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A UNIFIED APPROACH FOR THE ROBUST SPR PROPERTY OF A PARAMETRIC UNCERTAINTY SYSTEM

Chih-Yung Cheng* and Chao-Fong Chang**

Keywords: strict positive real, parametric uncertainty, value set.

ABSTRACT

In this paper, we adopt a unified approach based on the concept of value set to analyze the robust strict positive real (SPR) property of uncertain systems with interval, affine linear or multilinear uncertainty structures.

INTRODUCTION

Since the beginning of the 1980s, it was gradually recognized that the real issue of control engineering we were faced with was the difficulty of modelling accurately the plant to be controlled. The idea of taking model uncertainty into design consideration soon developed into the so-called robust control theory.

The issue of robustness in control system designs has been one of the main research interests over the past fifteen years. In the early stage, the small gain theorem is used as the principal tools for modelling plant uncertainty. H_∞ control timely supplied a powerful tool for robust control system designs. This approach is based on shaping of singular values, which is essentially a gain concept.

Gradually, the attention extends from the concept of "gain" to the concept of "phase", which is closely related to the concept of positive realness. A real rational function $G(s) = \frac{N(s)}{D(s)}$ is said to be *strict positive real* (SPR) if:

- $G(s)$ is analytic in the closed right half complex plane;
- $Re[G(s)] > 0$, for $Re[s] > 0$.

Therefore, for an SISO system, we have the very important *phase condition* that the polar plot of a SPR transfer function must lie in the right half of the complex plane. That is,

$$-\frac{\pi}{2} < arg[N(j\omega)] - arg[D(j\omega)] < \frac{\pi}{2},\ \forall \omega \in I\!R.$$

The strict positive realness can be thought of as a counterpart of small gain concept. Small gain theorem deals with the robust stability of a feedback loop consisting of a nominal system and a norm-bounded uncertainty block and it implies that the tolerance of uncertainty will increase if we can reduce the norm of the nominal system. Similar to the small gain theorem, the robust stability of a feedback loop consisting of a nominal system and a positive real uncertainty block will be retained if the nominal system is strictly positive real. The dual concept of small gain and positive real is shown in Figure 1. In the small gain case, system gain is restricted inside a finite number, say 1, while in the strict positive real case, system phase is restricted within ±90°.

In this paper, we focus on the system with parametric uncertainties and investigate the problem of *robust* strict positive realness. In the following discussions, the uncertain system will be modelled as a real rational function family

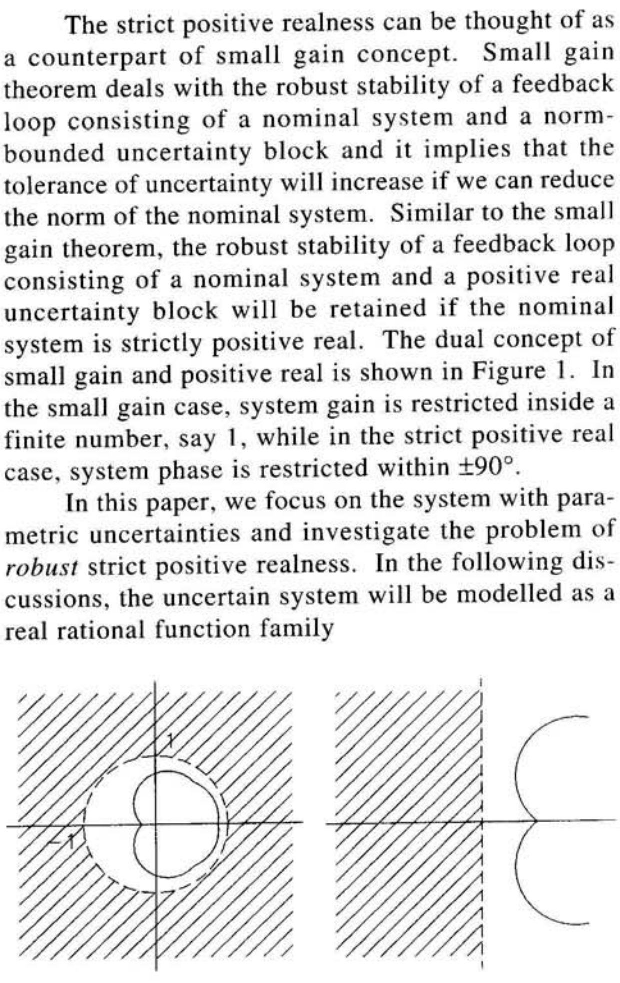

Fig. 1. small gain versus positive real.

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$$G(s) = \frac{N(s, a)}{D(s, b)},$$

where

$$N(s, a) = \sum_{i=0}^{m} a_i s^i,\ D(s, b) = \sum_{i=0}^{n} b_i s_i,$$

a_i, b_i are real uncertain coefficients.

The uncertain parameters could be of independent, affine linear or multilinear structures, which will be defined in later sections. And a given family of transfer functions is said to be *robust SPR* if every member of the family is SPR.

When analyzing a parameter uncertain system with its coefficients varying within a prescribed range, we are indeed dealing with a whole family of systems. Since there are infinitely many members in the family, it is not practical to check one by one whether all the members satisfy a certain property. Therefore, if we can find a finite subset of the family such that the whole family satisfy certain property if *and only if* this finite subset satisfy this property, we can reduce the computation drastically and this finite subset is usually called the *extreme point* set [2,3,6]. In this paper, we are going to provide an extreme point result for the robust SPR property for the parametric uncertainty systems.

Following the same line of idea, we have seen several related results in the literature. Bose et al. [4] and Shi [7] discussed the robust SPR problem for interval plant families. Bhattacharyya et al. [5] considered the robust SPR property in absolute stability of nonlinear system. Vicino et al. [9] provided a framework based on simple frequency domain geometric properties analyzing robust SPR for interval plant-controller families. Recently, Tang et al. [8] generalized the result of robust SPR for interval plant family to affine linear system and provided a method to synthesize the controller $C(s)$ such that the open loop system is SPR.

In the following sections, we will provide a unified geometric approach which includes some of the previous results as special cases and generalizes the robust SPR criterion to multilinear system.

DEFINITIONS AND NOTATIONS

In this section, essential terminologies and notations are introduced.

Definition 1 (SPR) A real rational function $G(s)$ of the complex variable s is said to be *strict positive real* (SPR) if:

- $G(s)$ is analytic in the closed right half plane,

and

- $Re[G(s)] > 0$, for $Re[s] > 0$.

The following property gives necessary and sufficient conditions for a real rational function to be SPR.

Property 1: A real rational $G(s) = \frac{N(s)}{D(s)}$ is SPR if and only if the following conditions hold:

- $D(s)$ is Hurwitz;
- $Re[G(j\omega)] > 0,\ \forall \omega \in \mathbb{R}$.

From the second point of Property 1, we obtain the following property.

Property 2: The polar plot of a SPR transfer function must lie in the right half of the complex plane. That is

$$-\frac{\pi}{2} < arg[N(j\omega)] - arg[D(j\omega)] < \frac{\pi}{2},\ \forall \omega \in \mathbb{R}.$$

In the paper, the robustness problems involving real parametric uncertainty will be considered. The real rational uncertain system is expressed as a ratio of two uncertain polynomials. Denote $p(s, q)$ with $q \in Q$ as an uncertain polynomial where q is a vector of uncertain parameters and Q is a bounding set for q where

$$Q = \{q \in \mathbb{R}^l : q_i^- \le q_i \le q_i^+ \text{ for } i = 1, 2, ..., l\}.$$

Similarly, $p(s, r)$ with $r \in R$ defines another uncertain polynomial with bounding set R.

Definition 2 (robust SPR) An uncertain system $\mathcal{G} = \frac{N(s, q)}{D(s, r)}$ with $q \in Q$, $r \in R$ is said to be robust SPR if $\frac{N(s, q)}{D(s, r)}$ is SPR for all $q \in Q$ and $r \in R$.

Whenever there is no danger of notation confusion, we will drop the uncertain parameter of $N(s, q)$ to $N(s)$ and $D(s, r)$ to $D(s)$.

The approach in this paper is mainly based on the geometry of value sets of uncertain polynomials, which is defined as follows.

Definition 3 (value set) Given an uncertain polynomial $p(s, q)$ with $q \in Q$. The value set at frequency $\omega \in \mathbb{R}$ is given by

$$p(j\omega, Q) = \{p(j\omega, q) : q \in Q\}.$$

That is, $p(j\omega, Q)$ is the image of Q under $p(j\omega, \bullet)$.

Since the value sets of uncertain polynomials associated with different parameter structures have certain special properties, we investigate the numerator and denominator uncertain polynomials separately and then combine them to get the phase properties of the uncertain systems.

ROBUST STRICT POSITIVE REALNESS OF INTERVAL SYSTEMS

Let $\mathcal{G}_I$ be a family of proper interval plants, i.e.,

$$\mathcal{G}_I = \{G(s): G(s) = \frac{N(s)}{D(s)},\ N(s) \in \mathcal{N}_I,\ D(s) \in \mathcal{D}_I\}, \quad (1)$$

where $\mathcal{N}_I$ and $\mathcal{D}_I$ are families of interval polynomials, i.e.,

$$\mathcal{N}_I = \{N(s): N(s) = \sum_{i=0}^{m} q_i s^i,\ q_i \in [q_i^-, q_i^+],\ i = 0, \ldots, m\}$$

$$\mathcal{D}_I = \{D(s): D(s) = \sum_{i=0}^{n} r_i s^i,\ r_i \in [r_i^-, r_i^+],\ i = 0, \ldots, n\}$$

and $n \geq m$.

For a given interval polynomial, we define the following subset and make specific reference to the numerator family, but similar definitions hold for the denominator family. The set $\mathcal{N}_K$ includes the four Kharitonov polynomials [2] representing four specific corners of the interval family in the coefficient space

$$\mathcal{N}_K = \{N_1(s), N_2(s), N_3(s), N_4(s)\},$$

where

$$N_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + \ldots,$$

$$N_2(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + \ldots,$$

$$N_3(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + \ldots,$$

$$N_4(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + \ldots.$$

Lemma 1: (Value set for interval polynomials) [2] For each frequency ω, the value sets in the complex plane of the numerator and denominator families of $\mathcal{G}_I$ are rectangles, denoted by $V_n(\omega)$ and $V_d(\omega)$, respectively. Furthermore, the previously defined Kharitonov polynomials correspond to the four vertices.

It will be shown that the SPR condition of an interval family of transfer functions can be simplified by checking "eight" extremal transfer functions of the family. The original form of this result has been given in [5]. In the following, a slightly generalized version will be given along with a straightforward proof.

Theorem 1: An interval family $\mathcal{G}_I$ of proper transfer functions defined in (1) is robust SPR if and only if the following *eight* transfer functions are SPR.

$$G_1 := \frac{N_4}{D_1},\quad G_2 := \frac{N_3}{D_1},\quad G_3 := \frac{N_1}{D_4},\quad G_4 := \frac{N_2}{D_4},$$

$$G_5 := \frac{N_1}{D_3},\quad G_6 := \frac{N_2}{D_3},\quad G_7 := \frac{N_4}{D_2},\quad G_8 := \frac{N_3}{D_2},$$

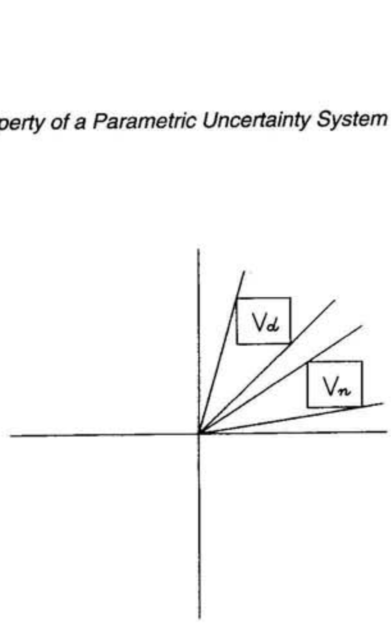

Fig. 2. Value sets for case 1.

Proof:

- Necessity: If the system is robust SPR, it means every member in the family is SPR. Therefore, the necessity holds obviously.
- Sufficiency: For an arbitrary $\omega \geq 0$, there are three possible alternative configurations.

– Case 1: (The value sets $V_n(\omega)$ and $V_d(\omega)$ lie entirely in the same quadrant but exclude the sides belonging to the coordinate axes). For example, if $V_n(\omega)$ and $V_d(\omega)$ both lie in the first quadrant. From Figure 2, the phase difference $\Delta\phi(\omega) = arg[N(j\omega)] - arg[D(j\omega)]$ satisfies the phase condition, i.e.,

$$-\frac{\pi}{2} < \Delta\phi(\omega) < \frac{\pi}{2}.$$

– Case 2: (The value sets $V_n(\omega)$ and $V_d(\omega)$ lie entirely in two adjacent quadrants of the complex plane including the coordinate axes). From simple geometric graph, the phase difference is achieved in correspondence to one out of the eight pairs of Kharitonov numerator and denominator polynomials. For example, if $V_d(\omega)$ lie in second or fourth quadrant and $V_n(\omega)$ lie in the first quadrant (see Figure 3), then $\Delta\phi(\omega)$ is achieved by the following configurations:

$$N(j\omega) = N_3;\ D(j\omega) = D_2 \rightarrow G_8,$$
$$N(j\omega) = N_4;\ D(j\omega) = D_1 \rightarrow G_1,$$
$$N(j\omega) = N_3;\ D(j\omega) = D_1 \rightarrow G_2,$$
$$N(j\omega) = N_4;\ D(j\omega) = D_2 \rightarrow G_7.$$

If $V_d(\omega)$ lie in the first or third quadrant and

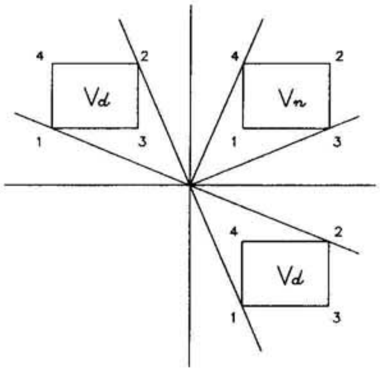

Fig. 3. Value sets for case 2.

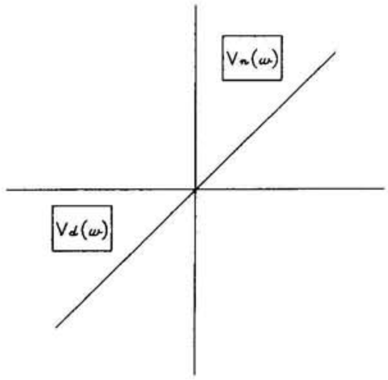

Fig. 5. Value sets for case 3.

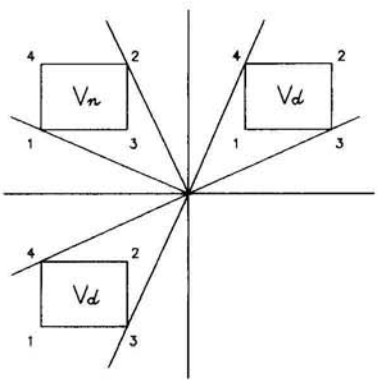

Fig. 4. Value sets for case.

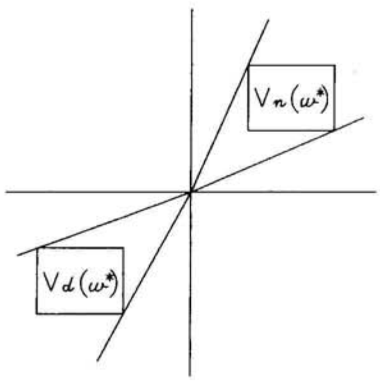

Fig. 6. Value sets for case 3.

$V_n(\omega)$ lie in the second quadrant (see Figure 3), then it is easy to check $\Delta\phi(\omega)$ is achieved by the following configurations:

$N(j\omega) = N_1;\ D(j\omega) = D_3 \rightarrow G_5,$
$N(j\omega) = N_2;\ D(j\omega) = D_4 \rightarrow G_4,$
$N(j\omega) = N_1;\ D(j\omega) = D_4 \rightarrow G_3,$
$N(j\omega) = N_2;\ D(j\omega) = D_3 \rightarrow G_6.$

The same method applies when considering other possible adjacent pairs of quadrants.

- Case 3: (The value sets $V_n(\omega)$ and $V_d(\omega)$ lie in three quadrants of the complex plane). There are two situations in this case: The first situation is the valuse sets $V_n(\omega)$ and $V_d(\omega)$ lie in the same half plane. For example, see Figure 5, it satisfies Property 2. So, this situation is similar to Case 2. The second situation, for example, if at some frequency ω^*, the value sets $V_n(\omega^*)$ and $V_d(\omega^*)$ lie in three quadrants is depicted in Figure 6, such that $|\Delta\phi(\omega^*)| \geq \frac{\pi}{2}$. Then it implies that there exist another $\omega < \omega^*$ such that $V_n(\omega)$ and $V_d(\omega)$ are in the situation described in Case 2, with $|\Delta\phi(\omega^*)| \geq \frac{\pi}{2}$. But, this situation violates the Property 2. Hence, if the eight special transfer functions satisfy the phase condition, this situation can not happen.

Note that the proof excludes the possibility of degree dropping more than one since it obviously violates Property 2. By the analysis above, we can conclude that if the eight transfer functions are SPR, then it implies the interval family of G_I is robust SPR. The proof is completed.

Remark: Several simplified conditions for the robust SPR property of families of low order transfer functions can be given as follows [9], which is similar to the case of the Kharitonov's theorem for polynomials of order $n \leq 4$ given in [1]. The following results can be easily obtained via the previous graphical analysis.

1. If $n = 2$, SPR of the two transfer functions G_2, G_5 implies robust SPR of the entire family.
2. If $n = 3$, SPR of the four transfer functions G_2, G_5, G_6, G_8 implies robust SPR of the entire family.
3. If $n = 4$, SPR of the six transfer functions G_2, G_5, G_6, G_8, G_4, G_7 implies robust SPR of the entire family.

Example 1: Consider the following stable family $G(s)$ of interval system whose generic element is given by

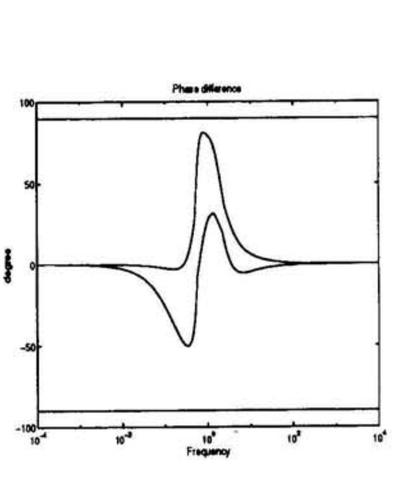

Fig. 7. Phase difference for example 1.

$$G(s) = \frac{s^3 + \beta s^2 + \alpha s + 1}{s^3 + \epsilon s^2 + \delta s + \gamma}.$$

Solution: We first check eight extremal plants as follows:

$$G_1(s) = \frac{s^3 + 4s^2 + 2s + 1}{s^3 + 4s^2 + 5s + 1}, \quad G_2(s) = \frac{s^3 + 3s^2 + s + 1}{s^3 + 4s^2 + 5s + 1},$$

$$G_3(s) = \frac{s^3 + 4s^2 + s + 1}{s^3 + 4s^2 + 6s + 1}, \quad G_4(s) = \frac{s^3 + 3s^2 + 2s + 1}{s^3 + 4s^2 + 6s + 1},$$

$$G_5(s) = \frac{s^3 + 4s^2 + s + 1}{s^3 + 3s^2 + 5s + 2}, \quad G_6(s) = \frac{s^3 + 3s^2 + 2s + 1}{s^3 + 3s^2 + 5s + 1},$$

$$G_7(s) = \frac{s^3 + 4s^2 + 2s + 1}{s^3 + 3s^2 + 6s + 2}, \quad G_8(s) = \frac{s^3 + 3s^2 + 2s + 1}{s^3 + 3s^2 + 6s + 2}.$$

Since these eight plants are SPR and by Theorem 1, $G(s)$ is robust SPR. In order to verify the result, a computer simulation result plotting the outer envolope of the phase angle is shown in Figure 7. For $\omega \in [10^{-4}, 10^4]$, its phase difference satisfies

$$|arg[N(j\omega)] - arg[D(j\omega)]| < \frac{\pi}{2};$$

therefore, $G(s)$ is robust SPR.

ROBUST STRICT POSITIVE REALNESS OF AFFINE LINEAR SYSTEMS

In this section, let us start with the definition of an affine linear uncertain polynomial.

Definition 4 (affine linear uncertain polynomial) Assume q is a vector composed of uncertain parameters. An uncertain polynomial $p(s, q) = \sum_{i=0}^{n} a_i(q)s^i$ is said to have an *affine linear uncertainty structure* if each of the coefficient function $a_i(q)$ is an affine function of q. That is, for each i, there exists a vector α_i and a scalar β_i such that

$$a_i(q) = \alpha_i^T q + \beta_i.$$

Let $\mathcal{G}_{aff}$ be a family of transfer functions with affine linear uncertainty structures, i.e.,

$$\mathcal{G}_{aff} = \{G(s) : G(s) = \frac{N(s)}{D(s)}, N(s) \in \mathcal{N}_{aff}, D(s) \in \mathcal{D}_{aff}\} \quad (2)$$

where both $\mathcal{N}_{aff}$ and $\mathcal{D}_{aff}$ are affine linear uncertain polynomials.

Affine linear cases can arise frequently in uncertain feedback system descriptions. For example, for an interval uncertain plant

$$P(s, q, r) = \frac{N(s, q)}{D(s, r)},$$

connected to a controller

$$C(s) = \frac{N_C(s)}{D_C(s)},$$

forming a unity-feeback system, the closed-loop transfer function

$$P_{CL} = \frac{N_C(s)N(s, q)}{N_C(s)N(s, q) + D_C(s)D(s, r)},$$

comes out naturally as an affine linear system.

In order to represent the value sets of affine linear polynomials, we first review some elementary materials from the theory of convex analysis. A set $C \subseteq I\!R^k$ is said to be *convex* if the line joining any two points c^1 and c^2 in C remains entirely within C, and its *convex hull*, denoted by *conv C*, is the smallest convex set which contains C. A polytope P in $I\!R^k$ is the convex hull of a finite set of points $\{p^1, p^2, ..., p^m\}$, i.e., $P = conv\ \{p^i\}$ and call $\{p^1, p^2, ..., p^m\}$ the set of generators. A point $p \in P$ is said to be an *extreme point* of P if it can not be expressed as a convex combination of two distinct points in P.

Lemma 2: (Value set for affine linear polynomials) [2] Let $\mathcal{P} = \{p(\bullet, q): q \in Q\}$ be an affine linear polynomial with uncertainty bounding set $Q = conv\{q^i\}$. Then, for a fixed $\omega > 0$, the value set $p(j\omega, Q)$ is a polygon with generating set $\{p(j\omega, q^i)\}$. That is

$$p(j\omega, Q) = conv\{p(j\omega, q^i)\}.$$

Furthermore, we introduce some new notations as follows:

$\mathcal{N}_\upsilon$: the subset of the vertex polynomial in $\mathcal{N}_{af\cdot f}$
$\mathcal{D}_\upsilon$: the subset of the vertex polynomial in $\mathcal{D}_{af\cdot f}$
$V_n(\omega)$: the value set of $\mathcal{N}_{aff}$,
$V_d(\omega)$: the value set of $\mathcal{D}_{aff}$,

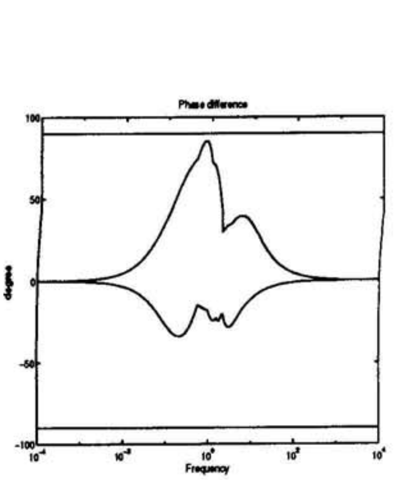

Fig. 8. Phase difference for example 2.

Suppose there are m vertex polynomials in $\mathcal{N}_{aff}$, the phase of the i-th extreme polynomial in $\mathcal{N}_{aff}$ at the $s = j\omega$ is denoted by $\theta_{ni}(\omega)$, $i = 1, 2, ..., m$, and $0 \leq \theta_{ni}(\omega) \leq 2\pi$. Then define

$$\theta_n^+(\omega) := \max_{i \in [1, m]} \theta_{ni}(\omega),$$

$$\theta_n^-(\omega) := \min_{i \in [1, m]} \theta_{ni}(\omega).$$

Similarly, $\theta_d^+(\omega)$ and $\theta_d^-(\omega)$ for $\mathcal{D}_v$ can be defined.

It will be shown in the following that the robust SPR condition of an affine linear family of transfer functions can be simplified by checking vertex transfer functions of the family. The original form of this result has been given in [8].

Theorem 2: A proper transfer function family $\mathcal{G}_{aff}$ defined in (2) is robust SPR if and only if the following conditions are satisfied:

- $Re[G(0)] > 0$, $\forall G(s) \in \mathcal{G}_{aff}$,
- $\mathcal{N}_{aff}$ and $\mathcal{D}_{aff}$ contain at least one Hurwitz stable polynomial, respectively;
- $\theta_n^+(\omega) - \theta_d^-(\omega) < \frac{\pi}{2}$, $\theta_n^-(\omega) - \theta_d^+(\omega) > -\frac{\pi}{2}$ for all $\omega \in \mathbb{R}$.

Proof: The proof will be included as a special case in multilinear case in the next section.

Example 2: Consider the following family $G(s)$ of an affine linear system whose generic element is given by

$$G(s) = \frac{s^3 + (q_1 + 4)s^2 + (q_1 + q_2 + 6)s + 1}{s^3 + (2q_3 + q_4 + 7)s^2 + (q_4 + 7)s + (2q_3 - q_4 + 5)},$$

and $q_1 \in [-1, 2]$, $q_2 \in [-2, 2]$, $q_3 \in [-1, 1]$, $q_4 \in [-2, 2]$. Check whether this system is robust SPR or not?

Solution:

1. $Re[G(0)] = \frac{1}{2q_3 - q_4 + 5}$ and $1 \leq 2q_3 - q_4 + 5 \leq 9$. So, the condition $Re[G(0)] > 0$ is satisfied.
2. Let q_1, q_2, q_3, q_4 be all zeros. Then the roots of numerator and denominator polynomials are $\{-1.9053 \pm j1.2837, -0.1895\}$ and $\{-0.5163 \pm j0.7558, -5.9674\}$, respectively. So, the numerator and denominator polynomials are both Hurwitz stable.
3. A computer simulation result plotting the phase difference is shown in Figure 8. The phase difference satisfies the third point of Theorem 2; therefore, the system is robust SPR.

ROBUST STRICT POSITIVE REALNESS OF MULTILINEAR SYSTEMS

In this section, let us start with the definition of a multilinear uncertain polynomial.

Definition 5 (multilinear uncertain polynomial) Assume q is a vector composed of uncertain parameters. An uncertain polynomial $p(s, q) = \sum_{i=0}^{n} a_i(q)s^i$ is said to have a *multilinear uncertainty structure* if each of the coefficient function $a_i(q)$ is multilinear. That is, if all but one component of the vector q is fixed, then $a_i(q)$ is affine linear in the remaining components of q.

Let $\mathcal{G}_{multi}$ be a family of transfer functions with multilinear uncertainty structures, i.e.,

$$\mathcal{G}_{multi} = \{G(s): G(s) = \frac{N(s)}{D(s)}, N(s) \in \mathcal{N}_{multi}, D(s) \in \mathcal{D}_{multi}\} \quad (3)$$

where both $\mathcal{N}_{multi}$ and $\mathcal{D}_{multi}$ are uncertain polynomials with multilinear structures.

Multilinear cases can arise frequently in uncertain system descriptions. The following are some examples.

1. In frequency domain description, for example, we only roughly know the two dominant poles location of a transfer function

$$G(s) = \frac{1}{(s + q_1)(s + q_2)},$$

and we can describe q_1 and q_2 to be in an interval to account for the uncertainty. Therefore we can rewrite it as

$$G(s) = \frac{1}{s^2 + (q_1 + q_2)s + q_1 q_2},$$

where it becomes a multilinear case.

2. In time domain description, for example, we represent a two-state system as

$$\dot{x} = Ax + bu,$$

$$= \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} x + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} u.$$

To accout for the uncertainty, which could be resulted from modelling inaccuracies, we allow q_1, q_2, q_3 and q_4 to lie in an interval. When computing its characteristic polynomial

$$det(s_I - A) = s^2 - (q_1 + q_4)s + (q_1q_4 - q_2q_3),$$

the multilinear structure came out naturally due to the operation of determinant.

As far as the value set is concerned, the multilinear case is much more complicated than the affine linear case. The following theorem provides a tight approximation of the value set of multilinear uncertain polynomial, which will be used in the later proof.

Lemma 3: (Mapping Theorem) [2] Suppose $Q \subset R^l$ is a box with extreme points $\{q^i\}$ and f: Q $\rightarrow R^k$ is multilinear. Let

$$f(Q) = \{f(q): q \in Q\}.$$

Then it follows that

$$conv\, f(Q) = conv\{f(q^i)\}.$$

And some new notations will be introduced as follows:

$\mathcal{N}_\upsilon$: the subset of the vertex polynomial in $\mathcal{N}_{multi}$,

$\mathcal{D}_\upsilon$: the subset of the vertex polynomial in $\mathcal{D}_{multi}$,

$V_n(\omega)$: the value set of $\mathcal{N}_{multi}$,

$V_d(\omega)$: the value set of $\mathcal{D}_{multi}$.

Suppose there are m vertex polynomials in $\mathcal{N}_{multi}$, the phase of the i-th extreme polynomial in $\mathcal{N}_{multi}$ at $s = j\omega$ is denoted by $\theta_{ni}(\omega)$, $i = 1, 2, ..., m$, and $0 \le \theta_{ni}(\omega) \le 2\pi$. Then define

$$\theta_n^+(\omega) := \max_{i \in [1, m]} \theta_{ni}(\omega),$$

$$\theta_n^-(\omega) := \min_{i \in [1, m]} \theta_{ni}(\omega).$$

Similarly, $\theta_d^+(\omega)$ and $\theta_d^-(\omega)$ for $\mathcal{D}_\upsilon$ can be defined.

Theorem 3: A proper transfer function family $\mathcal{G}_{multi}$ defined in (3) is robust SPR if and only if the following conditions are satisfied:

- $Re[G(0)] > 0$, $\forall G(s) \in \mathcal{G}_{multi}$,
- $\mathcal{N}_{multi}$ and $\mathcal{D}_{multi}$ contain at least one Hurwitz stable polynomial, respectively;
- $\theta_n^+(\omega) - \theta_d^-(\omega) < \frac{\pi}{2}$, $\theta_n^-(\omega) - \theta_d^+(\omega) > -\frac{\pi}{2}$ for all $\omega \in I\!R$.

Proof:

A basic difference between the analysis of affine linear and multilinear cases is the shapes of the corresponding value sets. The affine linear map has the property that transformation of straight lines remains straight lines. Therefore, in the affine linear case, the value set mapped from a polytope will be a polygon. However, in the multilinear case, due to the multilinear structure of uncertain parameters, the boundary of its value set is not necessarily composed of segments. Furthermore, it could even be a nonconvex set. This phenomenon highly complicates the problem of the multilinear systems. Typical value sets of affine linear and multilinear systems are shown in Figure 9. Fortunately, by Lemma 3, the value sets of multilinear system can be contained in the convex hull generated by its vertices and when calculating the phase angle no conservatism will be introduced by using the covering.

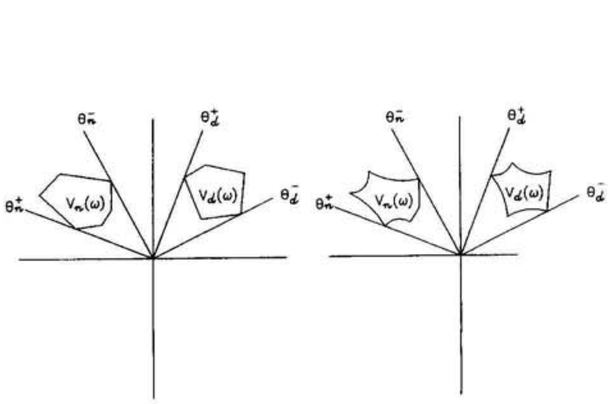

Fig. 9. Typical value sets for affine linear and multilinear systems.

• Necessity: If $\mathcal{G}_{multi}$ is robust SPR, it obviously implies that $Re[G(0)] > 0$, $\forall G(s) \in \mathcal{G}_{multi}$ by Property 1. From Definition 1 we can deduce following fact: a transfer function $G(s)$ is SPR if and only if $G^{-1}(s)$ is SPR. By above fact and Property 1, we know that $\mathcal{N}_{multi}$ and $\mathcal{D}_{multi}$ contain at least one Hurwitz stable polynomial, respectively. Furthermore, it naturally satisfies phase condition and implies that

$$\theta_n^+(\omega) - \theta_d^-(\omega) < \frac{\pi}{2},\ \theta_n^-(\omega) - \theta_d^+(\omega) > -\frac{\pi}{2},$$

for all $\omega \in I\!R$.

• Sufficiency: This part of proof is proceeded as follows. To have the robust SPR property, the frequency response of the whole family must stay inside the open right half complex plane. The first and second point of Theorem 3 can show that we have at least one such member. And we need to further show that the frequency response the whole family does not cross the imaginary axis. That is, $\forall r \in I\!R$, $\omega \in I\!R$, $N(s) \in \mathcal{N}_{multi}$, $D(s) \in \mathcal{D}_{multi}$

$$\frac{N(j\omega)}{D(j\omega)} \ne jr,$$

or

$D(j\omega) \neq -jrN(j\omega)$.

The left hand side is the value set of denominator uncertain polynomial and the right hand side is the $\pm\frac{\pi}{2}$ (depending on the sign of r) revolving value set of the nominator uncertain polynomial. In previous section, we have known that the value set for an affine linear uncertain polynomial is a polygon and all edges of the polygon are obtained from the edges of uncertainty box. For multilinear uncertain polynomial, its value set has more complex structure. The outer boundary of the value set is not only obtained from the edges of uncertainty box but also from the internal points of uncertainty box. For the sake of simplicity, the Mapping Theorem provides a conservative result. From Figure 9, the angles of $\theta_n^+(\theta_n^-)$ and $\theta_d^+(\theta_d^-)$ are determined by extreme polynomials in $\mathcal{N}_{multi}$ and $\mathcal{D}_{multi}$, respectively. It is easily shown the value set inequality $D(j\omega) \neq -jrN(j\omega)$ is equivalently to the phase condition

$$\theta_n^+(\omega) - \theta_d^-(\omega) < \frac{\pi}{2},\ \theta_n^-(\omega) - \theta_d^+(\omega) > -\frac{\pi}{2}\ \forall\omega \in I\!R.$$

Then it implies that the family $\mathcal{G}_{multi}$ is robust SPR. The proof is completed.

Remark: Note that the case of interval systems could also be treated similarly, but with the special property that its value set rectangle parallel to real and imaginary axis, more specific result can be obtained. This is exactly what Theorem 1 investigated.

Remark: The same framework and reasoning can be extended to include nonlinear parameter systems. However, in the nonlinear case, the extreme points needed to determine the phase angle may not happen on the vertices or edges of uncertainty space, which makes the nonlinear case much more difficult to compute.

Example 3: Consider the following family $G(s)$ of a multilinear system whose generic element is given by

$$G(s)=\frac{s_2+(q_1+q_2)s+q_1q_2}{q_4s^3+(q_3+q_4+q_4q_5)s^2+(q_3q_5+q_4q_5+q_3)s+q_3q_5}.$$

Solution:

1. $Re[G(0)] = \frac{q_1q_2}{q_3q_5}$. Due to q_1, q_2, q_3, q_4 are all positive numbers, $Re[G(0)] > 0$.
2. Let $q_1 = 0.4$, $q_2 = 0.6$, $q_3 = 1.5$, $q_4 = 1.5$, $q_5 = 1.5$. Then the roots of numerator and denominator polynomials are {-0.4, -0.6} and {-1, -1, -1.5}, respectively. So, the numerator and denominator polynomials are both Hurwitz stable.
3. A computer simulation result plotting the phase difference is shown in Figure 10. The phase difference satisfies the third point of Theorem 3; therefore, the system is robust SPR.

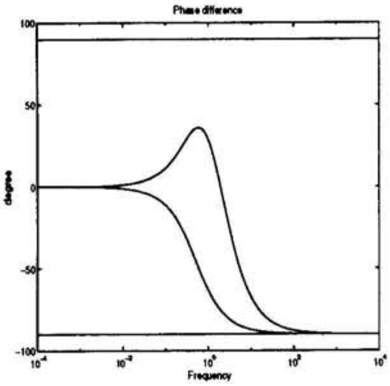

Fig. 10. Phase difference for example 3.

CONCLUSION

Positive realness is an important and fundamental notion in network and system theory. The concept has been widely used in the stability analysis of feedback systems, which includes lots of applications in adaptive control theory. In this paper, the problem of robust strict positive realness for uncertain systems defined by interval, affine linear and multilinear were discussed. Under a unified geometric framework, we can conclude that to guarantee the robust SPR property, we need only check some extreme points of the whole family of systems.

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參數不確定系統強健嚴格正實性質之整合分析

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摘 要

本文主要在研究參數不確定系統強健嚴格正實性質的判定。研究的對象分為區間結構、擬似線性結構以及多重線性結構的不確定系統。利用值集的幾何概念，本文提供一種整合的分析方法，使得強健嚴格正實性質的判斷準則能夠在統一的幾何架構下，適用於多種的不確定系統。