - 4U_{\frac{N}{2}-1,k+1} + 3U_{\frac{N}{2},k+1} \right) \Big/ 2h. \end{cases} \tag{3.3}$$

After simplifying the terms in (3.1), we obtain

$$\begin{cases} U_i^0 = p_0(x_i) & \text{for} \quad i = 0, \ldots, N \\ L_\varepsilon^{N,M} U_{i,k+1} = \tilde{f}_{i,k+1} & \text{for} \quad i = 1, \ldots, N-1 \\ U_{i,k+1} = p_1(x_i, t_{k+1}) & \text{for} \quad i = -N/2, \ldots, 0 \\ U_{N,k+1} = p_2(t_{k+1}) & \text{for} \quad k = 0, \ldots, M-1, \end{cases} \tag{3.4}$$

where

$$L_\epsilon^{N,M} U_{i,k+1} = \begin{cases} \left[r_i^- U_{i-1,k+1} + r_i^0 U_{i,k+1} + r_i^+ U_{i+1,k+1}\right] + \left[p_i^- U_{i-1,k} + p_i^0 U_{i,k} + p_i^+ U_{i+1,+k+1}\right] \\ \text{for} \quad i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1 \\ q_i^{-,2} U_{i-2,k+1} + q_i^{-,1} U_{i-1,k+1} + q_i^0 U_{i,k+1} + q_i^{+,1} U_{i+1,k+1} + q_i^{+,2} U_{i+2,k+1} \\ \text{for} \quad i = N/2 \end{cases} \tag{3.5}$$

and

$$\tilde{f}_{i,k+1} = \begin{cases} m_i^- f_{i-1,k+1} + m_i^0 f_{i,k+1} + m_i^+ f_{i+1,k+1} + s_i^+ p_1(x_i, t_{k+1}) + s_i^- U_{i-N/2,k+1} \\ \text{for} \quad i = 1, \ldots, N/2-1, N/2+1, \ldots, N-1 \\ 0 \quad \text{for} \quad i = N/2. \end{cases} \tag{3.6}$$

Here, for $i \in \{1 : N/4\}$

$$\begin{cases} r_i^- = \left(\dfrac{2\epsilon}{\widehat{h}_i h_i} - \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ r_i^0 = \left(\dfrac{-2\epsilon}{h_i h_{i+1}} + \dfrac{a_{i-1/2}}{h_i} - \dfrac{b_{i-1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ r_i^+ = \dfrac{2\epsilon}{\widehat{h}_i h_{i+1}}, \quad p_i^- = \dfrac{1}{2\Delta t}, \quad p_i^0 = \dfrac{1}{2\Delta t}, \\ p_i^+ = 0, \quad m_i^- = \dfrac{1}{2}, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = 0, \\ s_i^+ = c_{i-1/2}, \quad s_i^- = 0, \end{cases} \tag{3.7}$$

for $i \in \{N/4 + 1 : 3N/4 - 1\}/\{\frac{N}{2}\}$

$$\begin{cases} r_i^- = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_i} - \dfrac{a_i}{\widehat{h}_i}\right), \quad r_i^0 = \left(\dfrac{-2\epsilon}{h_i h_{i+1}} - b_i - \dfrac{1}{\Delta t}\right), \\ r_i^+ = \left(\dfrac{2\epsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_i}{\widehat{h}_i}\right), \quad p_i^- = 0, \quad p_i^0 = \dfrac{1}{\Delta t}, \\ p_i^+ = 0, \quad m_i^- = 0, \quad m_i^0 = 1, \quad m_i^+ = 0, \end{cases} \tag{3.8}$$

for $i \in \{N/4 + 1 : N/2 - 1\}$

$s_i^+ = c_i, \quad s_i^- = 0,$

for $i \in \{N/2 + 1 : 3N/4 - 1\}$

$s_i^+ = 0, \quad s_i^- = c_i,$

for $i \in \{3N/4 : N - 1\}$

$$\begin{cases} r_i^- = \dfrac{2\varepsilon}{\widehat{h}_i h_i}, \quad r_i^0 = \left(\dfrac{-2\varepsilon}{h_i h_{i+1}} - \dfrac{a_{i+1/2}}{h_{i+1}} - \dfrac{b_{i+1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ r_i^+ = \left(\dfrac{2\varepsilon}{\widehat{h}_i h_{i+1}} + \dfrac{a_{i+1/2}}{\widehat{h}_{i+1}} - \dfrac{b_{i+1/2}}{2} - \dfrac{1}{2\Delta t}\right), \\ p_i^- = 0, \ p_i^0 = \dfrac{1}{2\Delta t}, \ p_i^+ = \dfrac{1}{2\Delta t}, \\ m_i^- = 0, \quad m_i^0 = \dfrac{1}{2}, \quad m_i^+ = \dfrac{1}{2}, \quad s^+ = 0, \quad s^- = c_{i+1/2} \end{cases} \tag{3.9}$$

and lastly,

$$q_{N/2}^{-,2} = \frac{-1}{2h}, \quad q_{N/2}^{-,1} = \frac{2}{h}, \quad q_{N/2}^{0} = -\frac{3}{h}, \quad q_{N/2}^{+,1} = \frac{2}{h}, \quad q_{N/2}^{+,2} = \frac{-1}{2h}. \tag{3.10}$$

4. Error estimates

The difference operator $L_\varepsilon^{N,M}$ in (3.5) fails to fulfill the conditions of discrete maximum principle (as for this we need $q_i \geq 0$ to prove A to be an M-matrix defined in Equation (4.9)). As a consequence, we have to modify the equation (3.4), for $i = N/2$

$$q_{N/2}^{-,2} U_{N/2-2,k+1} + q_{N/2}^{-,1} U_{N/2-1,k+1} + q_{N/2}^{0} U_{N/2,k+1} + q_{N/2}^{+,1} U_{N/2+1,k+1} + q_{N/2}^{+,2} U_{N/2+2,k+1} = 0. \tag{4.1}$$

From (3.4), for $i = N/2 - 1$, we have

$$\left(\frac{2\varepsilon - h a_{N/2-1}}{2h^2}\right) U_{N/2-2,k+1} = f_{N/2-1,k+1} + c_{N/2-1} p_1(x_{-1}, t_{k+1}) - r_{N/2-1}^0 U_{N/2-1,k+1} + r_{N/2-1}^+ U_{N/2,k+1} + \frac{1}{\Delta t} U_{N/2-1,k} \tag{4.2}$$

and, for $i = N/2 + 1$

$$\left(\frac{2\varepsilon + h a_{N/2+1}}{2h^2}\right) U_{N/2+2,k+1} = f_{N/2+1,k+1} + c_{N/2+1} U(x_1, t_{k+1}) - r_{N/2+1}^0 U_{N/2+1,k+1} + r_{N/2+1}^+ U_{N/2,k+1} + \frac{1}{\Delta t} U_{N/2+1,k}. \tag{4.3}$$

Now putting the values of $U_{N/2-2,k+1}$ and $U_{N/2+2,k+1}$ from Equations (4.2) and (4.3) into (4.1), we get

$$q_{N/2}^{-,2}\left(\frac{2h^2}{2\varepsilon - h a_{N/2-1}}\right)\Big(f_{N/2-1,k+1} + c_{N/2-1} p_1 \left(x_{-1}, t_{k+1}\right) - r_{N/2-1}^0 U_{N/2-1,k+1} - r_{N/2-1}^+ U_{N/2,k+1} - \frac{1}{\Delta t} U_{N/2-1,k}\Big) + q_{N/2}^{-,1} U_{N/2-1,k+1} + q_{N/2}^{0} U_{N/2,k+1} + q_{N/2}^{+,1} U_{N/2+1,k+1}$$
$$+ q_{N/2}^{+,2}\left(\frac{2h^2}{2\varepsilon + h a_{N/2+1}}\right)\Big(f_{N/2+1,k+1} + c_{N/2+1} U\left(x_1, t_{k+1}\right) - r_{N/2+1}^0 U_{N/2+1,k+1}$$

$$-r^{+}_{N/2+1}U_{N/2,k+1}-\frac{1}{\Delta t}U_{N/2+1,k}\Big)=0. \quad (4.4)$$

After rearranging the terms in (4.4), the discrete problem thus reads

$$\begin{cases} U_{i,0}=p_0(x_i) & \text{for} \quad i=0,\dots,N \\ L^{N,M}_{\tau}U_{i,k+1}=\tilde{f}_{\tau,i,k+1} & \text{for} \quad i=1,\dots,N-1 \\ U_{i,k+1}=p_1(x_i,t_{k+1}) & \text{for} \quad i=-N/2,\dots,0 \\ U_{N,k+1}=p_2(t_{k+1}) & \text{for} \quad k=0,\dots,M-1, \end{cases} \quad (4.5)$$

where

$$L^{N,M}_{\tau}U_{i,k+1}=\begin{cases} (r_i^-U_{i-1,k+1}+r_i^0U_{i,k+1}+r_i^+U_{i+1,k+1}) & \\ +(p_i^-U^k_{i-1}+p_i^0U^k_i+p_i^+U_{i+1,k}) & if\, i=N/2 \\ L^{N,M}_{\epsilon}U_{i,k+1} & if\, i\neq N/2 \end{cases} \quad (4.6)$$

and

$$\tilde{f}_{\tau,i,k+1}=\begin{cases} \left[m_i^-f_{i-1,k+1}+m_i^0f_{i,k+1}+m_i^+f_{i+1,k+1}\right] & \\ +l_1p_1(x_{-1},t_{k+1})+l_2U(x_1,t_{k+1}) & \text{if } i=N/2 \\ \tilde{f}_{i,k+1} & \text{if } i\neq N/2, \end{cases}$$

where, for $i=N/2$

$$\begin{cases} r_i^-=\dfrac{1}{2h}\left(4-\dfrac{2\left(2\epsilon+h^2b_{i-1}+\dfrac{h^2}{\Delta t}\right)}{2\epsilon-ha_{i-1}}\right), \\ r_i^0=\dfrac{1}{2h}\left(-6+\dfrac{2\epsilon+ha_{i-1}}{2\epsilon-ha_{i-1}}+\dfrac{2\epsilon-ha_{i+1}}{2\epsilon+ha_{i+1}}\right), \\ r_i^+=\dfrac{1}{2h}\left(4-\dfrac{2\left(2\epsilon+h^2b_{i+1}+\dfrac{h^2}{\Delta t}\right)}{2\epsilon+ha_{i+1}}\right), \\ p_i^-=\dfrac{h}{(2\epsilon-ha_{i-1})\Delta t}, \quad p_i^0=0, \quad p_i^+=\dfrac{h}{(2\epsilon+ha_{i+1})\Delta t}, \\ m_i^-=\dfrac{h}{(2\epsilon-ha_{i-1})}, \quad m_i^0=0, \quad m_i^+=\dfrac{h}{(2\epsilon+ha_{i+1})}, \\ l_1=\dfrac{-hc_{i-1}}{2\epsilon-ha_{i-1}}, \quad l_2=\dfrac{-hc_{i+1}}{2\epsilon+ha_{i+1}} \end{cases}$$

and for $i\neq N/2$, these coefficients are defined in (3.7–3.9). Let $D^{N,M}=\overline{D}^{N,M}\cap D$ and $\Gamma=\overline{D}^{N,M}\setminus D^{N,M}$.

Lemma 4.1 Let $N\geq N_0$ be such that

$$\frac{N_0}{\ln N_0}\geq 2\sigma_0\gamma^* \quad \text{and} \quad (4.7)$$

$$\left(\|b\|_\infty+\Delta t^{-1}\right)\leq\frac{\gamma N_0}{2}. \quad (4.8)$$

Let Υ be the mesh function such that $\Upsilon\leq 0$ on $\Gamma^{N,M}$ and $L^{N,M}_{\tau}\Upsilon\geq 0$ in $D^{N,M}$. Then $\Upsilon\leq 0$ in $\overline{D}^{N,M}$.

Proof. Write $L^{N,M}_{\tau}$ as

$$-L^{N,M}_{\tau}\Upsilon_{i,k+1}=[A_{i,i-1}\Upsilon_{i-1,k+1}+A_{i,i}\Upsilon_{i,k+1}+A_{i,i+1}\Upsilon_{i+1,k+1}]$$
$$-[B_{i,i-1}\Upsilon_{i-1,k}+B_{i,i}\Upsilon_{i,k}+B_{i,i+1}\Upsilon_{i+1,k}], \quad (4.9)$$

where $A:=(A_{i,j})$ and $B:=(B_{i,j})$ are written as

$$A_{i,i-1}=-r_i^-, \quad A_{i,i}=-r_i^0, \quad A_{i,i+1}=-r_i^+,$$

$B_{i,i-1}=p_i^-,\quad B_{i,i}=p_i^0,\quad B_{i,i+1}=p_i^+.$

Clearly $B\geq 0$, since $p_i^-,p_i^0,p_i^+\geq 0$ and using (4.7) and (4.8) to show A is an M-matrix [11]. Remaining part is proved by induction. Assume

$Y_{i,k}\leq 0,\quad k=0,\ldots,N-1.$

Then, we can rewrite (4.9) as

$AY_{i,k+1}=BY_{i,k}-L^{N,M}Y_{i,k+1}.$

Since $B\geq 0$, $Y_{i,k}\leq 0$, $A^{-1}\geq 0$ and $L_\tau^{N,M}Y_{i,k+1}\geq 0$ by hypothesis. Then, $Y_{i,k+1}\leq 0$ in $D^{N,M}$.

We obtain the following estimate as an immediate consequence of Lemma 4.1.

Lemma 4.2 Let U be the solution of (4.5) and the conditions (4.7) and (4.8) hold true. Then

$$\|U\|_{\infty,\overline{D}^{N,M}}\leq\|U\|_{\infty,\Gamma^{N,M}}+\frac{1}{\gamma}\|\tilde{f}_\tau\|_{\infty,\overline{D}^{N,M}}$$

Proof. Let

$$\phi_i^{\pm,k}=-\|U\|_{\infty,\Gamma^{N,M}}-\begin{cases}x_i\dfrac{\|\tilde{f}_\tau\|_\infty}{\gamma}\mp U_{i,k} & for\ 1\leq i\leq N/2\\ (2-x_i)\dfrac{\|\tilde{f}_\tau\|_\infty}{\gamma}\mp U_{i,k} & for\ N/2<i\leq N\end{cases}.$$

Then $\phi_N^{\pm,k+1}\leq 0$, $\phi_i^{\pm,k+1}\leq 0$, $-N/2\leq i\leq 0$ and $\phi_i^{\pm,0}\leq 0$, $0\leq i\leq N$. For $i\neq N/2$, $L_\tau^{N,M}\phi_i^{\pm,k+1}\geq 0$. Further

$$L_\tau^{N,M}\phi_{N/2}^{\pm,k+1}=L_\tau^{N,M}\left\{-\|U\|_{\infty,\Gamma^{N,M}}-\frac{1}{\gamma}\|\tilde{f}_\tau\|_{\infty,\overline{D}^{N,M}}\mp U_{N/2,k+1}\right\}\geq(D_x^F-D_x^B)\phi_{N/2}^{\pm,k+1}$$

and

$$(D_x^F-D_x^B)\phi_{N/2}^{\pm,k+1}=\frac{1}{2h}\Big(-\phi_{N/2+2}+4\phi_{N/2+1}-6\phi_{N/2}-\phi_{N/2-2}+4\phi_{N/2-1}\Big)\geq 0.$$

$$U_{i,k+1}=\begin{cases}V_{L,i,k+1}+W_{L,i,k+1}, & 0\leq i\leq N/2-1\\ V_{L,i,k+1}+W_{L,i,k+1}=V_{R,i,k+1}+W_{R,i,k+1}, & i=N/2\\ V_{R,i,k+1}+W_{R,i,k+1}, & N/2+1\leq i\leq N-1.\end{cases}\tag{4.13}$$

Consequently, the required result follows from Lemma 4.1.

Next, we decompose the solution into smooth and singular component. We write $U_{i,k+1}=V_{i,k+1}+W_{i,k+1}$. Here the mesh functions V_L and V_R satisfy

$$\left.\begin{array}{l}L_\tau^{N,M}V_{L,i,k+1}=\tilde{f}_{\tau,i,k+1}\ \text{ for }\ i=1,\ldots,N/2-1,\\ V_{L,i,k+1}=v(x_i,t_{k+1})\ \text{ for }\ i=-N/2,\ldots,0,\\ V_{L,N/2,k+1}=v(1^-,t_{k+1}),\ \ k\geq 0,\\ V_{L,i,0}=v(x_i,0)\ \text{ for }\ i=1,\ldots,N/2\end{array}\right\}$$

and

$$\left.\begin{array}{l}L_\tau^{N,M}V_{R,i,k+1}=\tilde{f}_{\tau,i,k+1}\ \text{ for }\ i=N/2+1,\ldots,N-1,\\ V_{R,N/2,k+1}=v(1^+,t_{k+1}),\ \ V_{R,N,k+1}=v(2,t_{k+1}),\ \ k\geq 0,\\ V_{R,i,0}=v(x_i,0)\ \text{ for }\ i=N/2,\ldots,N.\end{array}\right\}\tag{4.11}$$

The mesh function W_L and W_R satisfy

$$\begin{array}{l}L_\tau^{N,M}W_{L,i,k+1}=0\ \text{ for }\ i=1,\ldots,N/2-1,\\ L_\tau^{N,M}W_{R,i,k+1}=0\ \text{ for }\ i=N/2+1,\ldots,N-1,\\ W_{L,i,k+1}=0\ \text{ for }\ i=-N/2,\ldots,0,\\ W_{R,N,k+1}=0,\ \ k\geq 0,\\ W_{L,i,0}=0\ \text{ for }\ i=0,\ldots,N/2,\\ W_{R,i,0}=0\ \text{ for }\ i=N/2,\ldots,N,\\ W_{R,N/2,k+1}+V_{R,N/2,k+1}=W_{L,N/2,k+1}+V_{L,N/2,k+1},\\ D_x^FW_{R,N/2,k+1}+D_x^FV_{R,N/2,k+1}=D_x^BW_{L,N/2,k+1}\\ +D_x^BV_{L,N/2,k+1},k\geq 0.\end{array}\tag{4.12}$$

Numerical solution U satisfies

Lemma 4.3 Let V_L and V_R are the solutions of (4.10), (4.11) and v is the solution of (2.4). Then under the assumptions (4.7) and (4.8), we have

$$\begin{cases} |V_{L,i,k+1}-v(x_i,t_{k+1})| \le C\left(N^{-2}+\Delta t\right)x_i & \text{for } i=1,\ldots,N/2-1 \\ |V_{R,i,k+1}-v(x_i,t_{k+1})| \le C\left(N^{-2}+\Delta t\right)(2-x_i) & \text{for } i=N/2+1,\ldots,N-1. \end{cases}$$

Proof. Consider $\Psi^k_{L,i} = -C(N^{-2}+\Delta t)x_i, \quad i\in\{0,\ldots,N/2\}$. Then

$$L_\tau^{N,M}(V_{L,i,k+1}-v(x_i,t_{k+1})) = \left(L_\varepsilon - L_\tau^{N,M}\right)v(x_i,t_{k+1}).$$

We observe that for $i\in\{1,\ldots,N/2-1\}$

$$\left|L_\tau^{N,M}(V_{L,i,k+1}-v(x_i,t_{k+1}))\right|$$

$$\le \begin{cases} C\left[(\epsilon+h_i)(h_i+h_{i+1})\left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty + h_i^2\left(\left\|\dfrac{\partial^2 v}{\partial x^2}\right\|_\infty + \left\|\dfrac{\partial v}{\partial x}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], & 1\le i\le N/4 \\ C\left[h^2\left(\epsilon\left\|\dfrac{\partial^4 v}{\partial x^4}\right\|_\infty + \left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], & N/4+1\le i\le N/2-1. \end{cases}$$

Using Theorem 2.4, conditions $h_i \le CN^{-1}$ and $\varepsilon \le N^{-1}$ to obtain

$$\left|L_\tau^{N,M}(V_{L,i,k+1}-v(x_i,t_{k+1}))\right| \le C\left(N^{-2}+\Delta t\right) \le L_\tau^{N,M}\psi^{k+1}_{L,i}.$$

Further the required result follows from Lemma 4.1

$$|V_{L,i,k+1}-v(x_i,t_n)| \le C\left(N^{-2}+\Delta t\right)x_i, \quad 1\le i\le N/2-1.$$

Consider

$$\psi_{R,i} = -C\left(N^{-2}+\Delta t\right)(2-x_i), \quad i\in\{N/2,\ldots,N\}.$$

Similarly, for $i\in\{N/2+1,\ldots,N-1\}$ we obtain

$$\left|L_\varepsilon^{N,M}(V_{R,i,k+1}-v(x_i,t_{k+1}))\right|$$

$$\le \begin{cases} C\left[h^2\left(\varepsilon\left\|\dfrac{\partial^4 v}{\partial x^4}\right\|_\infty + \left\|\dfrac{\partial^3 v}{\partial x^3}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], & N/2+1\le i\le 3N/4-1 \\ C\left[(\varepsilon+h_{i+1})(h_i+h_{i+1})\ \|\dfrac{\partial^3 v}{\partial x^3}\|_\infty + h_{i+1}^2\left(\left\|\dfrac{\partial^2 v}{\partial x^2}\right\|_\infty + \left\|\dfrac{\partial v}{\partial x}\right\|_\infty\right) + \Delta t\left\|\dfrac{\partial^2 v}{\partial t^2}\right\|_\infty\right], \\ 3N/4\le i\le N-1. \end{cases}$$

Arguing similarly for $(V_R - v)$, we obtain the following estimate for $i\in\{N/2+1,\ldots,N-1\}$

$$|V_{R,i,k+1}-v(x_i,t_{k+1})| \le C\left(N^{-2}+\Delta t\right)(2-x_i).$$

Now, we define the following two mesh functions on $S_x^N = \{x_i\}_0^N$:

$$S_i = \prod_{j=1}^{i}\left(1+\frac{\alpha h_j}{\epsilon}\right), 1\le i\le N/2, \quad \text{and}$$

$$Q_i = \prod_{j=1}^{N-i}\left(1+\frac{\alpha h_j}{\epsilon}\right), \quad N/2\le i\le N-1$$

so that $S_0 = 1$ and $Q_N = 1$, where α is a positive constant.

Lemma 4.4 Let $\alpha \le \gamma/2$, then the functions S_i and Q_i satisfy

$$-L_\tau^{N,M}S_i \ge \begin{cases} \dfrac{C}{\varepsilon+\alpha H}S_i & \text{if } i=1,\ldots,N/4 \\ \dfrac{C}{\varepsilon+\alpha h}S_i & \text{if } i=N/4+1,\ldots,N/2-1 \end{cases}$$

and

$$-L_\tau^{N,M}Q_i \geq \begin{cases} \dfrac{C}{\varepsilon+\alpha h}Q_i & \text{if} \quad i=N\Big/2+1,\ldots,3N\Big/4-1 \\ \dfrac{C}{\varepsilon+\alpha H}Q_i & \text{if} \quad i=3N\Big/4,\ldots,N-1. \end{cases}$$

Proof. For $i = 1,\ldots,N/4$

$$-L_\tau^{N,M}S_i = -\frac{2\varepsilon}{\hat{h}_i}\left[\left(\frac{S_{i+1}-S_i}{h_{i+1}}\right)-\left(\frac{S_i-S_{i-1}}{h_i}\right)\right] - a_{\frac{i-1}{2}}\left(\frac{S_i-S_{i-1}}{h_i}\right)+b_{\frac{i-1}{2}}S_{\frac{i-1}{2}}.$$

Since $S_i = (1+\frac{\alpha h_i}{\varepsilon})S_{i-1}$, $S_i - S_{i-1} = \frac{\alpha h_i}{\varepsilon}S_{i-1}$ and $a_i \leq -\gamma_1 \leq -2\alpha$. Thus

$$-L_\tau^{N,M}S_i = -\frac{2\alpha}{\hat{h}_i}(S_i-S_{i-1}) - a_{\frac{i-1}{2}}S_{i-1}+b_{\frac{i-1}{2}}S_{\frac{i-1}{2}} \geq \frac{C}{\varepsilon+\alpha h_i}S_i.$$

Next, for $i = N/4+1,\ldots,N/2-1$

$$-L_\tau^{N,M}S_i = -\varepsilon\delta_x^2S_i - a_iD_x^0S_i + b_iS_i + D_t^-S_i \geq \frac{C}{\varepsilon+\alpha h}S_i.$$

Now, for $i = N/2+1,\ldots,3N/4-1$

$$-L_\tau^{N,M}Q_i = -\varepsilon\delta_x^2Q_i - a_iD_x^0Q_i + b_iQ_i + D_t^-Q_i$$

$$= -\frac{\varepsilon}{h}\left[\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right)-\left(\frac{Q_i-Q_{i-1}}{h_i}\right)\right] - a_i\left(\frac{Q_{i+1}-Q_{i-1}}{\hat{h}_i}\right) + b_iQ_i.$$

Since $Q_{i+1}-Q_i = \frac{-\alpha h_{N-i}}{\varepsilon}Q_{i+1}$ and $a_i \geq \gamma_2 \geq 2\alpha$. Thus

$$-L_\tau^{N,M}Q_i \geq \frac{\alpha}{h}(Q_{i+1}-Q_i) + \frac{a_i}{2}\frac{\alpha}{\varepsilon}(Q_{i+1}-Q_{i-1}) \geq \frac{C}{(\varepsilon+\alpha h)}Q_i.$$

For $i = 3N/4,\ldots,N-1$

$$-L_\tau^{N,M}Q_i \geq -\frac{2\varepsilon}{\hat{h}_i}\left[\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right)-\left(\frac{Q_i-Q_{i-1}}{h_i}\right)\right] - a_{\frac{i+1}{2}}\left(\frac{Q_{i+1}-Q_i}{h_{i+1}}\right)$$

and $Q_{i+1}-Q_i = \frac{-\alpha h_{N-i}}{\varepsilon}Q_{i+1}$, $a_i \geq \gamma_2 \geq 2\alpha$. Therefore

$$-L_\tau^{N,M}Q_i \geq \frac{2\alpha}{\hat{h}_i}(Q_{i+1}-Q_i) + a_{\frac{i+1}{2}}\frac{\alpha}{\varepsilon}Q_{i+1} \geq \frac{C}{(\varepsilon+\alpha h_{N-i})}Q_i.$$

Lemma 4.5 Let $\sigma_0 \geq 2/\alpha$. Then for $\{x_i\}_0^N$, the following inequalities are satisfied:

$$\left(\frac{S_i}{S_{N/2}}\right) \leq CN^{\frac{-4(1-2i)}{N}}, \quad i=N/4,\ldots,N/2-1 \tag{4.14}$$

and

$$\left(\frac{Q_i}{Q_{N/2}}\right) \leq CN^{-4\left(\frac{2i}{N-1}\right)}, \quad i=N/2+1,\ldots,3N/4. \tag{4.15}$$

Proof. Since $h_i = h$ for $i = N/4,\ldots,N/2-1$, thus

$$\left(\frac{S_i}{S_{N/2}}\right) = \left(1-\frac{\alpha h}{\varepsilon+\alpha h}\right)^{(N/2-i)}.$$

Taking log on both sides to obtain

$$\left(\frac{S_i}{S_{N/2}}\right) \leq \exp\left((N/2-i)\left(\frac{-\alpha h}{\varepsilon+\alpha h}\right)\right) \leq CN^{-4(1-2i/N)}.$$

Since sequence $N^{\frac{8(\alpha\sigma_0)^2(1-2i/N)(N^{-1}\ln N)}{(1+4\alpha\sigma_0N^{-1}\ln N)}}$ is bounded. Further, for $i = N/2+1,\ldots,3N/4$

$$\left(\frac{Q_i}{Q_{N/2}}\right) = \frac{\prod_{j=1}^{N-i}\left(1+\frac{\alpha h}{\varepsilon}\right)}{\prod_{j=1}^{N/2}\left(1+\frac{\alpha h}{\varepsilon}\right)} = \left(1+\frac{\alpha h}{\varepsilon}\right)^{-(i-N/2)}.$$

Similarly, it is easy to follow that $\left(\frac{Q_i}{Q_{N/2}}\right) \leq CN^{-4(2i/N-1)}$.

Now, we will calculate the errors for the layers components W_L and W_R in $((0,1-\sigma]\cup[1+\sigma,2))\times(0,T]$.

Lemma 4.6 Let $\alpha \leq \gamma/2$ and $\sigma_0 \geq 2/\alpha$. Then under the assumptions (4.7) and (4.8), the errors associated to the layer components satisfy

$$\begin{cases} \left|W_{L,i,k+1} - w(x_i,t_{k+1})\right| \leq CN^{-2} & \text{for} \quad i=1,\ldots,N/4 \\ \left|W_{R,i,k+1} - w(x_i,t_{k+1})\right| \leq CN^{-2} & \text{for} \quad i=3N/4,\ldots,N-1. \end{cases}$$

Proof. From (4.13), we have

$$U_{N/2,k+1} = V_{R,N/2,k+1} + W_{R,N/2,k+1} = V_{L,N/2,k+1} + W_{L,N/2,k+1}$$

$$\left|U_{N/2,k+1}\right| \leq \left|V_{L,N/2,k+1}\right| + \left|W_{L,N/2,k+1}\right|.$$

Using Theorem 2.4 and Lemma 4.2 to obtain $\left|W_{N/2,k+1}\right| \leq C$. Consider

$$\phi_{L,i}^{k+1} = -C\left(\frac{S_i}{S_{N/2}}\right) \quad \text{for} \quad i=0,\ldots,N/2.$$

For $i = 0,\ldots,N/2$ and using Lemma 4.4, we have

$$L_\tau^{N,M}\left(\phi_{L,i}^{k+1} \pm W_{L,i,k+1}\right) = L_\tau^{N,M}\phi_{L,i}^{k+1} \pm L_\tau^{N,M} W_{L,i,k+1} \geq \frac{C}{\varepsilon+\alpha H}\frac{S_i}{S_{N/2}} \geq 0$$

and $W_{L,0,k+1} = 0 = W_{L,i,0}$ for $i \leq N/2$. Thus

$$\phi_{L,0}^{k+1} \pm W_{L,0,k+1} = \phi_{L,0}^{k+1} = \frac{-CS_0}{S_{N/2}} \leq 0.$$

Similarly, we can show $\phi_{L,i}^{0} \pm W_{L,i,0} \leq 0$. Therefore using Lemma 4.1, we get

$$|W_{L,i,k+1}| \leq C\left(\frac{S_i}{S_{N/2}}\right) \quad \text{for} \quad i = 1,\ldots,N/2-1. \tag{4.16}$$

Further

$$|W_{L,i,k+1}| \leq C\left(\frac{S_i}{S_{N/2}}\right) \leq C\left(\frac{S_{N/4}}{S_{N/2}}\right)$$

and from Lemma 4.5, we have

$$|W_{L,i,k+1}| \leq CN^{-2}. \tag{4.17}$$

Since $\alpha \leq \gamma/2 \leq \gamma$, $\sigma = \sigma_0 \varepsilon \ln N$, $\sigma_0 \geq 2/\alpha$ and using Theorem 2.4, we have

$$|w(x_i,t_{k+1})| \leq C\exp\left(\frac{-(1-x_i)\gamma_1}{\varepsilon}\right) \leq CN^{-2} \text{ for } i=1,\ldots,N/4. \tag{4.18}$$

Combine (4.17) and (4.18) to obtain

$$|W_{L,i,k+1} - w(x_i,t_{k+1})| \leq CN^{-2} \quad \text{for} \quad i = 1,\ldots,N/4.$$

Now, consider $\phi_{R,i}^{k+1} = -C\left(\frac{Q_i}{Q_{N/2}}\right)$ for $i = N/2,\ldots,N$. Then using Lemma 4.4, for $i = 3N/4,\ldots,N-1$ to obtain

$$L_\tau^{N,M}\left(\phi_{R,i}^{k+1} \pm W_{R,i,k+1}\right) = L_\tau^{N,M}\phi_{R,i}^{k+1} \pm L_\tau^{N,M} W_{R,i,k+1} \geq \frac{C}{\varepsilon+\alpha H}\frac{Q_i}{Q_{N/2}} \geq 0$$

and $W_{R,N,k+1} = 0 = W_{R,i,0}$. Thus $\phi_{R,N}^{k+1} \pm W_{R,N,k+1} = \phi_{R,N}^{k+1} \leq 0$.

Similarly, we can show $\phi_{R,i}^{0} \pm W_{R,i,0} \leq 0$. Thus using Lemma 4.1, we have $\phi_{R,i}^{k+1} \pm W_{R,i,k+1} \leq 0$ for $i = N/2,\ldots,N$. Thus

$$\left|W_{R,i,k+1}\right| \leq \left|\phi_{R,i}^{k+1}\right| \leq C\left(\frac{Q_i}{Q_{N/2}}\right) \text{ for } i = N/2+1,\ldots,N-1$$
$$\leq C\left(\frac{Q_{N/4}}{Q_{N/2}}\right) \text{ for } i = 3N/4,\ldots,N-1$$

since Q is decreasing and from Lemma 4.5, we have $\left(\frac{Q_{N/4}}{Q_{N/2}}\right) \leq CN^{-2}$. Thus

$$|W_{R,i,k+1}| \leq CN^{-2}. \tag{4.19}$$

Now, using Theorem 2.4 and doing the same calculation as we do for W_L, we have

$$|w(x_i,t_{k+1})| \leq C\exp\left(\frac{-(1-x_i)\gamma_2}{\varepsilon}\right) \leq CN^{-2}. \tag{4.20}$$

Using (4.19) and (4.20), we obtain $\left|W_{R,i,k+1} - w(x_i, t_{k+1})\right| \leq CN^{-2}$.

Next, we will state and prove some lemmas required to obtain ε uniform error bounds.

Lemma 4.7 The following inequalities hold true:

$$\exp(-\alpha(1-x_i)/\varepsilon) \leq \left(\frac{S_i}{S_{N/2}}\right), \quad i = 1,\ldots,N/2-1 \tag{4.21}$$

and

$$\exp(-\alpha(x_i-1)/\varepsilon) \leq \left(\frac{Q_i}{Q_{N/2}}\right), \quad i = N/2+1,\ldots,N-1. \tag{4.22}$$

Proof. Follows from Lemma 2.5 in [40], for each j, we obtain

$$\exp\left(\frac{-\alpha h_j}{\varepsilon}\right) = \left(\exp\left(\frac{\alpha h_j}{\varepsilon}\right)\right)^{-1} \leq \left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1}. \tag{4.23}$$

Multiplying (4.23), for $j = i+1,\ldots,N/2$ to obtain

$$\prod_{i+1}^{N/2}\exp\left(\frac{-\alpha h_j}{\varepsilon}\right) \leq \prod_{i+1}^{N/2}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \leq \frac{S_i}{S_{N/2}}.$$

Further, multiplying (4.23), for $j = N-(i+1),\ldots,N/2$, we have

$$\prod_{N-(i+1)}^{N/2}\exp\left(\frac{-\alpha h_j}{\varepsilon}\right) \leq \prod_{N-(i+1)}^{N/2}\left(1+\frac{\alpha h_j}{\varepsilon}\right)^{-1} \leq \frac{Q_i}{Q_{N/2}}.$$

Lemma 4.8 The difference operators D_x^B and D_x^F satisfy the following inequalities:

$$D_x^B S_{N/2} \geq \frac{C}{\varepsilon + \alpha h} S_{N/2}, \quad -D_x^F Q_{N/2} \geq \frac{C}{\varepsilon + \alpha h} Q_{N/2}.$$

Proof. Since $(S_i - S_{i-1}) = \frac{\alpha h_i}{\varepsilon} S_{i-1}$, we have

$$D_x^B S_{N/2} = \frac{1}{2h}\left(S_{N/2-2} - 4S_{N/2-1} + 3S_{N/2}\right)$$
$$\geq \frac{\alpha}{\varepsilon} \frac{S_{N/2}}{\left(1+\frac{\alpha h}{\varepsilon}\right)^2}\left(1+\frac{\alpha h}{\varepsilon}\right) \geq \frac{C}{\varepsilon + \alpha h} S_{N/2}.$$

Since $Q_{i+1} - Q_i = \frac{-\alpha h_{N-i}}{\varepsilon} Q_{i+1}$, we have

$$-D_x^F Q_{N/2} = \frac{1}{2h}\left(Q_{N/2+2} - 4Q_{N/2+1} + 3Q_{N/2}\right)$$
$$\geq \frac{\alpha}{\varepsilon} \frac{\left(1+\frac{\alpha h}{\varepsilon}\right)}{\left(1+\frac{\alpha h}{\varepsilon}\right)^2} Q_{N/2} \geq \frac{C}{\varepsilon + \alpha h} Q_{N/2}.$$

Theorem 4.9 Let u and $U_{i,k+1}$ be the solution of (2.1) and (4.5), respectively. Then under the assumptions (4.7), (4.8) and $\alpha \leq \gamma/2$ and $\sigma_0 \geq 2/\alpha$, we get

$$|U_{i,k+1} - u(x_i,t_{k+1})| \leq \begin{cases} C\left(N^{-2}+\Delta t\right) & \text{for} \quad i = 1,\ldots,N/4, 3N/4,\ldots,N-1 \\ C\left(N^{-2}\ln^2 N + \Delta t\right) & \text{for} \quad i = N/4+1,\ldots,3N/4-1. \end{cases} \tag{4.24}$$

Proof. We compute the error separately in the layer region and outside the layer region.

Case 1: For $i = 1,\ldots,N/4, 3N/4,\ldots,N-1$.Follows from the triangle inequality to $U-u$ and using Lemma 4.3 and Lemma 4.6 to obtain

$$|U_{i,k+1} - u(x_i,t_{k+1})| \leq |V_{L,i,k+1} - v(x_i,t_{k+1})|$$
$$+ |W_{L,i,k+1} - w(x_i,t_{k+1})|$$
$$\leq C\left(N^{-2}+\Delta t\right).$$

Case 2: Here, we need to find out the error estimate $|U_{i,k+1} - u(x_i,t_{k+1})|$ for $i = N/4+1,\ldots,3N/4-1$. For $i = N/4, 3N/4$, From Case 1

$$|U_{i,k+1} - u(x_i,t_{k+1})| \leq C\left(N^{-2}+\Delta t\right). \tag{4.25}$$

Consider

$$L_\tau^{N,M}\left(U_{i,k+1} - u(x_i,t_{k+1})\right) = L_\tau^{N,M} U_{i,k+1} - L_\tau^{N,M} u(x_i,t_{k+1})$$
$$= \left(\varepsilon\left(\frac{\partial^2}{\partial x^2} - \delta x^2\right) + a\left(\frac{\partial}{\partial x} - D_x^0\right) - \left(\frac{\partial}{\partial t} - D_t^-\right)\right) u(x_i,t_{k+1}).$$

Using Theorem 2.4, we have

$$\left|L_\tau^{N,M}\left(U_{i,k+1} - u(x_i,t_{k+1})\right)\right| \leq h \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon\left|\frac{\partial^4 u}{\partial x^4}\right| + \left|\frac{\partial^3 u}{\partial x^3}\right|\right) dx$$
$$+ C\Delta t \left\|\frac{\partial^2 u}{\partial t^2}\right\|_\infty$$
$$\leq C\left[h^2 + \frac{h}{\varepsilon^2}\exp\left(\frac{-(1-x_i)\gamma_1}{\varepsilon}\right)\sinh\left(\frac{h\gamma_1}{\varepsilon}\right)\right] + \Delta t. \tag{4.26}$$

From (3.5), we have

$$2\sigma_0\gamma^* \leq \frac{N}{\ln N} 4\sigma_0\gamma_1 \leq \frac{2N}{\ln N}, \ \frac{4\sigma\gamma_1}{\varepsilon \ln N} \leq \frac{2N}{\ln N} \text{ and } \frac{h\gamma_1}{\varepsilon} \leq 2.$$

Since $\sinh x \leq cx, \quad x \in [0, \quad 2]$. This implies $\sinh\left(\frac{h\gamma_1}{\varepsilon}\right) \leq C\left(\frac{\gamma_1 h}{\varepsilon}\right)$. Therefore (4.26) becomes

$$\left|L_\tau^{N,M}\left(U_{i,k+1} - u(x_i,t_{k+1})\right)\right|$$
$$\leq C\left[\left(h^2 + \frac{h^2}{\varepsilon^3}\exp\left(\frac{-(1-x_i)\gamma_1}{\varepsilon}\right)\right) + \Delta t\right]. \tag{4.27}$$

Similarly, for $i = N/2+1,\ldots,3N/4-1$

$$\left|L_\tau^{N,M}\left(U_{i,k+1} - u(x_i,t_{k+1})\right)\right|$$
$$\leq C\left[\left(h^2 + \frac{h^2}{\varepsilon^3}\exp\left(\frac{-(x_i-1)\gamma_2}{\varepsilon}\right)\right) + \Delta t\right]. \tag{4.28}$$

Further at $x_{N/2} = 1$

$$\left|L_\tau^{N,M}\left(U_{N/2,k+1} - u(x_{N/2},t_{k+1})\right)\right|$$
$$= \left|\tilde{f}_{\tau,N/2,k+1} - L_\tau^{N,M} u_{N/2,k+1}\right| \leq C\left(\frac{h^2}{\varepsilon^3} + \Delta t\right).$$

Define the following discrete function

$$\Theta_i^k = \begin{cases} -C\left(N^{-2}+\Delta t\right)\left(1+\left(x_i-(1-\sigma)\right)\right) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{S_i}{S_{N/2}}\right) & \text{for} \quad i = N/4, \ldots, N/2 \\ -C\left(N^{-2}+\Delta t\right)\left(1+\left((1+\sigma)-x_i\right)\right) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{Q_i}{Q_{N/2}}\right) & \text{for} \quad i = N/2+1, \ldots, 3N/4. \end{cases}$$

Then

$$L_\tau^{N,M}\Theta_i^{k+1} = \begin{cases} -C\left(N^{-2}+\Delta t\right)L_\tau^{N,M}(x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M}S_i}{S_{N/2}}\right) & \text{for} \quad i = N/4, \ldots, N/2 \\ C\left(N^{-2}+\Delta t\right)L_\tau^{N,M}(x_i) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M}Q_i}{Q_{N/2}}\right) & \text{for} \quad i = N/2+1, \ldots, 3N/4 \end{cases}$$

$$= \begin{cases} -C\left(N^{-2}+\Delta t\right)\left(a_i-b_ix_i\right) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M}S_i}{S_{N/2}}\right) & \text{for} \quad i = N/4, \ldots, N/2 \\ C\left(N^{-2}+\Delta t\right)\left(a_i-b_ix_i\right) - C\dfrac{h^2}{\varepsilon^2}\left(\dfrac{L_\tau^{N,M}Q_i}{Q_{N/2}}\right) & \text{for} \quad i = N/2+1, \ldots, 3N/4. \end{cases}$$

Using assumption $\alpha \leq \gamma/2$, Lemma 4.6 and Lemma 4.7 to obtain

$$L_\tau^{N,M}\Theta_i^{k+1} \geq \begin{cases} C\gamma_1\left(N^{-2}+\Delta t\right) + C\dfrac{h^2}{\varepsilon^2}\exp\left(-(1-x_i)\gamma_1/\varepsilon\right) & \text{for} \quad i = N/4, \ldots, N/2 \\ C\gamma_2\left(N^{-2}+\Delta t\right) + C\dfrac{h^2}{\varepsilon^2}\exp\left(-(x_i-1)\gamma_2/\varepsilon\right) & \text{for} \quad i = N/2+1, \ldots, 3N/4. \end{cases}$$

Assumption (4.7) implies $\frac{h}{\varepsilon} \leq \frac{2}{\gamma^*}$ and using Lemma 4.8, we get

$$L_\tau^{N,M}\Theta_i^{k+1} \geq \left(D_x^F - D_x^B\right)\Theta_i^{k+1} \geq 2C\left(N^{-2}+\Delta t\right) + 2C\frac{h^2}{\varepsilon^3}. \tag{4.29}$$

Therefore, it follows from (4.25)–(4.29) that

$$\begin{cases} L_\tau^{N,M}\Theta_i^{k+1} \geq \left|L_\tau^{N,M}(U_{i,k+1} - u(x_i, t_{k+1}))\right| & \text{for } i = N/4+1, \ldots, 3N/4-1 \\ -\Theta_i^{k+1} \geq |U_{i,k+1} - u(x_i, t_{k+1})| & \text{for } i = N/4, 3N/4 \text{ and} \\ -\Theta_i^0 \geq |U_{i,0} - u(x_i, t_0)|. & \end{cases}$$

Then applying discrete maximum principle to $\Theta_i^{k+1} \pm (U_{i,k+1} - u(x_i, t_{k+1}))$ over the domain $\overline{D}^{N,M} \cap ([1-\sigma, 1+\sigma] \times [0, T])$ we obtain for $i = N/4+1, \ldots, 3N/4-1$

$$|U_{i,k+1} - u(x_i, t_{k+1})| \leq C\left(\frac{h^2}{\varepsilon^2} + \Delta t\right) \leq C\left(N^{-2}\ln^2 N + \Delta t\right).$$

5. Numerical illustration

The performance of the proposed method is examined in this section and the theoretical estimates are numerically verified. We consider two test problems for numerical computations.

Example 5.1 Consider the following singularly perturbed problem of class (2.1) – (2.3)

$$\begin{cases} \varepsilon u_{xx} + a(x)u_x - x(2-x)u(x,t) - u_t = f(x,t) + u(x-1,t), & (x,t) \in (0,2) \times (0,2], \\ u(x,0) = 0, \quad x \in [0,2], \\ u(x,t) = t^2, \quad (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0, \quad t \in (0,2], \end{cases}$$

where

$$a(x) = \begin{cases} -(2 + x(2-x)), & x \in [0,1] \\ (2 + x(2-x)), & x \in (1,2] \end{cases}$$

and

$$f(x,t) = \begin{cases} 2(1+x^2)t^2, & (x,t) \in [0,1] \times [0,2] \\ 3(1+x^2)t^2, & (x,t) \in (1,2] \times [0,2] \end{cases}$$

Example 5.2 Consider the following singularly perturbed problem of class (2.1) – (2.3)

$$\begin{cases} \varepsilon u_{xx} + a(x)u_x - 5u(x,t) - u_t = f(x,t) + 2u(x-1,t), & (x,t) \in (0,2) \times (0,2], \\ u(x,0) = 0, \quad x \in [0,2], \\ u(x,t) = 0, \quad (x,t) \in [-1,0] \times [0,2], \\ u(2,t) = 0, \quad t \in (0,2], \end{cases}$$

where

$$a(x) = \begin{cases} -(4+x^2), & x \in [0,1] \\ (6-x^2), & x \in (1,2] \end{cases}$$

and

$$f(x,t) = \begin{cases} 4xt^2\exp(-t), & (x,t) \in [0,1] \times [0,2] \\ 4(2-x)t^2\exp(-t), & (x,t) \in (1,2] \times [0,2] \end{cases}$$

The exact solutions for the problems are unknown for comparison. Therefore, we use the double mesh principle to estimate the error. The maximum point–wise error ($E_\varepsilon^{N,\triangle t}$) and order of convergence ($R_\varepsilon^{N,\triangle t}$) are calculated using:

$$E_\varepsilon^{N,\triangle t} := \max\left|U^{N,\triangle t}(x_i, t_{k+1}) - \tilde{U}^{2N,\triangle t/2}(x_i, t_{k+1})\right|,$$

$$R_\varepsilon^{N,\triangle t} := \log_2\left(\frac{E_\varepsilon^{N,\triangle t}}{E_\varepsilon^{2N,\triangle t/2}}\right),$$

where $U^{N,\triangle t}(x_i, t_{k+1})$ and $\tilde{U}^{2N,\triangle t/2}(x_i, t_k)$ are the approximate solutions obtained on the mesh $D^{N,M}$ and $D^{2N,2M}$, respectively. When, the perturbation parameter approaches zero, the problem's solution exhibits turning point behaviour (Figs. 1–4).

Maximum absolute error and order of convergence for problems 5.1 and 5.2 are tabulated in Tables 1 and 2. Moreover, the maximum absolute errors for problems 5.1 and 5.2 are plotted in Figs. 5 and 6, respectively. The surface plot of the numerical solution for problems 5.1 and 5.2 are plotted in Figs. 1 and 3, respectively. Also, the numerical solutions at final time step ($t = 2$) for different values of ε are displayed in Figs. 2 and 4.

The numerical results tabulated in Tables 1 and 2 do not clearly depict the theoretical order of convergence for spatial discretization. It is to be noted that the error in numerical solution is due to spatial and temporal discretization. As a consequence, the errors given in Tables 1 and 2 are a combination of temporal and spatial errors, with the layer regions playing a significant role.

The hybrid difference scheme improves accuracy in space only. To verify this, we performed numerical experiments for $M = N^2$ and numerical results are tabulated in Table 3. In Table 4, we have fixed $\varepsilon = 2^{-6}$ and $N = 512$ and reduce Δt by half and the associated errors are presented at different values of x. It can be observed that the errors reduce by almost half which confirms the first-order convergence in time. Figures 7 and 8 have also been illustrated to show the errors in layer region and outside layer region. This demonstrates that the numerical method is second-order spatially accurate outside of the interior layer and the errors are reduced in the layer region as claimed in Theorem 4.9. The implementation of the proposed method is done in MATLAB R2015b (The Mathworks, Inc.). The program code is uploaded in GitHub and the URL of the source code is http://shorturl.at/gjlCS.

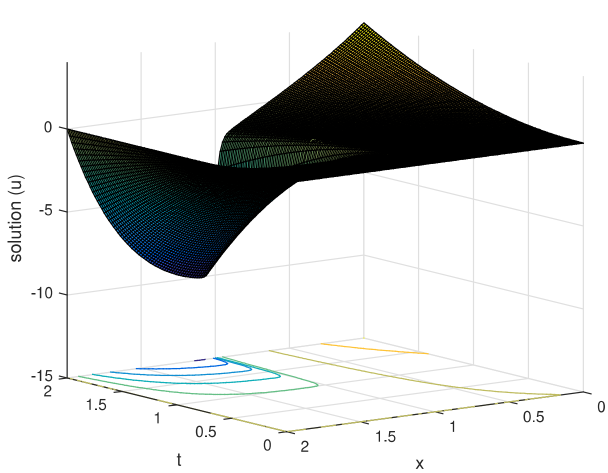

Fig. 1. Numerical solution of Example 5.1 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

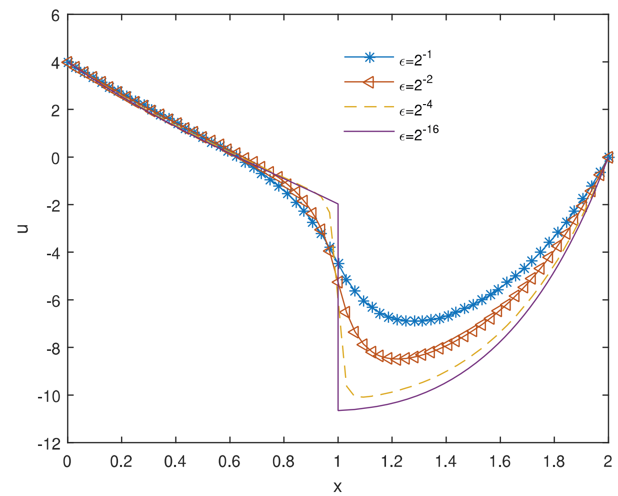

Fig. 2. Numerical solution of Example 5.1 at $t = 2$ for different values of ε when $N = 128$.

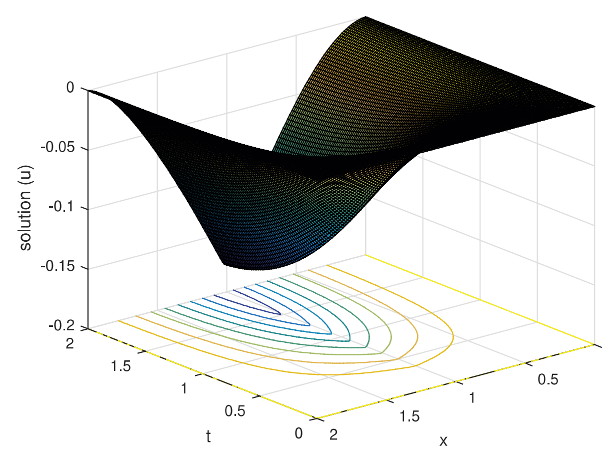

Fig. 3. Numerical solution of Example 5.2 for $\varepsilon = 2^{-4}$ when $M = N = 128$.

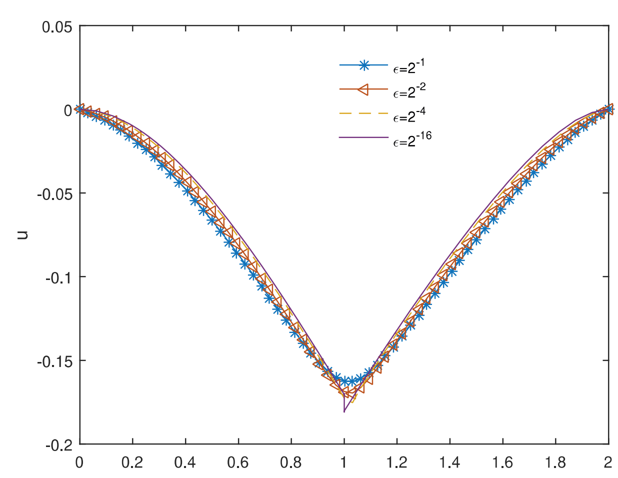

Fig. 4. Numerical solution of Example 5.2 at $t = 2$ for different values of ε when $N = 128$.

Table 1. Maximum absolute error and order of convergence for Example 5.1 for different values of ε, M and N when M = N.

N	$\varepsilon = 2^{-2}$	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
32	1.099e-01	8.697e-01	8.723e-01	8.712e-01	8.709e-01	8.708e-01
	1.6274	1.5675	1.2466	1.2572	1.2601	1.2609
64	3.558e-02	2.934e-01	3.676e-01	3.644e-01	3.635e-01	3.633e-01
	1.3320	2.0478	1.5866	1.6018	1.6060	1.6071
128	1.413e-02	7.096e-02	1.223e-01	1.200e-01	1.194e-01	1.192e-01
	1.1428	1.9332	1.7309	1.7302	1.7308	1.7311
256	6.400e-03	1.858e-02	3.687e-02	3.619e-02	3.598e-02	3.592e-02
	1.0336	1.7107	1.6093	1.6294	1.6359	1.6376
512	3.126e-03	5.676e-03	1.208e-02	1.169e-02	1.157e-02	1.154e-02
	1.0066	1.5189	1.5926	1.6132	1.6207	1.6241

Table 2. Maximum absolute error and order of convergence for Example 5.2 for different values of ε, M and N when M = N.

N	$\varepsilon = 2^{-2}$	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
32	1.886e-03	4.150e-03	2.485e-03	2.146e-03	2.288e-03	2.323e-03
	1.4071	1.6239	1.1244	1.0741	1.0986	1.1044
64	7.113e-04	1.346e-03	1.139e-03	1.019e-03	1.068e-03	1.080e-03
	1.5199	1.6981	1.5245	1.3442	1.3740	1.3811
128	2.480e-04	4.149e-04	3.961e-04	4.015e-04	4.122e-04	4.149e-04
	1.5620	1.4068	1.5615	1.6111	1.6576	1.6714
256	8.400e-05	1.565e-04	1.342e-04	1.313e-04	1.304e-04	1.302e-04
	1.5102	1.5403	1.5137	1.6794	1.7007	1.7062
512	2.949e-05	5.380e-05	4.700e-05	4.101e-05	4.014e-05	3.991e-05
	1.4117	1.5972	1.4798	1.4726	1.4368	1.4272

6. Concluding remark

Singularly perturbed parabolic functional differential equation with discontinuous coefficient and source term is numerically solved. The problem's solution takes into account not just the present state of the physical system but also its history. The simultaneous presence of discontinuous data and delay makes the problem stiff. In the limiting case, the solution of the problem exhibits multi-scale character. There are narrow regions where solution derivatives grow exponentially and exhibit turning point behaviour, leading to sharp interior layers across discontinuities.

A hybrid numerical scheme composed of a central difference scheme in the layer region and a midpoint upwind scheme outside the layer region is used to discretize space variable over a specially generated mesh. Whereas an implicit finite difference scheme is used to discretize the time variable. The mesh has been chosen so that most of the mesh

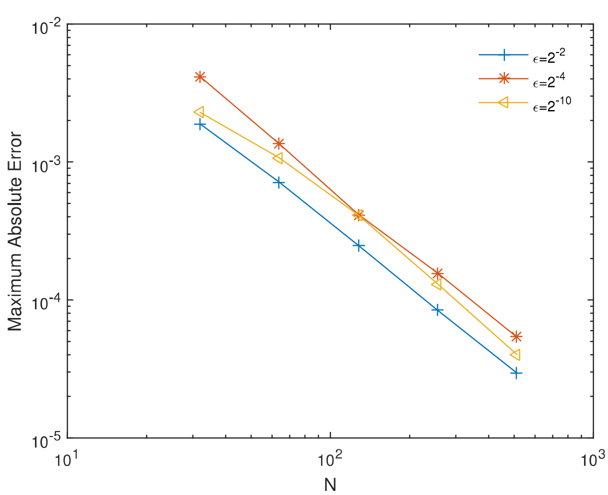

Fig. 5. Error plot for Example 5.1.

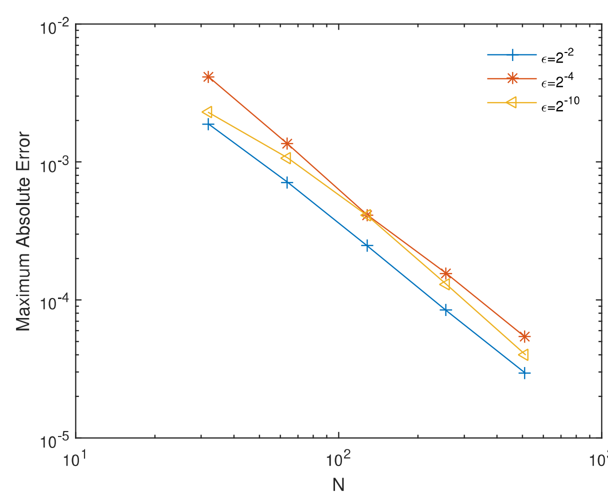

Fig. 6. Error plot for Example 5.2.

Table 3. Maximum absolute error and order of convergence for Example 5.1 and 5.2 for different values of M and N when $M = N^2$ and $\varepsilon = 2^{-10}$.

	For Example 5.1			For Example 5.2		
N	*left region* $[0, 1-\sigma]$	*interior layer region* $(1-\sigma, 1+\sigma)$	*righ tregion* $[1+\sigma, 2]$	*left region* $[0, 1-\sigma]$	*interior layer region* $(1-\sigma, 1+\sigma)$	*right region* $[1+\sigma, 2]$
32	7.602e-03	9.073e-01	2.237e-02	1.863e-04	2.215e-03	6.044e-05
	1.9665	1.3111	1.9344	1.9878	1.1088	1.8829
64	1.973e-03	3.656e-01	5.687e-03	4.652e-05	1.026e-03	1.632e-05
	1.9760	1.6430	1.9033	1.9624	1.3972	1.7645
128	5.083e-04	1.170e-01	1.478e-03	1.182e-04	3.895e-04	4.790e-06
	1.9592	1.7816	1.8513	1.9592	1.7332	1.5841
256	1.323e-04	3.404e-02	3.992e-04	3.064e-06	1.186e-04	1.594e-06
	1.9438	1.9236	1.8205	1.9382	1.8560	1.5632

Table 4. Maximum absolute error and order of convergence for Example 5.2 for different values of M and x when N = 512 and $\varepsilon = 2^{-6}$.

x	$M = 32$	64	128	256	512
$x_{N/2+1}$	3.837e-04	2.058e-04	1.071e-04	5.486e-05	2.773e-05
	0.8988	0.9423	0.9651	0.9843	0.9981
$x_{N/2+4}$	3.999e-04	2.122e-4	1.093e-04	6.530e-05	3.225e-05
	0.9142	0.9571	0.7432	1.0178	1.0184

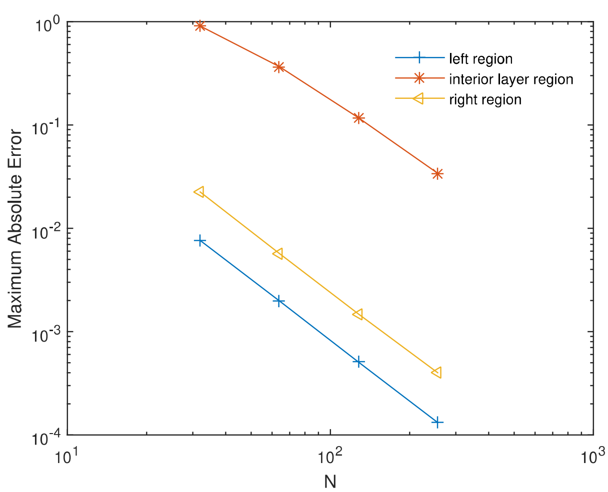

Fig. 7. Error plot of the spatial order of convergence for example 5.1.

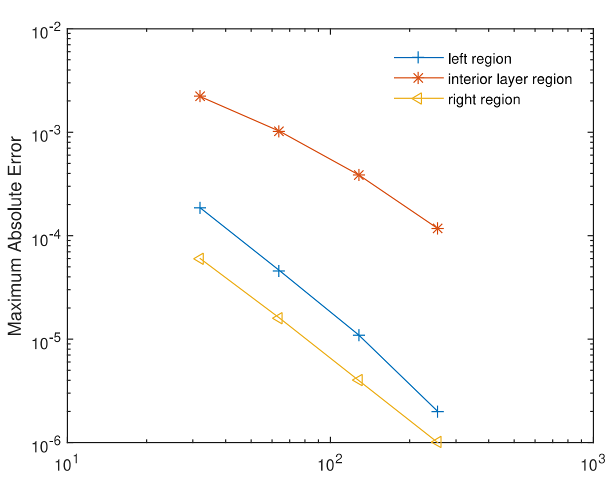

Fig. 8. Error plot of the spatial order of convergence for example 5.2.

points remain in the regions with rapid transitions. The proposed numerical method has been analyzed for consistency, stability and convergence. Extensive theoretical analysis is performed to obtain consistency and error estimates. It is found that the method proposed is unconditionally stable, and the convergence obtained is parameter uniform. Numerical illustrations are presented for two test examples that demonstrate the effectiveness of the technique. Convergence obtained in practical satisfies theoretical predictions.

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