A Representation of GM-Variation in Waves by the Volterra System

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A REPRESENTATION OF GM-VARIATION IN WAVES BY THE VOLTERRA SYSTEM

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Keywords: GM-variation, RoRo-ship, Transfer function, Volterra System.

ABSTRACT

As known, the variation of the metacentric height of a ship in irregular waves is not a pure linear process, particular when a ship has large beam to draught ratio and large flare near the waterline at bow and stern. This kind of unconventional hull form is usually adopted for modern RoRo-ship, cruise-ship etc. which allows large cargo space and high service speed. In this paper, the GM-variation is derived into a function series with respect to the variation order and represented by the Volterra system. The transfer functions for the different orders are integrated numerically or analytically through expressing the sectional beam, area and moment in Taylor’s series as function of the momentary water line. Thereby the explicit relationship between the hull form and GM-variation can be obtained. The numerical result has shown the significant effect of the second order term in the Volterra system on the GM-variation in waves. Hence, the non-linear characteristics of the GM-variation in an irregular wave can be easily analyzed by means of available nonlinear probability theories or Monte-Carlo simulation technique.

INTRODUCTION

In her service life, a ship will involuntarily experience a lot occasions of storm weather and rough seas, during which some dynamic problems can happen to the ship. For example, roll motion in resonance with the wave excitation, roll motion due to stability reduction or loss in combination with wind- or wave-induced excitation moment, and parametrically excited roll motion. The GM-variation of a ship in wave is an important evaluation factor in the latter two problems, and that is why it has been an interesting object for research internationally during the past years [1-3].

It has been always desired to explain the basic mechanisms behind different ship dynamic stability problems by means of some basic parameters, which can not only provide explicit description of relationship between the ship main particulars and its dynamic behavior, but also result in design criteria for dynamic stability. However, owing to the limited computation capacity before 1970’s, the analyses for the effect of GM-variation on the dynamic stability problems were mostly qualified by assuming linear relationship on the wave amplitude, see [1] by Dunwoody. He derived an explicit formulation for the GM-variation related to the hull form as a linear response to the wave elevation along a ship, therefore, the spectrum theory could be applied for description of the GM-variation of a ship in long crested seas. It provides the possibility for study of ship roll behavior by means of an idealized single differential equation. When calculating the heave and pitch motions of the quasi-static equilibrium is assumed, i.e., only the Froude-Kryloff forces are taken into account.

Nevertheless, the GM-variation has nonlinear characteristics in relation to the wave amplitude and the degree of nonlinearity is dependent on the hull form. The numerical investigation into the GM-variation in regular and irregular waves by Palmquist [4] in 1994 has shown that the GM-variation in an irregular wave can be described by a sum of a linear Gaussian process and a nonlinear process due to the higher order influence. Besides, the numerical result has also shown that the nonlinear process is strongly coupled with the evolution of the linear Gaussian process.

In this paper, the GM-variation has been derived into a series with respect to the variation order and then represented by the Volterra system. The transfer functions for the different orders are integrated analytically by means of expressing the sectional beam and moment into Taylor’s series as a function of the momentary draft change around the mean one. As similar as a function, \( f(x) \), can be expressed in a Taylor’s series:

\[
\begin{align*}
f(h + x) &= f(h) + f'(h) \cdot x + \frac{1}{2!} \cdot f''(h) \cdot x^2 + \cdots \nonumber \\
&+ \frac{1}{(n + 1)!} \cdot f^{n-1}(h) \cdot x^{n-1} + R_n.
\end{align*}
\] (1)
a nonlinear system can be written in the form of Volterra system [5],

\[ y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} h_n(\sigma_1, \ldots, \sigma_n) \cdot u(t - \sigma_1) \ast \cdots \ast u(t - \sigma_n) \cdot d\sigma_1 \cdots d\sigma_n, \]

where \( u(t) \) is the input signal and also the wave elevation in our case, and \( h_n(\sigma_1, \ldots, \sigma_n) \) are kernel functions for the nonlinear system. When the input signal \( u(t) \) is a stochastic variable represented by a power spectrum, the system can be rewritten in the following way

\[ y(t) = \sum_{n=1}^{\infty} H_n(\omega_n) \cdot e^{i\omega_n t} + \beta_n \]

for a linear system, and

\[ y(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{mn}(\omega_m, \omega_n) \cdot e^{i(\omega_m - \omega_n) t} + \beta_{mn} \]

for a second order system, and so on. Where \( \omega_n \) and \( \omega_m \) in the above Eqs. denote frequency. \( \beta_{mn} \) and \( \beta_n \) are phase angles. \( i = \sqrt{-1} \). The transfer functions, \( H_1(\omega) \) and \( H_2(\omega_1, \omega_2) \), are the Fourier’s transforms of \( h_1(\sigma) \) and \( h_2(\sigma_1, \sigma_2) \) in Eq. (2). Our task in the following is to derive the transfer functions \( H_1(\sigma) \) and \( H_2(\sigma_1, \sigma_2) \) to represent the GM-variation of a ship in waves. If needed, \( h_1(\sigma) \) and \( h_3(\sigma_1, \sigma_2) \) can be obtained by inverse Fourier’s transforms of \( H_1(\sigma) \) and \( H_2(\sigma_1, \sigma_2) \).

**GM-VARIATION AS AN INTEGRATION SERIES**

Mathematically, the section beam of a ship as a function of draft can be expanded around the mean draft \( T(x) \) as a Taylor’s series

\[ B(x, T(x) + z) = B(x, T(x)) + \frac{\partial B}{\partial c} \cdot z + \frac{1}{2!} \frac{\partial^2 B}{\partial c^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 B}{\partial c^3} \cdot z^3 + \ldots, \quad \text{(5)} \]

where \( z \) is a variable for the sectional draft change. As well for the sectional moment with respect to the keel line,

\[ M(x, T(x) + z) = M(x, T(x)) + \frac{\partial M}{\partial c} \cdot z + \frac{1}{2!} \frac{\partial^2 M}{\partial c^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 M}{\partial c^3} \cdot z^3 + \ldots, \quad \text{(6)} \]

and the sectional area,

\[ A(x, T(x) + z) = A(x, T(x)) + \frac{\partial A}{\partial c} \cdot z + \frac{1}{2!} \frac{\partial^2 A}{\partial c^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 A}{\partial c^3} \cdot z^3 + \ldots \quad \text{(7)} \]

For the sake of simplicity, Eqs. (5) to (7) can be respectively expressed as

\[ B(x, T(x) + z) = B(x, T(x)) + c_1(x) \cdot z + c_2(x) \cdot z^2 + c_3(x) \cdot z^3 + \ldots; \quad \text{(8)} \]

\[ M(x, T(x) + z) = M(x, T(x)) + d_1(x) \cdot z + d_2(x) \cdot z^2 + d_3(x) \cdot z^3 + \ldots; \quad \text{(9)} \]

\[ A(x, T(x) + z) = A(x, T(x)) + e_1(x) \cdot z + e_2(x) \cdot z^2 + e_3(x) \cdot z^3 + \ldots; \quad \text{(10)} \]

where

\[ c_n(x) = \frac{\partial B}{\partial c}, \quad c_2(x) = \frac{1}{2!} \frac{\partial^2 B}{\partial c^2}, \quad c_3(x) = \frac{1}{3!} \frac{\partial^3 B}{\partial c^3} + \ldots; \]

\[ d_n(x) = \frac{\partial M}{\partial c}, \quad d_2(x) = \frac{1}{2!} \frac{\partial^2 M}{\partial c^2}, \quad d_3(x) = \frac{1}{3!} \frac{\partial^3 M}{\partial c^3} + \ldots; \]

\[ e_n(x) = \frac{\partial A}{\partial c}, \quad e_2(x) = \frac{1}{2!} \frac{\partial^2 A}{\partial c^2}, \quad e_3(x) = \frac{1}{3!} \frac{\partial^3 A}{\partial c^3} + \ldots. \]

It should be noted that \( c_n(x), d_n(x), e_n(x) \) can be obtained numerically by using the piecewisely polynomial functions fitting the section beam, section moment and section area along the ship.

The initial metacentric height \( GM_0 \) at the mean draft in still water can be calculated as followed:

\[ GM_0 = KB + BM - KG, \quad \text{(11)} \]

where \( KB \) is the height of buoyancy center \( B \) above the keel \( K \) and defined as

\[ KB = \frac{1}{V} \int_{L} M(x, T(x)) \, dx, \quad \text{(12)} \]

\( BM \) is the height of transverse metacenter \( M \) above buoyancy center \( B \) and defined as

\[ BM = \frac{1}{12} \frac{1}{V} \int_{L} B^3(x, T(x)) \, dx, \quad \text{(13)} \]

and \( KG \) is the height of mass center \( G \) above the keel \( K \). In the above two equations, \( V \) and \( L \) denote displacement volume and length of the target ship respectively. The initial GM-variation of a ship in a longitudinal regular or irregular wave then becomes by neglecting the Smith-effect

\[ \partial GM = \frac{1}{V} \left[ \frac{B^3(x, T(x) + r(x))}{12} + M(x, T(x) + r(x)) \right] \frac{4}{12} + \left[ A(x, T(x) + r(x)) KG(x) \right] dx \quad \text{GM}_0 \quad \text{(14)} \]

where the sectional mass center above the keel,

\[ KG(x) = KG + x \cdot (\eta_t - \alpha_{trim}). \]
\[ \eta_5 \text{ in (15) is the wave-induced pitch angle and } \alpha_{\text{trim}} \text{ the trim angle in still water.} \]

First replacing the variable \( z \) in Eqs. (8) to (10) with \( \eta \), the relative motion of the wave surface against ship, and then substituting Eqs. (8) to (10) into Eq. (14), the following expression can be yielded

\[ \partial GM = \sum_i \partial GM_i, \]

where

\[ \partial GM_1 = \frac{1}{V_L} \int L \left[ \frac{B^2(x, T) \kappa_1(x)}{4} + d_j(x) - K Ge(x) \right] \eta(x) dx, \]

\[ \partial GM_2 = \frac{1}{V_L} \int L \left[ \frac{3B^2(x, T) \kappa_2(x)}{12} + 3B(x, T) \kappa_3^2(x) + d_j(x) - K Ge_2(x) \right] \eta^2(x) dx - \frac{1}{V_L} \int L x e_j(x) \eta_i \eta_j(x) dx, \]

and so on. Let

\[ G_1(x) = \frac{1}{V_L} \int L \left[ \frac{B^2(x, T) \kappa_1(x)}{4} + d_j(x) - K Ge_1(x) \right], \]

\[ G_2(x) = \frac{1}{V_L} \int L \left[ \frac{3B^2(x, T) \kappa_2(x)}{12} + 3B(x, T) \kappa_3^2(x) + d_j(x) - K Ge_2(x) \right], \]

\[ R_3(\omega, x) = -\frac{x e_j(x) \eta_5}{\sqrt{V}} = -\frac{x B(x, T) \eta_5}{\sqrt{V}}, \]

and so on, such that

\[ \partial GM_i = \int L G_i(x) \eta_i(x) dx + \int L R_i(x) \eta_i^{-1}(x) dx. \]

It clearly shows that \( G_i(x) \) can be considered as the characteristic function describing the hull geometry near the water line with respect to the GM-variation of a ship in waves. \( R_i(x)(i \geq 2) \) is a function linearly depending upon the pitch response and contributes to the higher order GM-variation.

**AN EXAMPLE OF THE GM-VARIATION OF A SHIP IN REGULAR FOLLOWING WAVES**

The target ship in the example is an ordinary RoRo-ship. Similar ships were constructed mostly un-seveneties. The ship has a aspect ratio of the beam to draught equal to three. Actually, the aspect ratio of modern RoRo-ship or cruise ships are often around four. The main particulars of the ship are shown in Table 1 and the hull form is shown in Fig. 1. In Fig. 2, \( G_1(x) \) is shown along the ship. For comparison, the contributions from the first, second and third term in Eq. (17) are also demonstrated. Apparently, the contribution from the first term has the dominant portion in \( G_1(x) \), particular in the aft body. Because large relative motion usually takes place in the fore and aft bodies, the conclusion can be drawn that the first order GM-variation of the ship in following wave is mainly due to the large flares around the warterline areas in the fore and aft bodies. Fig. 3 shows the \( G_1(x), G_2(x) \) and \( G_3(x) \) along the ship. As seen, \( G_1(x) \) is the greatest, \( G_2(x) \) is in one order less than \( G_1(x) \), and \( G_3(x) \) one order less than

---

**Table 1. The ship’s main particulars**

<table>
<thead>
<tr>
<th>( L_{pp} ) (mm)</th>
<th>B (m)</th>
<th>T (m)</th>
<th>( C_b ) (m)</th>
<th>KG (m)</th>
<th>( GM_0 ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>27.3</td>
<td>9.1</td>
<td>0.64</td>
<td>11.2</td>
<td>0.81</td>
</tr>
</tbody>
</table>

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**Fig. 1. The hull form.**

**Fig. 2.** \( G_1(x) \) along the ship and the contributions from the first, second and third term in Eq. (17) and (18).

\[ + d_j(x) - K Ge_2(x), \]

\[ R_3(\omega, x) = -\frac{x e_j(x) \eta_5}{\sqrt{V}}, \]

and so on, such that

\[ \partial GM_i = \int L G_i(x) \eta_i(x) dx + \int L R_i(x) \eta_i^{-1}(x) dx. \]
The first order GM-variation of the ship in following regular waves calculated using the actual method is compared with the one using a simulation method similar to the one used by Palmquist [4] in 1994. The result without consideration to the Smith-effect shows good agreements as shown in Fig. 4. While in Fig. 5, the comparison is shown with consideration to the Smith-effect. It should be noted that in the actual method, the Smith-effect is taken into account only in the calculation of heave and pitch motion in regular waves.

**VOLterra SYSTEM REPRESENTATION OF GM-VariATION IN IRREGULAR WAVES**

The elevation equation for a regular wave is

\[
\eta(x, t) = \frac{1}{2} a \cdot \left[ e^{i(kx - \omega t + \beta)} + e^{-i(kx - \omega t + \beta)} \right].
\]  

(23)

An irregular wave is a wave system consisting of a series harmonic wave components and its elevation is written as followed:

\[
\eta(x, t) = \frac{1}{2} \sum_{n=1}^{N} a_n \cdot \left[ e^{i(k_n x - \omega_n t + \beta_n)} + e^{-i(k_n x - \omega_n t + \beta_n)} \right],
\]  

(24)

\(\omega\) and \(\omega_n\) in the above denote wave frequency, \(k = \frac{\omega^2}{g}\), and \(k_n = \frac{\omega_n^2}{g}\) both denote the wave number. Random wave phase \(\beta\) and \(\beta_n\) follow the uniform distribution between \(\pi - \pi\).

Assuming that a ship’s heave and pitch motion in regular waves follow the linear relation to the wave amplitude, the relative motion, as shown in Fig. 6, of the ship against the irregular wave can then be written as followed,

\[
r(x, t) = \frac{1}{2} \sum_{n=1}^{N} a_n \cdot \left[ e^{i(k_n x - \omega_n t + \beta_n)} + e^{-i(k_n x - \omega_n t + \beta_n)} \right].
\]
\[ f_1(\omega) = \int_{L} G_1(x) \cdot \tilde{u}(\omega, x) \cdot dx \] (32)

such that

\[ \partial GM_1(t) = \sum_{n=1}^{N} a_n \cdot [f_1(\omega_n) \cdot e^{-i(\omega_n t + \beta_n)} + \tilde{f}_1(\omega_n) \cdot e^{-i(\omega_n t + \beta_n)}]. \] (33)

The transfer functions of the second order GM-variation become

\[ u_2(\omega_m, \omega_n) = \int_{L} [G_2(x) \cdot \tilde{u}(\omega_m, x) \cdot \tilde{u}(\omega_n, x) + R_2(\omega_m, x) \cdot \tilde{u}(\omega_n, x)] dx \]
\[ \tilde{u}_2(\omega_m, \omega_n) = \int_{L} [G_2(x) \cdot \tilde{u}(\omega_m, x) \cdot \tilde{u}(\omega_n, x) + R_2(\omega_m, x) \cdot \tilde{u}(\omega_n, x)] dx \]
\[ \nu_2(\omega_m, \omega_n) = \int_{L} [G_2(x) \cdot \nu(\omega_m, x) \cdot \nu(\omega_n, x) + R_2(\omega_m, x) \cdot \nu(\omega_n, x)] dx \]
\[ \tilde{\nu}_2(\omega_m, \omega_n) = \int_{L} [G_2(x) \cdot \tilde{\nu}(\omega_m, x) \cdot \tilde{\nu}(\omega_n, x) + R_2(\omega_m, x) \cdot \tilde{\nu}(\omega_n, x)] dx \] (34)

thereby,

\[ \partial GM_2(t) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_m a_n \cdot \{ u_2(\omega_m, \omega_n) \cdot e^{-i[(\omega_m + \omega_n) t + \beta_m + \beta_n]} \]
\[ + \tilde{u}_2(\omega_m, \omega_n) \cdot e^{-i[(\omega_m + \omega_n) t + \beta_m + \beta_n]} \]
\[ + \nu_2(\omega_m, \omega_n) \cdot e^{-i[(\omega_m - \omega_n) t + \beta_m - \beta_n]} \]
\[ + \tilde{\nu}_2(\omega_m, \omega_n) \cdot e^{-i[(\omega_m - \omega_n) t + \beta_m - \beta_n]} \] (35)

and so on for the higher order GM-variation.

According to the definition in (34), \( \phi(\omega_m, \omega_n) \) is conjugate to \( u_2(\omega_m, \omega_n) \) and \( \nu_2(\omega_m, \omega_n) \) to \( \tilde{u}_2(\omega_m, \omega_n) \). Eq. (35) shows that the second order GM-variation consists of two parts, i.e. high frequency variation and \( \tilde{u}_2(\omega_m, \omega_n) \) slow varying part. \( u_2(\omega_m, \omega_n) \) represents the transfer functions for high frequency variation and \( \tilde{u}_2(\omega_m, \omega_n) \) slow varying. Fig. 7 and Fig. 8 show the real and imaginary part of \( u_2(\omega_m, \omega_n) \) and \( \tilde{u}_2(\omega_m, \omega_n) \) respectively. Fig. 9 and Fig. 10 show the real and imaginary part of \( \tilde{u}_2(\omega_m, \omega_n) \) respectively.

Fig. 11 shows an example of the first and second order GM-variation in an irregular wave. As seen in the figure, the second order GM-variation is positive all the time and gives in increase in the GM-variation. The
second order GM-variation is strongly related to the first order one, i.e. high second order GM-variation is associated with the high first order one.

CONCLUSION

The most significance of the derived Volterra system representation is that the effect of hull geometry on the GM-variation is explicitly expressed in a function series $G_i(x)$ which are characteristic functions for the different order GM-variations. Eq. (17) and Fig. 2 show clearly that the first order GM-variation is governed by the product of the hull side slope times the water line beam in quadratic and the relative motion along the ship, and that the second order one by the product of the hull side curvature times the water line beam in quadratic and the relative motion in quadratic along the ship. Hence, it becomes apparently why RoRo-ship have considerably GM-variation in waves due to its large breadth to draft ratio and large flares at fore and aft bodies.

The Volterra system approach is a mathematical...
theory for treatment of nonlinear electronic and mechanical problems. By the Volterra system representation, one has the possibility to analyze a problem in a mathematical sense, such as to determine the characteristics of the problem, to derive the correlation between the input parameters and output results, and the probabilistic nature of the results due to a stochastic input. For an example, by applying the second order probability theory formulated by Naess [6] it becomes possible to calculate the peak value distribution of the GM-variation of a ship in irregular waves. Monte-Carlo simulation technique is a practical approach for similar problems.

As shown by the numerical results, the Taylor’s series has been successfully used for the representation of the hull form in the vicinity of the draft line, and thereby provides the basis for the Volterra system representation of the GM-variation of a ship in following or heading waves. This technique should be extended for the description of the coupled sway, heave, pitch and yaw motion of a ship in following and quartering waves. It is of fundamental interest for the study of dynamic instability problems such as broaching-off, course instability coupled with the stability loss in quartering wave, etc. These problems can then be analyzed taking the higher order effect into account by means of the Volterra system theory. Further extension work shall be done in another paper.

REFERENCES