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ON A COMPUTATIONAL ALGORITHM TO THE HJE IN NONLINEAR H_∞ CONTROL

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ABSTRACT

The Hamilton-Jacobi equation (HJE) plays an essential role in both classical mechanics and nonlinear H_∞ control theory. In this paper, we propose a detailed successive algorithm for finding an approximate solution of the HJE by solving linear equations. A scalar example is given to compare the computational procedures and the accuracy for the proposed approach with some other methods. In addition, the nonlinear H_∞ controller design for the inverted pendulum is also included to show the superiority of the nonlinear H_∞ controller, as compared with the linear H_∞ controller, in the aspect of the robust performance/stability.

INTRODUCTION

As the linear H_∞ control techniques [2, 4] have been developed for several years, recently the much more complicated nonlinear H_∞ control has drawn attention to many investigators [1, 10, 13, 14] and has been solved based on the concept of energy dissipation. Ball, Helton, and Walker (BHW) [1] have successfully derived the nonlinear H_∞ controller formulas involving two Hamilton-Jacobi inequalities (HJIs) or equations (HJEs), in which the HJEs coincidentally have the similar forms as the ones in the classical mechanics due to they both can be derived from the concept of energy. In order to obtain a nonlinear H_∞ controller from BHW's controller formulas, one needs to solve the HJEs. Up to date, there is no computational algorithm for the exact explicit solution of HJE; however, an approximate solution can be obtained by using the successive computational methods [3, 9, 11, 13, 15]. Specifically, Lukes [11] derived a successive algorithm for finding an approximate solution of the HJE by the power series method. Glad [3] and van der Schaft [13] simplified Lukes approach for the nonlinear input-affine system. However, their works were presented only in conceptually, not showing the detailed procedures. Wise and Sedwick (WS) [15] presented a successive approximation approach in which some integral expressions are built and used to find an approximate solution of the HJE successively. For Huang and Lin's approach [9], a computational algorithm is provided, but the approach is too complicated to employ.

In this paper, we present a successive algorithm based on [3, 13] in a thorough manner to show how to find an approximate solution of the HJE easily and efficiently. The proposed successive algorithm assumes that the solution of the HJE and the state equations are in the form of power series. After plugging all the information to the original HJE, one will find the linear part from the algebraic Riccati equation (ARE) is vanished and hence the higher order terms are left to form a new equation. There are two approaches to obtain an approximate solution of the HJE from the equation. One is to compare the coefficients of both sides of the equation, forming a set of linear equations, and solve them to construct an approximate solution of the HJE. The other approach uses integration instead of comparing the said coefficients. Note that the approximate solutions obtained from both approaches are identical. A higher order approximate solution can be found successively by using the above approaches if the higher accuracy is required. A scalar example will be given to demonstrate the procedure of finding an approximate solution of the HJE by the linear equation, the integration approach, and the WS [15] approach. Finally, the nonlinear H_∞ controller design for the inverted pendulum is also included to demonstrate the design procedures and how to solve the HJE by the proposed algorithm. Simulations of the pendulum responses are also enclosed to show the superiority of the nonlinear H_∞ controllers, as compared with its linear counterpart.

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PRELIMINARIES

Nonlinear H_∞ Control Problem

Consider the following nonlinear input-affine generalized plant G:

$$G:\begin{cases}\dot{x}=f(x)+g_1(x)w+g_2(x)u\\ z=h_1(x)+D_{12}(x)u\\ y=h_2(x)+D_{21}(x)w\end{cases} \tag{1}$$

where $x\in\mathbf{R}^n$ is the state of the system, $z\in\mathbf{R}^{p1}$ is the controlled output, $w\in\mathbf{R}^{m1}$ is the exogenous input including all commands and disturbances, $u\in\mathbf{R}^{m2}$ represents the control input, and $y\in\mathbf{R}^{p2}$ is the measured output [1, 8, 10]. The problem is to find a controller

$$K:\begin{cases}\dot{\xi}=A_K(\xi)+B_K(\xi)y\\ u=C_K(\xi)\end{cases} \tag{2}$$

so that the closed-loop system is stable and γ-dissipative [5, 8] or, equivalently, the L_2-gain [13] of the system is less than or equal to γ, which is a prescribed positive number.

Nonlinear H_∞ Controller Formulas

The nonlinear H_∞ controller formulas in [1] are summarized in the following theorem.

Theorem 1 *Consider the nonlinear generalized plant defined in (1). If there exists a controller K of the form (2) such that the closed-loop system is stable and γ-dissipative, then we have the following:*

(i) There exist $X(x)$ and $Y_I(x)$ such that the following Hamilton-Jacobi inequalities:

$$HJX(x):=2X^T(x)H_A(x)+X^T(x)H_R(x)X(x)+H_Q(x)\le 0 \tag{3a}$$

and

$$HJY_I(x):=2Y_I^T(x)J_A(x)+Y_I^T(x)J_R(x)Y_H(x)+J_Q(x)\le 0 \tag{3b}$$

are satisfied for all x in the domain of interest where

$$H_A(x)=f(x)-g_2(x)E_1^{-1}(x)D_{12}^T(x)h_1(x) \tag{4a}$$

$$H_R(x)=\gamma^{-2}g_1(x)g_1^T(x)-g_2(x)E_1^{-1}(x)g_2^T(x) \tag{4b}$$

$$H_Q(x)=h_1^T(x)h_1(x)-h_1^T(x)D_{12}(x)E_1^{-1}(x)D_{12}^T(x)h_1(x) \tag{4c}$$

$$J_A(x)=f(x)-g_1(x)D_{21}^T(x)E_2^{-1}(x)h_2(x) \tag{4d}$$

$$J_R(x)=\gamma^{-2}g_1(x)g_1^T(x)-\gamma^{-2}g_1(x)D_{21}^T(x)E_2^{-1}(x)D_{21}(x)g_1^T(x) \tag{4e}$$

$$J_Q(x)=h_1^T(x)h_1(x)-\gamma^2h_2^T(x)E_2^{-1}(x)h_2(x) \tag{4f}$$

$$E_1(x)=D_{12}^T(x)D_{12}(x) \tag{4g}$$

$$E_2(x)=D_{21}(x)D_{21}^T(x) \tag{4h}$$

(ii) $Y_I(x)$ - $X(x)$ is the gradient of a positive function in the neighborhood of the equilibrium point.

(iii) A nonlinear γ-dissipative H_∞ controller can be constructed as:

$$A_K(\xi)=f(\xi)+\gamma^{-2}[g_1(\xi)-B_K(\xi)D_{21}(\xi)]g_1^T(\xi)X(\xi)+g_2(\xi)C_K(\xi)-B_K(\xi)h_2(\xi) \tag{5a}$$

$$C_K(\xi)=-E_1^{-1}(\xi)[g_2^T(\xi)X(\xi)+D_{12}^Th_1(\xi)] \tag{5b}$$

where $B_K(x)$ satisfies the following equation:

$$[Y_I(\xi)-X(\xi)]^TB_K(\xi)=[\gamma^2h_2^T(\xi)+Y_I^T(\xi)g_1(\xi)D_{21}^T(\xi)]E_2^{-1}(\xi_1) \tag{5c}$$

Linearized Model and ARIs

In order to construct a nonlinear H_∞ controller from (5), one needs to solve the HJI or HJE. There is no exact closed-form solution available for the HJE; however, some successive approximation approaches [3, 9, 11, 13, 15] can be employed to solve the HJE. The solution is in the form of power series in which the first term is constructed based on the corresponding ARE. Assume the equilibrium point is at $x=\mathbf{0}$, the linearized model of the nonlinear generalized plant described in (1) can be obtained as follows:

$$G(s)_{linear}:\begin{cases}\dot{x}=Ax+B_1w+B_2u\\ z=C_1x+D_{12}u\\ y=C_2x+D_{21}w\end{cases} \tag{6}$$

To use a successive approximation algorithm for the solution of the HJIs, the first step is to solve their ARIs, i.e., to find $X>0$ and $Y_I>0$ so that the following three inequalities are satisfied.

$$RicX:=(A-B_2E_1^{-1}D_{12}^TC_1)^TX+X(A-B_2E_1^{-1}D_{12}^TC_1)+X(\gamma^{-2}B_1B_1^T-B_2E_1^{-1}B_2^T)X+C_1^T(I-D_{12}E_1^{-1}D_{12}^T)C_1\le 0 \tag{7a}$$

$$RicY_I := (A - B_1 D_{21}^T E_2^{-1} C_2)^T Y_I + Y_I (A - B_1 D_{21}^T E_2^{-1} C_2) + Y_I \gamma^{-2} B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T Y_I + (C_1^T C_1 - \gamma^2 C_2^T E_2^{-1} C_2) \le 0 \qquad (7b)$$

$$Z := Y_I - X > 0 \qquad (7c)$$

where

$$E_1 = E_1(0) = D_{12}^T(0) D_{12}(0) \qquad (7d)$$

$$E_2 = E_2(0) = D_{21}(0) D_{21}^T(0) \qquad (7e)$$

HJE Approximate Solution by WS Method

The successive approximate solution method proposed by Wise and Sedwick (WS) [15] will be briefly reviewed. Assume the nonlinearity only occurs in the system function $f(x)$ defined in (1). Define

$$A_s = A - B_2 E_1^{-1} D_{12}^T C_1 \qquad (8a)$$

$$R_s = \gamma^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T \qquad (8b)$$

$$Q_s = C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 \qquad (8c)$$

$$F_c = A_s + R_s X \qquad (8d)$$

$$H_A(x) = A_s x + f_h(x) \qquad (8e)$$

where $X \ge 0$ is the solution of the ARE (7a) and $f_h(x) = O(x^2)$. The HJE (3a) can be rewritten as

$$V_x(x) H_A(x) + \frac{1}{4} V_x(x) R_s(x) V_x^T(x) + x Q_s x^T = 0 \qquad (9a)$$

where

$$V_x = \frac{\partial V}{\partial x} = 2X^T(x) \qquad (9b)$$

and the corresponding ARE (7a) becomes

$$A_s^T X + X A_s + X R_s X + Q_s = 0 \qquad (10)$$

The WS successive approach is summarized as follows. Define

$$J(t) = \int_0^t e^{F_c \xi} \left(\frac{1}{2} R_s\right) e^{F_c^T \xi} d\xi \qquad (11a)$$

$$q(z,p) = -2 \frac{\partial^T f_h(z)}{\partial z} X z - \frac{\partial^T f_h(z)}{\partial z} p - 2X f_h(z) \qquad (11b)$$

$$z(t) = e^{F_c t} z(0) + J(t) p(t) + e^{F_c t} \int_0^t e^{-F_c \tau} [f_h(z(\tau)) - J(\tau) q(z(\tau), p(\tau))] d\tau \qquad (11c)$$

$$p(t) = -e^{-F_c^T t} \int_0^t e^{F_c^T \tau} q(z(\tau), p(z)) d\tau \qquad (11d)$$

The successive computing procedure starting by setting the zero-th approximation as

$$z_{(0)}(t) = e^{F_c t} z \text{ and } p_{(0)}(t) = 0 \qquad (12)$$

then plugging (12) into (11) to obtain the first approximation. Repeat to compute (11) by using the latest $z(t)$ and $p(t)$ until generate the desired degree of approximation. The approximate solution of $V_x(x)$ can be computed by

$$V_x(x) = 2x^T X + p(t)|_{t=0} \qquad (13)$$

A DETAILED COMPUTATIONAL ALGORITHM FOR SOLVING THE HJE

In order to obtain a nonlinear H_∞ controller from Theorem 1, one needs to solve the HJEs. In [3, 13], the method of solving the HJE was only presented conceptually without showing detailed procedures. In this section, we will present a modified successive algorithm in a thorough manner to construct an approximate solution for the HJE which is the main contribution of the paper. The HJE (3a) can be rewritten as:

$$V_x(x) H_A(x) + \frac{1}{4} V_x(x) H_R(x) V_x^T(x) + H_Q(x) = 0 \qquad (14)$$

where $V_x(x)$ is defined in (9b). Define A_s, R_s, Q_s, F_c, and $H_A(x)$ as the same in (8). Let , $R_h(x) = O(x)$ and $Q_h(x) = O(x^3)$ be the high-order terms that satisfy the following,

$$\frac{1}{4} H_R(x) = \frac{1}{4} R_s + R_h(x) \qquad (15a)$$

$$H_Q(x) = x^T Q_s x + Q_h(x) \qquad (15b)$$

Recall that $(\bullet)^{(k)}$ denotes the k-th order term , $(\bullet)^{[k]}$ stands for the sum of the accumulated terms up to the k-th order term, and X is the positive semi-definite stabilizing solution of the ARE (10). Now, the k-th order approximate solution of (14) can be written as

$$V^{[k]}(x) = \sum_{m=2}^{k} V^{(m)}(x) = x^T X x + \sum_{m=3}^{k} V^{(m)}(x) \qquad (15c)$$

and its derivative will be

$$\frac{\partial V^{[k]}}{\partial x} = 2x^T X + \sum_{m=3}^{k} \frac{\partial V^{(m)}}{\partial x} \qquad (16)$$

Note that the order of $\frac{\partial V^{(m)}}{\partial x}$ is $m - 1$. Using (8e), (15a), and (15b), we approximate (14) to the following:

$$\frac{\partial V^{[K]}}{\partial x} [A_s x + f_h(x)] + \frac{1}{4} \frac{\partial V^{[K]}}{\partial x} [R_s + 4R_h(x)] \frac{\partial^T V^{[K]}}{\partial x}$$

$$+x^TQ_sx+Q_h(x)$$

$$=\frac{\partial V^{[K]}}{\partial x}A_sx+\frac{\partial V^{[K]}}{\partial x}f_h(x)+\frac{1}{4}\frac{\partial V^{[K]}}{\partial x}R_s\frac{\partial^T V^{[K]}}{\partial x}$$

$$+\frac{\partial V^{[K]}}{\partial x}R_h(x)\frac{\partial^T V^{[K]}}{\partial x}+x^TQ_sx+Q_h(x)=0 \qquad (17)$$

From (16), equation (17) can be expressed as

$$\underline{2x^TXA_sx}+\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}A_sx+\frac{\partial V^{[K]}}{\partial x}f_h(x)+\underline{x^TXR_sXx}$$

$$+\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}\frac{1}{4}R_s\frac{\partial V^{(m)}}{\partial x}+\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}R_sXx$$

$$+\frac{\partial V^{[K]}}{\partial x}R_h(x)\frac{\partial^T V^{[K]}}{\partial x}+\underline{x^TQ_sx}+Q_h(x)=0 \qquad (18)$$

Collecting all the underlined terms in (18) and from (10) we have

$$2x^TXA_sx + x^TXR_sXx + x^TQ_sx$$

$$= x^T(A^Tx + XA + XR_sX + Q)x = 0 \qquad (19)$$

From (19) and (8d), the equation (18) can be rearranged as

$$\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}F_cx+\frac{\partial V^{[K]}}{\partial x}f_h+\sum_{m=3}^{k-1}\frac{\partial V^{(m)}}{\partial x}\frac{1}{4}R_s\sum_{m=3}^{k-1}\frac{\partial^T V^{(m)}}{\partial x}$$

$$+\frac{\partial V^{[k]}}{\partial x}R_h\frac{\partial^T V^{[K]}}{\partial x}+Q_h=0 \qquad (20)$$

In order to find $\frac{\partial V^{(k)}}{\partial x}$ successively, one needs to put the first term of (20) in the left hand side of a new equation and all other terms in (20) using the solutions from the previous iteration to form the right hand side of the new equation, i.e.,

$$-\sum_{m=3}^{k}\frac{\partial V^{(m)}}{\partial x}F_cx=\frac{\partial V^{[k-1]}}{\partial x}f_h+\sum_{m=3}^{k-1}\frac{\partial V^{(m)}}{\partial x}\frac{1}{4}R_s\sum_{m=3}^{k-1}\frac{\partial^T V^{(m)}}{\partial x}$$

$$+\frac{\partial V^{[k-1]}}{\partial x}R_h\frac{\partial^T V^{[k-1]}}{\partial x}+Q_h \qquad (21)$$

Note that in (21) all the terms whose order is less than k will be vanished due to the cancellation of the previous successive procedures and all terms whose order are grater than k are ignored during the k-th order iteration and hence (21) becomes a pure k-th order equation and the approximate solutions of the HJE (3a) can be computed successively based on the following equation:

$$-\frac{\partial V^{(k)}}{\partial x}F_cx=\sum_{m=2}^{k-1}\frac{\partial V^{(m)}}{\partial x}f_h^{(k-m+1)}$$

$$+\sum_{m=3}^{k-1}\frac{\partial V^{(k-m+2)}}{\partial x}\frac{1}{4}R_s\frac{\partial^T V^{(m)}}{\partial x}$$

$$+\sum_{n=1}^{k-2}\sum_{m=2}^{k-n}\frac{\partial V^{(k-n-m+2)}}{\partial x}R_h^{(n)}\frac{\partial^T V^{(m)}}{\partial x}+Q_h^{(k)}:=H_m^{(k)}(x) \qquad (22)$$

where $k \geq 3$ is an integer. By comparing the coefficients on both sides of (22), a set of linear equations are established and employed to solve $V^{(k)}$. Then, based on (9b), an approximate solution of the HJE in (3a), $X(x)$, is constructed as follows,

$$X^{[k-1]}(x)=\frac{1}{2}\frac{\partial^T V^{[k]}}{\partial x}=X^{[k-2]}(x)+\frac{\partial^T V^{(k)}}{\partial x} \qquad (23)$$

Computational Procedure

(i) *The first-order approximate solution:*

$$X^{[1]}(x)=\frac{1}{2}\frac{\partial^T V^{(2)}}{\partial x}=Xx \qquad (24)$$

where X is defined in (10).

(ii) *The second-order approximate solution:*

Assume the number of possible third-order terms of x is n_3. Let $V^{(3)}(x)$ be the linear combination of these n_3 terms. For example, if $x = [x_1\ \ x_2\ \ x_3]^T$, then

$$V^{(3)}(x)=c_1x_1^3+c_2x_2^3+c_3x_3^3+c_4x_1^2x_2+c_5x_1^2x_3+c_6x_1x_2^2$$

$$+c_7x_1x_3^2+c_8x_1x_2x_3+c_9x_2^2x_3+c_{10}x_2x_3^2 \qquad (25)$$

and $n_3 = 10$. Equation (22) now becomes

$$-\frac{\partial V^{(3)}(x)}{\partial x}F_cx=2x^TXf_h^{(2)}(x)+4x^TXR_h^{(1)}(x)Xx+Q_h^{(3)}(x) \qquad (26)$$

Note that both sides of (26) consist of only third-order terms. By comparing the coefficients for both sides of (26), a set of n_3 linear equations are established to give the solution $V^{(3)}(x)$. Then the second-order approximate solution from (23) is

$$X^{[2]}(x)=Xx+\frac{1}{2}\frac{\partial^T V^{(3)}}{\partial x} \qquad (27)$$

(iii) *The third-order approximate solution:*

As before, assume $V^{(4)}(x)$ has n_4 forth-order terms of x. Equation (22) now is

$$-\frac{\partial V^{(4)}}{\partial x}F_cx=2x^TXf_h^{(3)}+\frac{\partial V^{(3)}}{\partial x}f_h^{(2)}$$

$$+\frac{\partial V^{(3)}}{\partial x}\frac{1}{4}(\gamma^{-2}B_1B_1^T-B_2B_2^T)\frac{\partial^T V^{(3)}}{\partial x}$$

$$+4\frac{\partial V^{(3)}}{\partial x}R_h^{(1)}Xx+4x^TXR_h^{(2)}Xx+Q_h^{(4)} \qquad (28)$$

Note that the only nonlinear terms involved in (28) are $f_h^{(2)}, f_h^{(3)}, R_h^{(1)}, R_h^{(2)}$, and $Q_h^{(4)}$; hence $V^{(4)}(x)$ can be found as a solution of the n_4 linear equations obtained from comparing the coefficients of (28). The third-order approximate solution $X^{[3]}(x)$ is

$$X^{[3]}(x) = X^{[2]}(x) + \frac{1}{2}\frac{\partial^T V^{(4)}}{\partial x} \tag{29}$$

The successive computation procedure can continue to produce higher order approximations if higher accuracy is required.

Remark 1

(i) From the above detailed procedure of finding an approximate solution of the HJE, we know that the number of terms, i.e. n_k, involved in $V^{(k)}(x)$ during the k-th order iteration is essential for applying the proposed successive algorithm. Actually the n_k can be simply represented [12] by $\begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix}$ where n is the size of the system.

(ii) One question arises on how to make sure that the approximate solution computed is correct. Here we provide a method to check the solution. Once the approximate solution $X^{[k-2]}(x)$ ($k \geq 3$) is obtained and plug it in (3a), you will see that the terms k whose order is less than k is vanished.

(iii) Another method for finding $V^{(k)}(x)$ is to use the integration method [11, 13]. Due to F_c being a stable matrix, $V^{(k)}(x)$ can be calculated from (22) by

$$V^{(k)}(x) = \int_0^\infty H_m^{(k)}(e^{F_c t}x)dt \tag{30}$$

where $H_m^{(k)}$ is defined in (22). Note that $V^{(k)}(x)$ should be identical for both the proposed linear equations method and the integration method (30).

A SCALAR EXAMPLE

In this section, a scalar example which was originally presented by WS [15] is given to demonstrate finding an approximate solution for the HJE by using the three approaches mentioned in the previous sections, i.e., the WS method, the proposed linear equations method, and the integration method. Consider the following HJE with the form of (14) as

$$V_x[-7x + f_h(x)] - \frac{1}{4}V_x^2 + 15x^2 = 0 \tag{31}$$

whose exact solution can be easily computed as

$$V_x = 2x\left(-7 + \frac{f_h(x)}{x} + \sqrt{(-7 + \frac{f_h(x)}{x})^2 + 15}\right) \tag{32a}$$

$$= 2x + \frac{1}{4}f_h + \frac{15}{8^3}\frac{f_h^2}{x} + \frac{105}{8^5}\frac{f_h^3}{x^2} + \cdots \tag{32b}$$

Assume $f_h(x) = ax^3$ where a is a constant, then (32b) becomes

$$V_x = 2x + \frac{1}{4}ax^3 + \frac{15}{8^3}a^2x^5 + \frac{105}{8^5}a^3x^7 + \cdots \tag{33}$$

The ARE in (10) for the linearized model is:

$$X^2 + 14X - 15 = 0 \tag{34}$$

whose positive solution is $X = 1$. From (8) and (10), it is easy to see that $R_s = -1$, $F_c = -8$, $R_h(x) = 0$, and $Q_h(x) = 0$.

(M1) The WS method

From (11), we can get

$$J(t) = \frac{1}{32}(e^{16t} - 1) \tag{35a}$$

$$a(x, p) = -8ax^3 - 3ax^2p \tag{35b}$$

$$(t) = e^{-8t}z(0) + \frac{1}{32}(e^{-16t} - 1)p(t) + ae^{-8t}\int_0^t e^{8\tau}[z^3 + \frac{1}{32}(e^{-16\tau} - 1)(8z^3 + 3z^2p)]d\tau \tag{35c}$$

$$p(t) = ae^{8t}\int_0^t e^{-8\tau}[8z^3 + 3z^2p]d\tau \tag{35d}$$

in which the zeroth approximation is assumed as

$$z_{(0)}(t) = e^{-8t}x \text{ and } p_{(0)}(t) = 0 \tag{36}$$

Plugging (36) into (35c,d) results in the first approximation as

$$z_{(1)}(t) = e^{-8t}x - ax^3(-\frac{7}{128}e^{-8t} + \frac{3}{64}e^{-24t} - \frac{1}{128}e^{-40t}) \tag{37a}$$

$$p_{(1)} = \frac{1}{4}ae^{-24t}x^3 \tag{37b}$$

Repeat the same procedure by plugging (37) into (35c,d) again, we have

$$p_{(2)}(t=0) = a[\frac{1}{4}x^3 + \frac{31}{2^{10}}ax^5 + \frac{289}{2^{15}\cdot 10}a^2x^7 + \cdots] \tag{38}$$

By using (13), $V_x(x)$ can be obtained as:

$$V_x(x) = 2x + \frac{1}{4}ax^3 + \frac{31}{2^{10}}a^2x^5 + \frac{289}{2^{15}\cdot 10}a^3x^7 + \cdots] \tag{39}$$

(M2) The linear equations method

Next, we consider the method based on the construction of the linear equations from comparing the coefficients of both sides of (22). Starting from $k = 3$, it is easy to see that $V^{(3)}(x) = 0$ because of $f_h^{(2)} = R_h^{(1)} = Q_h^{(3)} = 0$. Note that in this example, $V^{(2m+1)}(x) = 0$ for all $m \in \boldsymbol{N}$. For $k = 4$, we have $V^{(4)}(x) = b_4x^4$ where b_4 is a constant to be determined. Equation (22) can be rewritten as

$$-\frac{\partial V^{(4)}}{\partial x} F_c x = 2xXf_h^{(3)} \tag{40a}$$

Plugging all given numbers into (40a) results in

$$-4b_4x^3(-8)x = 2x(1)ax^3 = 2ax^4 := H_m^{(4)}(x) \tag{40b}$$

Comparing the coefficient of x^4 for both sides, we can construct one linear equation as $32b_4 = 2a$ which gives $b_4 = \frac{1}{16a}$ and hence $V_x^{(4)} = \frac{1}{4}ax^3$. For $k = 6$, by assuming $V^{(6)}(x) = b_6x^6$, Equation (18) can be rewritten as

$$-\frac{\partial V^{(6)}}{\partial x} F_c x = \frac{\partial V^{(4)}}{\partial x} f_h^{(3)} + \frac{1}{4}\left(\frac{\partial V^{(4)}}{\partial x}\right)^2 R_s \tag{41a}$$

$$-6b_6x^5(-8)x = \frac{15}{64}a^2x^6 := H_m^{(6)}(x) \tag{41b}$$

which gives $b_6 = \frac{15}{6\times 8^3}a^2$. The same procedure can be repeated to get $V^{(8)}(x) = b_8x^4$ with $b_8 = \frac{105}{8^6}$. Then we have the following approximate solution

$$V_x^{[8]}(x) = 2x + \frac{1}{4}ax^3 + \frac{15}{8^3}a^2x^5 + \frac{105}{8^5}x^7 \tag{42}$$

(M3) The integration method

The method is the same as M2 with the exception that the integration method is applied to find $V_x^{(k)}(x)$. As in M2, we have $V^{(2m+1)} = 0$ for $m \in N$. $V^{(4)}(x)$ can be calculated by (30) and (40b) as

$$V^{(4)}(x) = 2a\int_0^{\infty} (e^{-8t}x)^4 dt = \frac{1}{16}ax^4 \tag{43}$$

$V^{(6)}(x)$ can be calculated by (30) and (41b) as

$$V^{(6)}(x) = \frac{15}{64}a^2\int_0^{\infty} (e^{-8t}x)^6 dt = \frac{15}{6\times 8^3}a^2x^6 \tag{44}$$

Remark 2

(i) The solution from M3 is the same as the one from M2 because $V^{(k)}(x)$ is the unique solution by comparing the coefficient from (18). Actually, the solution (42) from M2 to the exact solution defined in (33) with the terms higher than seventh-order truncated. On the other hand, the solution (39) from M1 is slightly different from the truncated exact solution. In Fig. 1, the linear solution, the third-order approximate solution for both approaches, the seventh-order approximate solutions (39) by M1 and (42) by M2 are compared with the exact solution (32b). From Fig. 1, we can see that higher order solution is closer to the exact solution. In this particular example, the solution from M2 is more accurate than the one from M1.

(ii) Both M1 and M3 involve integration computations which may require impractical computing time when the matrix size of F_c is large. Method M2 requires only solving linear equations which is less completed than integration.

NONLINEAR H_∞ CONTROLLER DESIGN FOR INVERTED PENDULUM

The nonlinear model of the Inverted pendulum [5] is represented as:

$$\begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} := \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ 24.1314x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4.0219x_3^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2.4606 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.2303x_3^2 \end{bmatrix} u \tag{45a}$$

$$:= f(x) + g_2(x)u := Ax + f_h(x) + B_2u + g_{2h}(x)u$$

where r is the cart displacement and θ is the pendulum angle. The measured outputs are r and θ, so the output equations can be represented by

$$y = \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x := C_2x \tag{45b}$$

The objective is to design a (nonlinear H_∞) con-

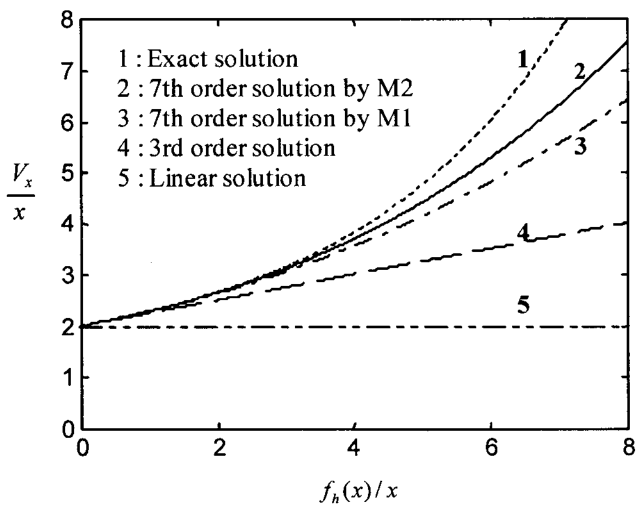

Fig. 1. Comparison of different approaches with the exact solution.

troller to drive the cart motor so that the cart can move back and forth maintaining the stick at a strictly vertical position (keeping $\theta = 0$). From the nonlinear H_∞ control problem formulation [6], we assume that a disturbance d is injected into the system via the state equation and the measurement is contaminated by the noise n. The weighted state vector, z_1, represents the disturbance response of interest to be minimized. The weighted control input, z_2, is employed to add control input constraint into the problem formulation. W_n, W_x, W_u, and W_n are appropriate constant weighting matrices. Let $w = [d \;\; n]^T$ and $z = [z_1 \;\; z_2]^T$, then the nonlinear generalized plant for the inverted pendulum control problem formulation is constructed as follows,

$$\dot{x} = f(x) + g_1(x)w + g_2(x)u$$

$$= f(x) + [W_d \;\; 0]w + g_2(x)\mathrm{u} :$$

$$= Ax + f_h(\mathrm{x}) + B_1 w + B_2 u + g_{2h}(x)u \qquad (46a)$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = h_1(x) + D_{12}(x)u = \begin{bmatrix} W_x \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ W_u \end{bmatrix} u \qquad (46b)$$

$$y = h_2(x) + D_{21}(x)w = C_2 x + [0 \;\; W_n]w \qquad (46c)$$

The weighting matrices W_x, W_d, W_n, and W_u are chosen as:

$$W_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10^{-6} \end{bmatrix}, \quad W_d = I_4,\; W_n = I_2,$$

$$\text{and } W_u = 1 \qquad (47)$$

The first step is to consider the linearized model of (46). The optimal H_∞ norm of the linear closed-loop system is computed [2] as $\gamma_{opt} = 52.42$. Choosing $\gamma = 55 > \gamma_{opt}$, one can find the solution for the ARE in (10) as

$$X = \begin{bmatrix} 1.8258 & 1.6660 & 5.3303 & 1.0859 \\ 1.6660 & 2.4959 & 8.6347 & 1.7594 \\ 5.3303 & 8.6347 & 69.6382 & 14.1508 \\ 1.0859 & 1.7594 & 14.1508 & 2.8797 \end{bmatrix} \qquad (48)$$

The first-order approximate solution of the HJE from (24) is

$$X^{[1]}(x) = Xx \qquad (49)$$

where X is given in (48). Due to $f_h^{(2)}(x) = R_h^{(1)}(x) = Q_h^{(3)}(x) = 0$ and from (26), we see that $V^{(3)}(x) = 0$. The second-order approximate solution is the same as the first-order one, i.e.,

$$X^{[2]}(x) = X^{[1]}(x) = Xx \qquad (50)$$

The construction of the third-order approximate solution is explained as follows. First, we compare the coefficients of the terms on both sides of (28) and set up 35 linear equations ($n_4 = 35$, see Remark 1 (i)), which in turn will be solved for $V^{(4)}(x)$. Then we have the third-order approximate solution $X^{[3]}(x)$ according to (29),

$$X^{[3]}(\mathrm{x}) = Xx + X^{(3)}(x) \qquad (51a)$$

where

$$X^{(3)}(x) := \begin{bmatrix} X_1^{(3)} \\ X_2^{(3)} \\ X_3^{(3)} \\ X_4^{(3)} \end{bmatrix} \qquad (51b)$$

$$X_1^{(3)}(x) = 0.07290x_1x_3^2 + 0.1240x_1x_3^2 + 2.3450x_3^3 + 0.6130x_3^2x_4 \qquad (51c)$$

$$X_2^{(3)}(x) = 0.1240x_1x_3^2 + 0.2055x_2x_3^2 + 3.9339x_3^3 + 1.0069x_3^2x_4 \qquad (51d)$$

$$X_3^{(3)}(x) = 0.2480x_1x_2x_3 + 0.2055x_2^2x_3 + 7.0348x_1x_3^2 + 11.8017x_2 + 98.6658x_3^3 + 1.2260x_1x_3x_4 + 2.0137x_2x_3x_4 + 37.5422x_3^2x_4 + 3.9021x_3x_4^2 + 0.07290x_1^2x_3 + 0.09527x_4^2 \qquad (51e)$$

$$X_4^{(3)}(x) = 0.6130x_1x_3^2 + 1.0069x_2x_3^2 + 12.5141x_3^3 + 3.9021x_3^2x_4 + 0.2858x_3x_4^2 + 0.04679x_2x_3x_4 \qquad (51f)$$

For simplicity, the coefficients with absolute value less than 0.04 were chopped. A nonlinear H_∞ controller can be obtained by plugging the third-order approximate solution (51) of the HJE to the controller formulas (5).

Simulations

The computer simulations for the closed-loop system will be performed. Let the initial conditions be $[0 \;\; 0 \;\; \theta_0 \;\; 0]^T$ and the exogenous input be

$$w = A_w[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]^T \qquad (52)$$

where A_w is a constant representing the disturbance amplitude. First, the simulation is performed under the condition of no disturbance. The pendulum responses with A_w = 0 and small initial angle θ_0 = 0.2 are plotted in Fig. 2. It shows that the pendulum angle θ converges to 0 after 3 seconds and the cart displacement r only deviates a little bit before it quickly returns to its equilibrium. Simulations for the linear H_∞ controller show that the performance of the linear controller is almost the same as that of the nonlinear controller at this point.

Next we increase the initial pendulum angular displacement to θ_0 = 0.5 and let the disturbance amplitude be A_w = 0.15. The θ responses for both the nonlinear H_∞ and linear H_∞ controllers are plotted in Fig. 3. It is obvious that the nonlinear H_∞ controller has better performance in keeping the stick straight. If A_w is increased even larger to 0.2, then the θ response for the linear H_∞ controller diverges while the response for the nonlinear H_∞ controller is able to converge to zero after only 4 seconds.

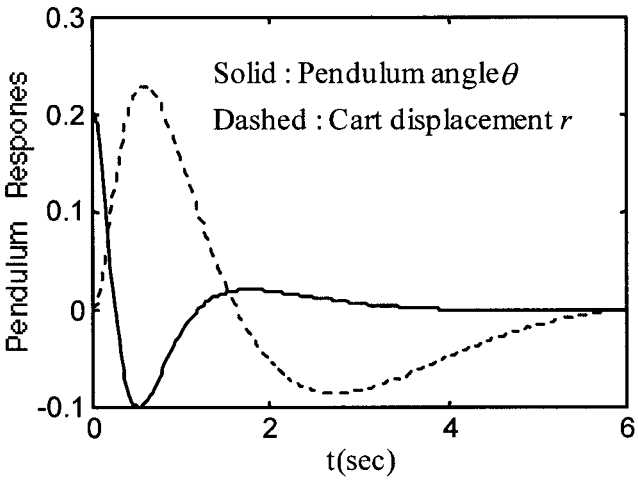

Fig. 2. Pendulum responses with the nonlinear H_∞ controller when θ_0 = 0.2 and A_w = 0.

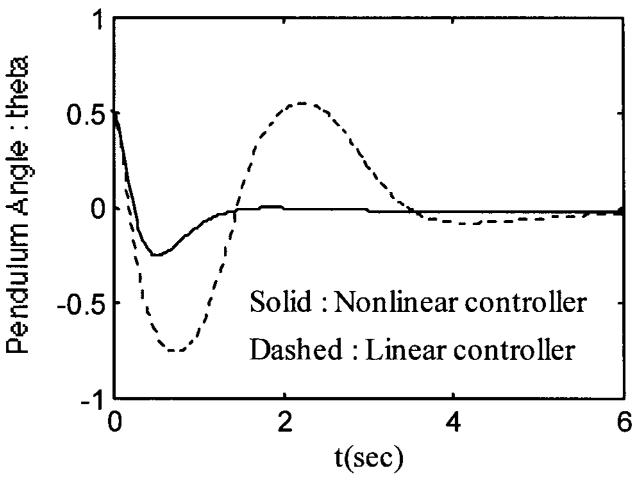

Fig. 3. Comparison of pendulum angle θ responses when θ_0 = 0.5 and A_w = 0.15.

CONCLUSIONS

In this paper, a detailed successive algorithm and computational procedure for finding an approximate solution of the HJE by solving the linear equations were presented. A scalar example was given to compare three approaches: the WS approach, the proposed linear equations method, and the integration method and it was found that the proposed successive algorithm - linear equations method (M2) is the best approach among the three methods. The proposed algorithm was also employed to find an approximate solution for the HJE leading to construct a nonlinear H_∞ controller for the inverted pendulum. Simulations of the closed-loop pendulum responses for both nonlinear and linear controllers were performed and it was found that the nonlinear H_∞ controller has better performance and robustness than the linear controller, which reveals the importance of the proposed algorithm of solving the Hamilton-Jacobi equation.

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NOMENCLATURE

ARE	algebraic Riccati equation
ARI	algebraic Riccati inequality
BHW	Ball, Helton, and Walker
HJE	Hamilton-Jacobi equation
HJI	Hamilton-Jacobi inequality
$O(x^m)$	the higher order terms including x^m
$\mathbf{R}^n$	n-dimensional Euclidean space
WS	Wise and Sedwick
$(\bullet)^{(k)}$	the k-th order term
$(\bullet)^{[k]}$	the sum of all terms up to the k-th order term

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求解HJE之計算法則並應用於非線性控制

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摘　要

在古典力學及非線性H_∞控制理論，Hamilton-Jacobi方程式(HJE)均扮演著重要的角色。在本論文中，我們提出了一個詳細的疊代法則及計算程序，以解線性方程式來求得HJE之近似解。文中附上一個純量的範例來比較所提之計算法則與其他方法之求解過程。最後，本論文探討一個控制的實例，即設計倒單擺之非線性H_∞控制器，並驗證其與線性H_∞控制器比較之強韌性能/穩定性的優越性，以顯示本計算法則之必要性。

關鍵詞：非線性H_∞控制、Hamilton-Jacobi方程式、強健控制。