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Wu-Ting Tsai

Professor, Department of Civil Engineering, National Chiao Tung University, Hsin Chu, 300, Taiwan.

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Recommended Citation

Tsai, Wu-Ting (2001) "Faraday Resonance for Stratified Two-Layer Flow," *Journal of Marine Science and Technology*. Vol. 9: Iss. 2, Article 8.

DOI: 10.51400/2709-6998.2443

Available at: <https://jmstt.ntou.edu.tw/journal/vol9/iss2/8>

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FARADAY RESONANCE FOR STRATIFIED TWO-LAYER FLOW

Wu-Ting Tsai

Keywords: faraday resonance, stratified flow, perturbation analysis.

ABSTRACT

The analysis developed in Tsai [1] is extended to consider the parametrically resonant motion of a stratified two-layer fluid in a two-dimensional rectangular basin forced to oscillate vertically. The modulation equation governing the slowly-varying amplitude of the interface is derived using method of multiple scales. For steady harmonic response, the present result compares well with available experimental measurements for the low frequency region. The discrepancy increases, however, for high-frequency response.

INTRODUCTION

The phenomenon of parametric resonance arises in a variety of free-surface problem (see e.g. Miles and Henderson [2]). In contrast to forced resonance, in which the forcing gives rise to an inhomogeneity in the governing equation of motion, parametric excitation appears as coefficient of the governing differential equation.

In this note, we extend the perturbation analysis developed in Tsai [1] for resonant motion of one-layer free-surface flow to the motion of a stratified two-layer fluid in a two-dimensional rectangular basin subject to a vertical oscillation $-a_e \cos \omega_e t$. The densities of the lower and upper fluids are ρ' and ρ'' respectively, with the lower layer heavier than the upper layer ($\rho' > \rho''$). For convenience, in what follows, all physical variables are non-dimensionalized by the half length of the basin L , and the timescale $2\omega_e$. A coordinate system fixed with the basin is chosen so that the origin and x -axis are in the undisturbed interface, z is positive upwards, the side walls of the basin are at $x = \pm 1$, and the lower bottom and the upper lid are at $z = -h'$ and $z = h''$ respectively.

Paper Received May 1, 2001. Author for Correspondence: Wu-Ting Tsai.
*Professor, Department of Civil Engineering, National Chiao Tung University, Hsin Chu, 300, Taiwan.

BOUNDARY VALUE PROBLEM

For ideal, incompressible and irrotational fluids, the velocity potentials of lower and upper flows, $\Phi'(x, z, t)$ and $\Phi''(x, z, t)$, satisfy the Laplace equations with the solid boundary conditions on the side walls, bottom and lid. The kinematic boundary conditions on the interface $z = \zeta(x, t)$ are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} \frac{\partial \Phi}{\partial x} - \frac{\partial \Phi}{\partial z} = 0, \quad (1)$$

where $\Phi = \Phi'$ and Φ'' . The dynamic boundary condition of the interface is

$$\begin{aligned} & \frac{\partial \Phi'}{\partial t} + \frac{1}{2} \nabla \Phi' \cdot \nabla \Phi' + 4(N^2 \mu + \varepsilon \cos 2t) \zeta \\ & = \rho \left[\frac{\partial \Phi''}{\partial t} + \frac{1}{2} \nabla \Phi'' \cdot \nabla \Phi'' + 4(N^2 \mu + \varepsilon \cos 2t) \zeta \right], \end{aligned} \quad (2)$$

where the density ratio $\rho = \rho''/\rho' < 1$, the nondimensional amplitude of excitation $\varepsilon = a_e/L \ll O(1)$, $\mu = (\mu' + \rho\mu'')/(1 - \rho)$, $\mu' = (m\pi \tanh m\pi h')^{-1}$, and $\mu'' = (m\pi \tanh m\pi h'')^{-1}$. We consider here the 1/2-subharmonic resonance of the m th-mode standing wave, so that $N = \Omega/\omega_e = 1/2 + \lambda\varepsilon$, where $\Omega = (g/(L\mu))^{1/2}$ is the dimensional natural frequency of the m th-mode internal standing wave, and λ is the detuning parameter.

MULTIPLE-SCALE ANALYSIS

Multiple-scale analysis of the boundary value problem is processed with the same long timescale $\tau = \varepsilon t$ and the asymptotic expansions of Φ' , Φ'' and ζ as for the free-surface Faraday resonance (e.g. Tsai [1] and Ockendon and Ockendon [3]) or for the cross wave in a rectangular tank (e.g. Tsai and Yue [4]): $\Phi = \sum_{i=1}^{\infty} \varepsilon^{i/2} \Phi_i$ and $\zeta = \sum_{i=1}^{\infty} \varepsilon^{i/2} \zeta_i$. For the Laplace equation and the homogeneous Neumann conditions on the side wall, bottom and lid, the equations are linear and the velocity potentials Φ' and Φ'' satisfy the same form of linear equations for all asymptotic orders. Expanding the kinematic and dynamic interfacial conditions (1) and (2) in Taylor series about the mean interface $z = 0$ and

substituting the asymptotic series for ζ , Φ' and Φ'' , we obtain, at each order i , linearized interfacial condition for ζ_i , Φ'_i and Φ''_i . The boundary value problem at first and second orders can then be solved readily. At the leading order, $O(\epsilon^{1/2})$, the velocity potentials Φ_1 and Φ_1''

$$\Phi_1' = \frac{1}{2}[B(\tau)e^{-it} + \text{c.c.}] \cos kx \frac{\cosh k(z+h')}{k \sinh kh'}, \quad (3a)$$

$$\Phi_1'' = -\frac{1}{2}[B(\tau)e^{-it} + \text{c.c.}] \cos kx \frac{\cosh k(z-h')}{k \sinh mkh''}, \quad (3b)$$

where the complex amplitude $B = C + iD$ and $k = m\pi$. At the third order, combining the kinematic and dynamic interfacial conditions for Φ_3' and Φ_3'' gives rise to the nonhomogeneous equation at $z = 0$:

$$\frac{\partial^2 \Phi_3'}{\partial t^2} + \mu \frac{\partial \Phi_3'}{\partial z} - \rho \left[\frac{\partial^2 \Phi_3''}{\partial t^2} + \mu \frac{\partial \Phi_3''}{\partial z} \right] = F(\Phi_1', \Phi_1''), \quad (4)$$

where $F(\Phi_1', \Phi_1'')$ represents nonhomogeneous forcing function of the third-order interfacial condition. Suppressing the secular terms in (4) to avoid resonance yields the evolution equation for the amplitude $B(\tau)$ as

$$\mu \frac{dB}{d\tau} + i2\mu\lambda B - i\beta B^* - i\Gamma B^2 B^* = 0, \quad (5)$$

where

$$\begin{aligned} \beta &= 1, \quad \Gamma = \frac{1}{1-\rho} (\Gamma' - \rho\Gamma''), \\ \Gamma' &= -\frac{p_2'}{8} (k^2\mu'^2 + k^2\mu\mu' - \frac{\mu'}{\mu} - 1) - \frac{p_2''}{4} (k^2\mu\mu' - 1) \\ &\quad + \frac{q_2'}{4} (k^2\mu\mu' + 1) + \frac{k^2}{64} (\mu' - \mu), \\ \Gamma'' &= -\frac{p_2''}{8} (k^2\mu''^2 - k^2\mu\mu'' + \frac{\mu''}{\mu} - 1) + \frac{p_2'}{4} (k^2\mu\mu'' + 1) \\ &\quad - \frac{q_2''}{4} (k^2\mu\mu'' - 1) - \frac{k^2}{64} (\mu'' + \mu), \\ p_2' &= \frac{3k^4\mu'\mu''(\rho\mu''^2 - \mu'^2) + k^2(\rho - 1)\mu'\mu''}{16(\rho\mu' + \mu'')}, \\ q_2' &= -\frac{k^2(\rho\mu''^2 - \mu'^2) + (\rho - 1)}{8(\rho\mu' + \mu'')}, \\ p_2'' &= \frac{3k^4\mu'\mu''(\rho\mu''^2 - \mu'^2) + k^2\mu'[(3 - \rho)\mu'' + 2\rho\mu']}{8(\mu'' + \rho\mu')}, \\ p_2'' &= \frac{3k^4\mu'\mu''(\rho\mu''^2 - \mu'^2) + k^2\mu''[(1 - 3\rho)\mu' - 2\mu'']}{8(\mu'' + \rho\mu')}. \end{aligned}$$

The limiting case, $\rho = 0$, of (5) corresponds to the evolution equation of free-surface Faraday wave as in Tsai [1], Ockendon and Ockendon [3] and Miles [5].

STATIONARY RESPONSES

The stationary solutions of (5) and the linear stability analyses results are: For $\mu\lambda < -\beta/2$ when $\Gamma > 0$, and $\mu\lambda > \beta/2$ when $\Gamma < 0$, the only critical point is a stable center at $C = 0, D = 0$. For $|\mu\lambda| < \beta/2$, the zero solution $C = 0, D = 0$ becomes an unstable saddle point and the stable centers are at $C = 0, D = \pm[(2\mu\lambda + \text{sgn}(\Gamma)\beta)/\Gamma]^{1/2}$. For $\mu\lambda > \beta/2$ when $\Gamma > 0$, and $\mu\lambda < -\beta/2$ when $\Gamma < 0$, there are three stable centers: $C = 0, D = 0$ and $C = 0, D = \pm[(2\mu\lambda - \text{sgn}(\Gamma)\beta)/\Gamma]^{1/2}$ and two unstable saddle points: $C = \pm[(2\mu\lambda - \text{sgn}(\Gamma)\beta)/\Gamma]^{1/2}, D = 0$. Note that $\mu\lambda = \beta/2$ and $-\beta/2$ correspond to sub- (super-) and supercritical (pitchfork) bifurcation points for $\Gamma > 0$ ($\Gamma < 0$).

In Fig. 1, the *stable* stationary (harmonic) responses of the present analysis, $D = \pm[(2\mu\lambda + \text{sgn}(\Gamma)\beta)/\Gamma]^{1/2}$, are compared with the experimental measurements of Sekerzh-Zen'kovich and Kalinichenko [6]. Note that the experiments were carried out with a free surface on the upper lid, but the free surface was not perturbed within the range of excitation frequency as indicated in Sekerzh-Zen'kovich and Kalinichenko [6]. The rectangular tank, with width $2L = 11.2$ cm, was filled with distilled water ($\rho = 1$ g/cm³) in lower layer and kerosene ($\rho = 0.782$ g/cm³) in upper layer, each with equal thickness (6.8 cm). In Fig. 1, the dimensional amplitudes of harmonic responses (cm) are plotted versus the excitation frequency (Hz) for $m = 1, 3/2$ and 2 internal standing-wave modes. The comparisons are fairly good for the first-mode ($m = 1$) wave, but for higher modes the theoretical results overpredict the

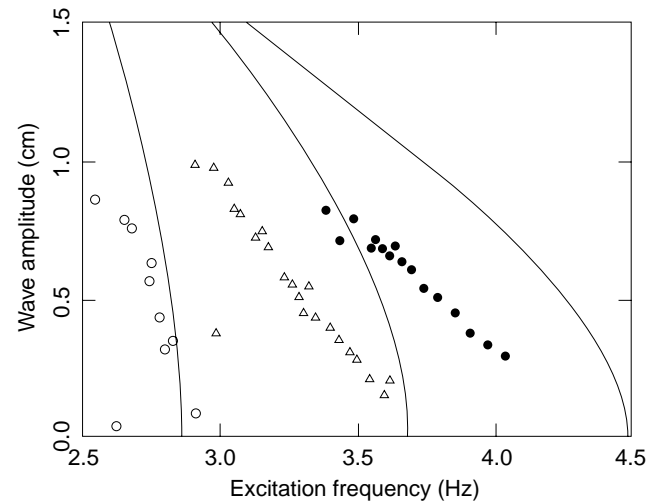


Fig. 1. Comparisons between the present theory (—) and experimental measurements ($m=1$ (o), $3/2$ (Δ) and 2 (\bullet)) of Sekerzh-Zen'kovich and Kalinichenko [6] for the harmonic response of the stratified two-layer Faraday problems as a function of the excitation frequency.

responses. The discrepancy for the higher modes might arise from the relatively higher wave slopes for which the weakly nonlinear analysis above become invalid.

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二層層變流之法拉第共振

蔡 武 廷

國立交通大學土木工程學系

摘 要

本文延伸Tsai [1]之理論分析，探討在二維方形槽之二層層變流體介面受垂直振動所產生的共振波動。利用多尺度之微擾動分析，我們推導了介面波動之振幅演化方程式。由振幅演化方程式之定常解，我們可得介面波動之簡諧運動振幅。在低頻波動範圍此理論結果吻合實驗值，然兩者間之差異在高頻波動時增大。