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DUAL BOUNDARY INTEGRAL EQUATIONS FOR HELMHOLTZ EQUATION AT A CORNER USING CONTOUR APPROACH AROUND SINGULARITY

I-Lin Chen*, Ming-Te Liang**, Shyh-Rong Kuo** and Jeng-Tzong Chen**

Key words: dual boundary element method, corner, limiting process, regularized method.

ABSTRACT

A dual integral formulation for the Helmholtz equation problem at a corner is derived by means of the contour approach around the singularity. It is discovered that employing the contour approach the jump term comes half and half from the free terms in the L and M kernel integrations, respectively, which differs from the limiting process from an interior point to a boundary point where the jump term is descended from the L kernel only. Thus, the definition of the Hadamard principal value for hypersingular integration at the collocation point of a corner is extended to a generalized sense for both the tangent and normal derivative of double layer potentials as compared to the conventional definition. The free terms of the six kernel functions in the dual integral equations for the Helmholtz equation at a corner have been examined. The kernel functions of the Helmholtz equation are quite different from those of the Laplace equation while the free terms of the Helmholtz equation are the same as those of the Laplace equation. It is worth to point out that the Laplace equation is a special case of the Helmholtz equation when the wave number approaches zero.

INTRODUCTION

Chen [1] developed a dual integral formulation for crack problems. This work was published in 1988 [3] and was extended to the Laplace equation with a degenerate boundary [3, 4, 5]. In the numerical implementation, it was termed the dual boundary element method by Portela *et al*. [6]. The formulations have been chiefly applied to problems with a smooth boundary. However, a nonsmooth boundary often happens in the description of many engineering problems, so the ability to manage this situation is not trivial. The nonsmooth boundary presents a corner or edge, which makes the normal vector and normal flux at a corner undefined. How to exactly simulate the potential and potential gradient near a corner has received much attention in the boundary element method. Banerjee and Butterfield [7] discussed the double node technique for a corner problem. Alarcon *et al*. [8] applied the transformation of tangent flux and normal flux to establish the constraint equation to secure a unique solution. Walker and Fenner [9] indicated that error will be present in calculating the normal flux independently of the BEM, so they provided a nonlinear relationship for the tangent and normal fluxes. However, if the interior angle is close to ninety degrees, the ill-condition will happen. Therefore, a hypersingular equation has been utilized to furnish a constraint at a corner in an analytical way. Gray and Manne [10] have used the hypersingular equation as an additional constraint to secure a unique solution by a limiting process from an interior point to a corner. Gray and Lutz [11] extended this technique to the three dimensional case. From point of view of dual integral equations, the singular and hypersingular equations can furnish sufficient constraints for a singular system with a corner. On a nonsmooth boundary, *e.g*., a corner point, the jump terms of singular and hypersingular integral equations are the same in the former derivations as described by Lutz *et al*. [12] and Chen and Hong. [13] Mansur *et al*. [14] presented the hypersingular formulation for Laplace's equation in two-dimensional problem, using vector approach in tensor form to obtain the free terms which are independent of any coordinate system. A dual integral formulation for the Laplace equation problem at a corner using the contour surrounding the singularity was derived by Chen and Hong [15]. The Laplace equation problem can be treated as a special case of the Helmholtz equation. The Helmholtz equation is often present in engineering, *e.g*., vibration problem and acoustics. Chen and Chen [16] used the dual integral formulation for the Helmholtz equation to solve the

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acoustic modes of a two-dimensional cavity with a degenerate boundary. The jump properties of the potentials resulting from the four kernel functions in the dual integral equation were examined in [16].

In this paper, a dual integral formulation for the Helmholtz equation problem at a corner by using contour approach around singularity will be studied. Following the same notation [3, 15] of U, V, L and M kernels for single layer kernel and its normal derivative, double layer kernel and its normal derivative, respectively, three alternatives for constraint equations can be favorite: (1) by the U, V equation and the L^-, M^- equation with the collocation point before the corner; (2) by the U, V equation and the L^+, M^+ equation with the collocation point after the corner; and (3) by the L^-, M^- equation and the L^+, M^+ equation with the collocation point by using different normal vectors before and after the corners. In order to avoid the boundary effect, one regularization technique will be proposed.

DUAL INTEGRAL FORMULATION OF BEM FOR HELMHOLTZ EQUATION WITH A CORNER

The dual boundary integral equations for the potential u can be derived as

$$0 = \int_{B_\alpha + B'} \{V(\hat{x}, x)\, u(\hat{x}) - U(\hat{x}, x)\, v(\hat{x})\}\, dB(\hat{x}) \tag{1}$$

$$0 = \int_{B_\alpha + B'} \{M(\hat{x}, x)\, u(\hat{x}) - L(\hat{x}, x)\, v(\hat{x})\}\, dB(\hat{x}) \tag{2}$$

$$0 = \int_{B_\alpha + B'} \{M^t(\hat{x}, x)\, u(\hat{x}) - L^t(\hat{x}, x)\, v(\hat{x})\}\, dB(\hat{x}) \tag{3}$$

where x and $\hat{x}$ denotes the field point and source point, respectively. The $u(\hat{x})$ and $v(\hat{x})$ indicate the potential and its normal flux on the boundary point $\hat{x}$, respectively, B' and B_α are the contour integration path not containing the singularity inside the domain, D, as shown in Fig. 1, and U, V, L, M, L^t and M^t are the six kernel functions [5] in the dual integral equations with the properties shown explicit form in Table 1. The U and M kernels are weakly singular and hypersingular, respectively, whereas the V and L kernels are strongly singular. Aliabadi *et. al*. [17, 18] have used Taylor's expansion to reduce the singularity order for the single and double layer kernels. Eqs. (2) and (3) are distinguishable in the direction of derivative on the collocation point x. The superscript 't' in Eq. (3) expresses the tangent vector. The B_α integration path in Fig. 1 marks the contour integration surrounding the singularity with radius ε, and $B' + B^+ + B^-$ is just the definition of the integration region of the Cauchy principal value. B^+ and B^- indicate two of the elements in the B' boundary near singularity as shown in Fig. 1. First of all, we integrate the B_α path

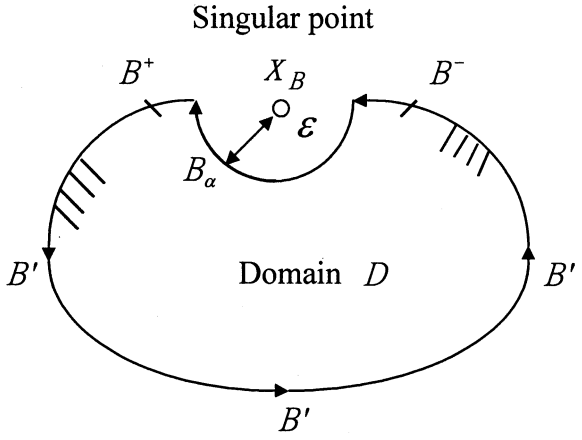


Fig. 1. The considered boundary integration path.

integration to obtain the free terms for the six kernel functions.

Without loss of generality, there are the following symbols in Fig. 2:

$$x = (0, 0) \tag{4}$$

$$\hat{x} = (\varepsilon \cos(\theta), -\varepsilon \sin(\theta)) \tag{5}$$

$$r = \left| x - \hat{x} \right| \tag{6}$$

$$y_1 = -\varepsilon \cos(\theta) \tag{7}$$

$$y_2 = \varepsilon \sin(\theta) \tag{8}$$

$$n(\hat{x}) = (n_1, n_2) = (-\cos(\theta), \sin(\theta)) \tag{9}$$

$$n(x) = (\overline{n}_1, \overline{n}_2) = (0, 1) \textit{ for normal derivative} \tag{10}$$

$$t(x) = (\overline{n}_1, \overline{n}_2) = (1, 0) \textit{ for tangent derivative} \tag{11}$$

$$u(\hat{x}) = u(x) + \frac{\partial u}{\partial x_1} \varepsilon \cos(\theta) - \frac{\partial u}{\partial x_2} \varepsilon \sin(\theta) \tag{12}$$

$$v(\hat{x}) = -\frac{\partial u}{\partial x_1} \cos(\theta) - \frac{\partial u}{\partial x_2} \sin(\theta) \tag{13}$$

where $t(x)$ Eq. (11) indicates the tangent vector on the collocation point x with components (1, 0) as shown in Fig. 2. Since the corner is considered in order to decide the two normal vectors before and after the corners, we define the following notations:

$$v^- = \frac{\partial u}{\partial x_2} \tag{14}$$

$$v^+ = -\frac{\partial u}{\partial x_1} \sin(\alpha) - \frac{\partial u}{\partial x_2} \cos(\alpha) \tag{15}$$

$$v^{-'} = \frac{\partial u}{\partial x_1} \tag{16}$$

Table 1. Properties of different kinds of potentials across smooth boundary

Kernel function $K(\hat{x}, x)$ direct method	$U(\hat{x}, x)$	$V(\hat{x}, x)$	$L(\hat{x}, x)$	$M(\hat{x}, x)$	$L^t(\hat{x}, x)$	$M^t(\hat{x}, x)$
Kernel function $K(x, \hat{x})$ indirect method	$U(x, \hat{x})$	$U^*(x, \hat{x})$	$V(x, \hat{x})$	$V^*(x, \hat{x})$	$V^t(x, \hat{x})$	$V^{*t}(x, \hat{x})$
Singularity 1D	$O(r)$	$O(1)$	$O(1)$	$O(\delta(r))$	$O(1)$	$O(\delta(r))$
Singularity 2D	$O(\ln(r))$	$O(1/r)$	$O(1/r)$	$O(1/r^2)$	$O(1/r)$	$O(1/r^2)$
Singularity 3D	$O(1/r)$	$O(1/r^2)$	$O(1/r^2)$	$O(1/r^3)$	$O(1/r^2)$	$O(1/r^3)$
Density function $\mu(\hat{x})$ direct method	$-\upsilon$	u	$-\upsilon$	u	$-\upsilon$	u
Density function $\mu(\hat{x})$ indirect method	$-\phi$	ψ	$-\phi$	ψ	$-\phi$	ψ
Potential type $\int K(\hat{x}, x)\,\mu(\hat{x})\,d\hat{x}$	single layer	double layer	normal derivative of single layer potential	normal derivative of double layer potential	tangent derivative of single layer potential	tangent derivative of double layer potential
Continuity across boundary	conti-nuous	disconti-nuous	disconti-nuous	pseudo conti-nuous	conti-nuous	disconti-nuous
Free term method(1) [20]	no jump	πu	$-\pi\upsilon$	no jump	no jump	$\pi \frac{\partial u}{\partial x}$
Free term method (2) direct method	no jump	πu	$-\frac{1}{2}\pi\upsilon$	$\frac{1}{2}\pi\upsilon$	$-\frac{1}{2}\pi \frac{\partial u}{\partial x}$	$\frac{1}{2}\pi \frac{\partial u}{\partial x}$
Free term Lamb method direct method	no jump	$\pi\phi$	$\pi\psi$	0	$\pi \frac{\partial \phi}{\partial x}$	0
Jump term method(1)	no jump	$2\pi u$	$-2\pi\upsilon$	no jump	no jump	$2\pi \frac{\partial u}{\partial x}$
Jump term method(2)	no jump	$2\pi u$	$-\pi\upsilon$	$-\pi\upsilon$	$\pi \frac{\partial u}{\partial x}$	$\pi \frac{\partial u}{\partial x}$
Principal value sense	R.P.V.	C.P.V.	C.P.V.	H.P.V.	C.P.V.	H.P.V.

where U* and V* are kernels of single layer and double layer potentials.

$$\upsilon^{+'} = -\frac{\partial u}{\partial x_1}\cos(\alpha) - \frac{\partial u}{\partial x_2}\sin(\alpha) \tag{17}$$

where α is the interior angle of the corner, υ^- and υ^+ are normal derivatives on the boundary point before a corner and after a corner, respectively, and $u^{-'}$ and $u^{+'}$ are tangent derivatives along the boundary before and after a corner, respectively.

In according to the related notations in Fig. 3, the free terms of the six kernels will be derived as in the following.

(1). Single layer potential resulted from

$$x = (0,0)$$
$$\hat{x} = (\varepsilon\cos(\theta), -\varepsilon\sin(\theta))$$
$$y_1 = x_1 - \hat{x}_1 = -\varepsilon\cos(\theta)$$
$$y_2 = x_2 - \hat{x}_2 = \varepsilon\sin(\theta)$$
$$n(\hat{x}) = (n_1, n_2) = (-\cos(\theta), \sin(\theta))$$

$$n(x) = (\bar{n}_1, \bar{n}_2) = (0,1) \quad \text{, for normal derivative}$$
$$t(x) = (\bar{n}_1, \bar{n}_2) = (1,0) \quad \text{, for tangent derivative}$$
$$u^{+'} = \frac{\partial u}{\partial x_1}\cos(\alpha) - \frac{\partial u}{\partial x_2}\sin(\alpha)$$
$$u^{-'} = \frac{\partial u}{\partial x_1}$$

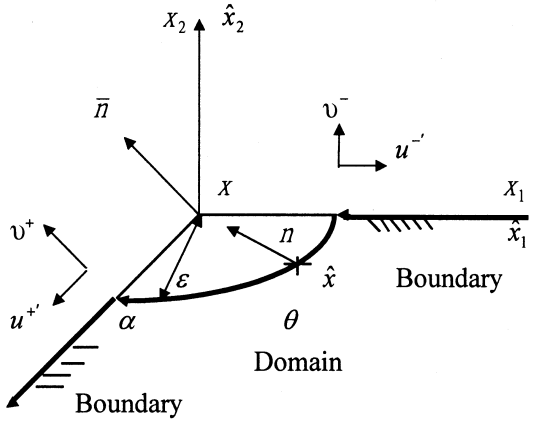


Fig. 2. Notations of the integration path around a corner.

$U(\hat{x}, x) = -\frac{i\pi}{2} H_0^{(1)}(kr)$:

$$\int_{B_\alpha} U(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = \varepsilon\,[-\frac{i\pi}{2} H_0^{(1)}(k\varepsilon)\,(\upsilon^+ + \upsilon^-)]$$

(*finite value*) (18)

The free term is zero since $H_0^{(1)}(k\varepsilon)$ approaches zero as the radius ε approaches zero.

(2). Double layer potential owing to $V(\hat{x}, x) = -\frac{ik\pi}{2} H_1^{(1)}(kr)\frac{y_i n_i}{r}$:

$$\int_{B_\alpha} V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = -\alpha u(x) + \varepsilon\,(\upsilon^+ + \upsilon^-)]$$

(*finite value*) (19)

As ε approaches zero, the free term is $-\alpha u(x)$.

(3). Normal derivative of single layer potential resulted from $L(\hat{x}, x) = \frac{ik\pi}{2} H_1^{(1)}(kr)\frac{y_i \overline{n}_i}{r}$:

$$\int_{B_\alpha} L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = c\,\upsilon^-(x) + d u^{-'}(x) \tag{20}$$

where

$$c = \frac{(-\sin(2\alpha) + 2\alpha)}{4} \tag{21}$$

$$d = \frac{(\cos(2\alpha) - 1)}{4} \tag{22}$$

As ε approaches zero, the free term is $c\ \upsilon^-(x) + d\ u^{-'}(x)$.

(4). Normal derivative of double layer potential resulted from

$$M(\hat{x}, x) = -\frac{ik\pi}{2}[-k\frac{H_2^{(1)}(kr)}{r^2} y_i y_j n_i \overline{n}_j + \frac{H_1^{(1)}(kr)}{r} n_i \overline{n}_i]$$

$$\int_{B_\alpha} M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = -c\,\upsilon^-(x) - d u^{-'}(x) + \textit{Boundary term} \tag{23}$$

where the boundary term $B(\varepsilon)$ is

$$B(\varepsilon) = \frac{1 - \cos(\alpha)}{\varepsilon} u(x) \tag{24}$$

It is interesting to discover that the free terms from the L and M kernels are the same but different by a minus sign. The free terms comprise the boundary term, which is infinite as ε approaches zero. By joining together the Cauchy principal value of the M kernel integration over B' comprising B^+ and B^- as shown in Fig. 1, the finite part can be extracted, and the infinity can be cancelled out. Therefore, the Hadamard principal value in the contour integration at a corner for M kernel can be defined by

$$H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = C.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) + \frac{1 - \cos(\alpha)}{\varepsilon} u(x) \tag{25}$$

(5). Tangent derivative of single layer potential resulted from $L^t(\hat{x}, x) = \frac{ik\pi}{2} H_1^{(1)}(kr)\frac{y_i \overline{n}_i}{r}$:

Since the tangent derivative in place of the normal derivative is considered

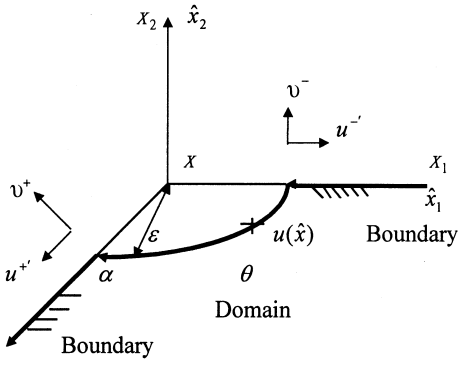
$$n(x) \to t(x) \tag{26}$$

$$\int_{B_\alpha} L^t(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = c' u^{-'}(x) + d\,\upsilon^-(x) \tag{27}$$

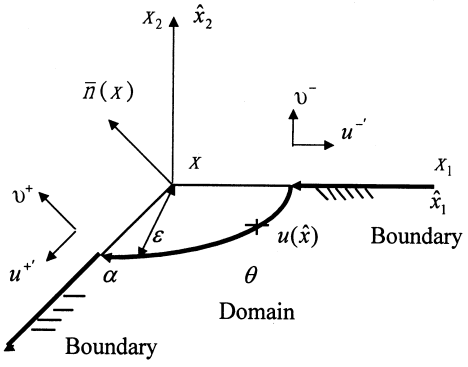
where

$$c' = \frac{(\sin(2\alpha) + 2\alpha)}{4}$$

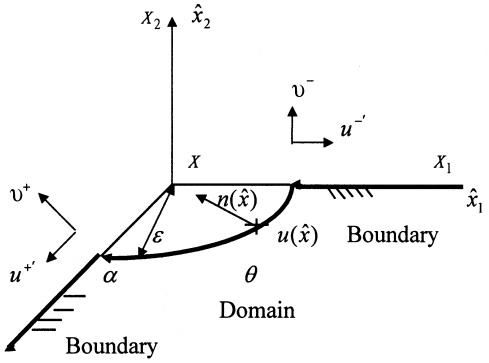
(1)$U(\hat{x},x)$



(3)$L(\hat{x},x)$



(2)$V(\hat{x},x)$



(4)$M(\hat{x},x)$

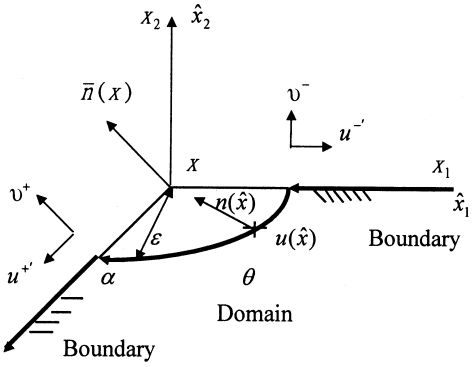


Fig. 3. Related symbols around a corner.

As ε approaches zero, the free term is $c'u^{-'}(x) + d\upsilon^-(x)$.

(6). Tangent derivative of double layer potential resulted from

$$M(\hat{x},x) = \frac{ik\pi}{2}\left[-k\frac{H_2^{(1)}(kr)}{r^2}y_i y_j n_i \bar{n}_j + \frac{H_1^{(1)}(kr)}{r} n_i \bar{n}_i\right]$$

In a way similar to Eq. (26), only alter the normal derivative to tangent derivative as

$$n(x) \to t(x) \tag{28}$$

$$\int_{B-\alpha} M^t(\hat{x},x)\,u(\hat{x})\,dB(\hat{x}) = -c'u^{-'}(x) - d\upsilon^-(x) + \textit{Boundary term} \tag{29}$$

where the boundary term $B(\varepsilon)$ is

$$B(\varepsilon) = -\frac{\sin(\alpha)}{\varepsilon}u(x) \tag{30}$$

The free terms include the boundary term, which is infinite as ε approaches zero. By joining together the Cauchy principal value of the M^t kernel integration over B' containing B^+ and B^-, the finite part can be extracted, and the infinity can be cancelled out. Therefore, the Hadamard principal value in the contour integration at a corner for M^t kernel can be defined by

$$H.P.V.\int_B M^t(\hat{x},x)\,u(\hat{x})\,dB(\hat{x}) = C.P.V.\int_B M^t(\hat{x},x)\,u(\hat{x})\,dB(\hat{x}) + \frac{\sin(\alpha)}{\varepsilon}u(x) \tag{31}$$

Since the basic unknowns in the BEM are the potential and the normal derivative of potential on the boundary, the tangent derivative, $u^{-'}$, in the present formulation would be better transformed to the combination of the normal derivative before and after the corners in Fig. 4 as follows:

$$u^{-'} = \frac{-1}{\sin(\alpha)}\left[\upsilon^+ + \cos(\alpha)\,\upsilon^-\right] \tag{32}$$

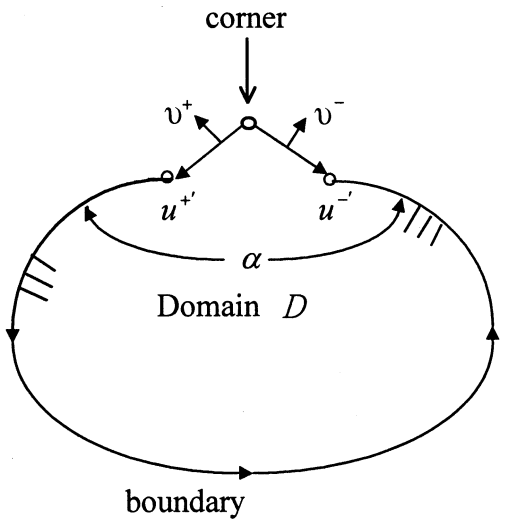


Fig. 4. Transformation of flux at a corner.

Therefore, the free term of the L kernel of Eq. (20) can be represented as

$$c\upsilon^-(x) + du^{-'}(x) = \frac{1}{2}\alpha\upsilon^- + \frac{1}{2}\sin(\alpha)\upsilon^+ \tag{33}$$

In the implementation of the BEM, it is put forward that this transformation be considered since the state variables are υ^+ and υ^- instead of $u^{-'}$ and υ^- although $u^{-'}$ can be represented in the phraseology peculiar to the numerical derivative of the nodal variables of u. For the free terms of the L^t and M^t kernel, this transformation is not necessary since the tangent derivative of potential can be represented in the phraseology peculiar to superposition of all the state variables, which comprise the potential and the normal derivative of potential on the boundary just solved by the U,V and L, M equations. Therefore, we can derive the following dual boundary integral equations employing the transformation of Eq. (32) and the representation for the tangential flux along the boundary:

$$\alpha u(x) = C.P.V.\int_B V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - R.P.V.\int_B U(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{34}$$

$$\alpha\upsilon^-(x) + \sin(\alpha)\upsilon^+(x) = H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{35}$$

$$2c'u^{-'}(x) = -2d\upsilon^-(x) + H.P.V.\int_B M^t(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L^t(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{36}$$

after employing

$$\int_{B'} U(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = R.P.V.\int_B U(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{37}$$

$$\int_{B'} V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = C.P.V.\int_B V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) \tag{38}$$

$$\int_{B'} L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = C.P.V.\int_B L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{39}$$

$$\int_{B'} M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - \frac{1-\cos(\alpha)}{\varepsilon}\, u(x) \tag{40}$$

$$\int_{B'} L^t(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) = C.P.V.\int_B L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{41}$$

$$\int_{B'} M^t(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = H.P.V.\int_B M^t(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) + \frac{\sin(\alpha)}{\varepsilon}\, u(x) \tag{42}$$

DISCUSSION ON THE DUAL BOUNDARY INTEGRAL EQUATIONS AT A CORNER

1. It is interesting to discover that the hypersingular equation for the collocation point after the corner has a similar representation as compared to the equation collocated at the point before the corner but for the change of υ^- and υ^+ as follows:

$$\alpha\upsilon^-(x) + \sin(\alpha)\upsilon^+(x) = H.P.V.\int_B M^-(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L^-(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{43}$$

$$\alpha\upsilon^+(x) + \sin(\alpha)\upsilon^-(x) = H.P.V.\int_B M^+(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L^+(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{44}$$

where M^- and M^+ indicate the M kernels with different normal vectors collocated before and after the corner, respectively. Similarly, the same significances of L^- and L^+ are employed. Eqs. (43) and (44) can be used to solve the corner problem with the Dirichlet conditions since they are linearly independent. In the literature, the $\sin(\alpha)\upsilon^-(x)$ term is neglected by Lutz *et al.* [12], and by Chen and Hong [13].

2. It is worthy of noticing that the Cauchy principal value of the L kernel integration at the corner, joining together the Hadamard principal value of the M kernel integration containing the two elements of B^+ and B^- in Fig. 1, exists under the requirement of C^1 continuity for u. The coefficients of $\ln(\varepsilon)$ owing to L and M kernels can be summed to zero as shown below:

$$\{-\cos(\alpha)u^{-'} + \sin(\alpha)v^{-} + u^{+'}\}\ \ln(\varepsilon) = 0 \qquad (45)$$

after employing the definition of $u^{+'}$ by Eq. (17).

3. For the case of a smooth boundary, Eqs. (43) and (44) bring into

$$\pi u(x) = C.P.V.\int_B V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - R.P.V.\int_B U(\hat{x}, x)\, v(\hat{x})\, dB(\hat{x}) \qquad (46)$$

$$\pi v(x) = H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L(\hat{x}, x)\, v(\hat{x})\, dB(\hat{x}) \qquad (47)$$

The expression of the tangent derivative of the boundary potential of Eq. (36) is

$$\pi u(x) = H.P.V.\int_B M^{t}(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L^{t}(\hat{x}, x)\, v(\hat{x})\, dB(\hat{x})$$

by substituting $\alpha = \pi$. At the tip of the degenerate boundary, the dual boundary integral equations can be brought into

$$0 = C.P.V.\int_B V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - R.P.V.\int_B U(\hat{x}, x)\, v(\hat{x})\, dB(\hat{x}) = H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) - C.P.V.\int_B L(\hat{x}, x)\, v(\hat{x})\, dB(\hat{x})$$

since $\alpha = 0$. The two equations supply additional constraints for the potential at the tip on the degenerate boundary to be $u^{+} = u^{-}$.

4. For a smooth boundary, the definition of the Hadamard principal value for M kernel integration is reduced to

$$H.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) = C.P.V.\int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) + \frac{2}{\tilde{c}}\, u(x)$$

we can judge the classical definition of the Hadamard principal value in the literature [19] as a special case of the present formulation by putting

$$M(\hat{x}, x) = -\frac{1}{(x-\hat{x})^2} \qquad (48)$$

$$dB(\hat{x}) = d\hat{x} \qquad (49)$$

Therefore, the Hadamard principal value of the M kernel integration is brought into the conventional one as follows:

$$H.P.V.\int_B \frac{-u(\hat{x})}{(x-\hat{x})^2}\, d\hat{x} = C.P.V.\int_B \frac{-u(\hat{x})}{(x-\hat{x})^2}\, d\hat{x} + \frac{2}{\tilde{c}}\, u(x) \qquad (50)$$

5. If a smooth boundary is considered, the interior angle α is π, and the property of the free term and the jump term can be brought into the same result of classical potential theory as shown in Table 1. The singularity orders for the six kernel functions in one, two and three dimensional problems are shown in the third row. The eighth row shows the free term derived by method (1) shown in Fig. 5(a) by means of a limiting process from the interior point to the boundary point employing an analytical integration [4, 20]. The ninth row indicates the free term by using contour

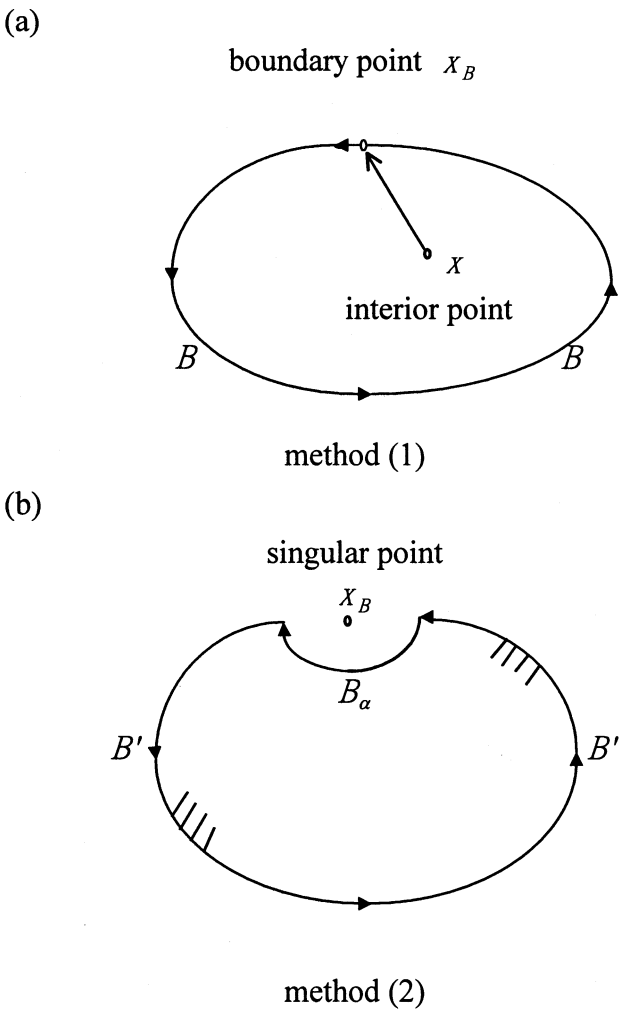


Fig. 5. (a) Limiting process from an interior point to a boundary point. (b) Contour around singularity.

integration surrounding singularity by the direct method (2) as shown in Fig. 5(b). The final results are the same after joining together the contributions from the L and M kernels, although the intermediate products are different. However, do the jump properties in the indirect method have the same manner? Since the direct method and the indirect method differ in the free terms, the unknown densities (u and υ for the direct method, ϕ and ψ for the indirect method as shown in Table. 1, where ϕ and ψ are functions defined on the boundary only) for the same problem are not the same. As shown in Appendix, the free terms derived by the indirect method are different from those of direct method.

6. Although the boundary B_α will contract to zero radius in the derivation, the $u(\hat{x})$ field along B_α can not be expressed by $u(x)$; therefore, care should be taken in employing the contour approach surrounding the singularity in the following computation:

$$\int_{B_\alpha} M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x}) \neq u(x) \int_{B_\alpha} M(\hat{x}, x)\, dB(\hat{x})$$
$$= -\, u(x) C.P.V. \int_B M(\hat{x}, x)\, dB(\hat{x}) \tag{51}$$

In the same means, the $\upsilon(\hat{x})$ field along B_α can not be expressed by $\upsilon(x)$; so there is

$$\int_{B_\alpha} L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \neq \upsilon(x) \int_{B_\alpha} L(\hat{x}, x)\, dB(\hat{x}) \tag{52}$$

The nonequal sign stems from the loss of free terms, $-c\upsilon(x) - du'(x)$. This finding will be very important to the order analysis in the following derivation for regularized version.

A REGULARIZED VERSION OF DUAL BOUNDARY INTEGRAL EQUATIONS AT A CORNER

Although the dual integral equations have been derived, the *C.P.V.* and *H.P.V.* concepts must be defined. In order to reduce the order of singularity, one regularized version of dual integral equations derived by employing order analysis is furnished. By joining together the unregularized version, one version of dual boundary integral formulations for a boundary corner point is summarized as follows:

$$\alpha u(x) = C.P.V. \int_B V(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x})$$
$$- R.P.V. \int_B U(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{53}$$

$$\alpha \upsilon^-(x) + \sin(\alpha)\upsilon^+(x) = H.P.V. \int_B M(\hat{x}, x)\, u(\hat{x})\, dB(\hat{x})$$
$$- C.P.V. \int_B L(\hat{x}, x)\, \upsilon(\hat{x})\, dB(\hat{x}) \tag{54}$$

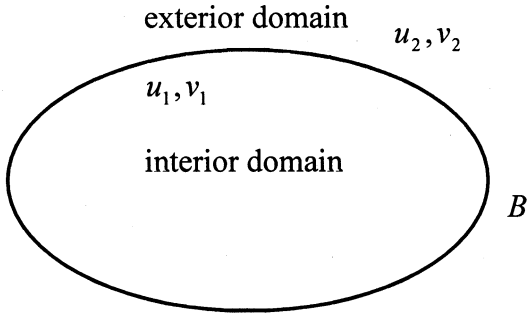


Fig. 6. Interior domain and exterior domain.

Regularized form with respect to u:

Based on the non-singular boundary integral equation derived by Koo *et al.* [21], the strong singularity can be removed by the following procedure.

$$0 = R.P.V. \int_B V(\hat{x}, x)[u(\hat{x}) - \hat{u}(x)]\, dB(\hat{x})$$
$$- R.P.V. \int_B U(\hat{x}, x)\, \upsilon(\hat{x})\, t(\hat{x})\, dB(\hat{x}) \tag{55}$$

$$\alpha \upsilon^-(x) + \sin(\alpha)\upsilon^+(x) = C.P.V. \int_B M(\hat{x}, x)[u(\hat{x})$$
$$- \hat{u}(x)]\, dB(\hat{x}) - C.P.V. \int_B L(\hat{x}, x)\, \upsilon(\hat{x})\, t(\hat{x})\, dB(\hat{x}) \tag{56}$$

where $\hat{u}(x) = \overline{u}\, e^{i[k(x-\eta)\bullet a]}$ is the one-dimensional wave equation satisfying the Helmholtz equation, in which η and a refer to an arbitrary reference point and the unit vector of wave propagation, respectively, the amplitude $\overline{u}$ is a fixed value, and k is wave number. For a point $\hat{x}$ on D near boundary, we can select the fixed value $\overline{u}$ as $u(\eta)$ and the point η as the source point $\hat{x}$. The density function has the following property

$$u(\hat{x}) - \hat{u}(x) = O(r)$$

where it is noted that the strong singularity in Eq. (55) can be removed, and only weak singularity is present.

On the authority of the one version of expression, it is discovered that the lower the order of regularization applied, the more free terms will be present. By employing Eq. (51), regularized form can be reformulated as unregularized form. It is found that no distinguished difference is made for the boundary point and the interior point in Eq. (55) since only the reference potential is subtracted.

DISCUSSIONS ON THE LAPLACE AND HELMHOLTZ EQUATIONS AT A CORNER

The wave equation is

$$-\nabla^2 u(x,t)+\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}=Q(x,t) \tag{57}$$

where D is the domain of interest, x is the domain point, u is the velocity potential, t is time and $Q(x, t)$ is a source term. By applying the Fourier transform to Eq. (57), we obtain the Helmholtz equation

$$\nabla^2\overline{u}+k^2\overline{u}=\overline{Q}(x) \tag{58}$$

where the bar over the symbol means the complex transform, and k is the wave number defined by $k = \omega/c$, ω is frequency and c is wave velocity. In the case of static problem, frequency ω is very small. Thus k is also very small and can be negligible. When both the value of k approaches zero and no source term exists, the Helmholtz equation (58) reduces to

$$\nabla^2\overline{u}=0 \tag{59}$$

Eq. (59) is the Laplace equation. Obviously, the Laplace equation is a special case of the Helmholtz equation [22].

A dual integral formulation for the Laplace equations problem at a corner was derived by Chen and Hong [15] and in this paper by using the contour approach surrounding the singularity, respectively. It is found that the six kernels to both the Laplace and Helmholtz equations are, respectively, quite different. However, for the case of dual integral formulation of BEM at a corner, the free terms of the six kernel functions in the dual boundary integral equations at a corner are all same to both the Laplace and Helmholtz equations when the value of k approaches zero.

CONCLUSIONS

The dual boundary integral equations for the two-dimensional Helmholtz equation at a corner using contour approach around singularity has been derived in this paper. The free terms of the six kernel functions in the dual integral equation for the Helmholtz equation at a corner have been examined. Both the Laplace and Helmholtz equation have the different kernel functions while they have the same free terms. It is worth to point out that the Laplace equation is a special case of the Helmholtz equation when the value of wave number approaches zero.

APPENDIX

The indirect method can represent the solution in terms of single layer or double layer sources only, on the boundary. Let u_1 be the velocity-potential occupying a certain region as shown in Fig. 6, and let u_2 now denote the velocity-potential through the rest of infinite space. Then, if the field point x be internal to the first region, and therefore external to be the complement of the first region. For a smooth boundary, we have

$$2\pi u_1(x)=\int_B V^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x})-\int_B U^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x}) \tag{60}$$

$$0=\int_B V^e(\hat{x},x)\,u_2(\hat{x})\,dB(\hat{x})-\int_B U^e(\hat{x},x)\,v_2(\hat{x})\,dB(\hat{x}) \tag{61}$$

$$2\pi v_1(x)=\int_B M^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x})-\int_B L^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x}) \tag{62}$$

$$0=\int_B M^e(\hat{x},x)\,u_2(\hat{x})\,dB(\hat{x})-\int_B L^e(\hat{x},x)\,v_2(\hat{x})\,dB(\hat{x}) \tag{63}$$

where B denotes the real boundary and the superscripts i and e denote the interior and exterior, respectively. By employing Eqs. (18)~(20) and Eq.(23), we have

$$\int_B U^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x})=R.P.V.\int_B U^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x}) \tag{64}$$

$$\int_B V^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x})=\pi u_1 + C.P.V.\int_B V^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x}) \tag{65}$$

$$\int_B L^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x})=-\frac{1}{2}\pi v_1 + C.P.V.\int_B L^i(\hat{x},x)\,v_1(\hat{x})\,dB(\hat{x}) \tag{66}$$

$$\int_B M^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x})=\frac{1}{2}\pi v_1 + H.P.V.\int_B M^i(\hat{x},x)\,u_1(\hat{x})\,dB(\hat{x}) \tag{67}$$

In the reference [23], the relationship of the influence matrix between the interior and exterior domain can be found as shown below:

$$U^i_{pq}=U^e_{pq}, \tag{68}$$

$$M^i_{pq}=M^e_{pq}, \tag{69}$$

$$V^i_{pq}=\begin{cases}-V^e_{pq}, & if\ p\neq q,\\ V^e_{pq}, & if\ p=q,\end{cases} \tag{70}$$

$$L^i_{pq}=\begin{cases}-L^e_{pq}, & if\ p\neq q,\\ L^e_{pq}, & if\ p=q,\end{cases} \tag{71}$$

for the exterior domain since the direction of normal vector of boundary point $\hat{x}$ is inverse of interior domain,

we have

$$\int_B U^e(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) = R.P.V. \int_B U^i(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) \tag{72}$$

$$\int_B V^e(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) = -\pi u_2 + C.P.V. \int_B V^e(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) = -\pi u_2 - C.P.V. \int_B V^i(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) \tag{73}$$

$$\int_B L^e(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) = \frac{1}{2}\pi\upsilon_2 + C.P.V. \int_B L^e(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) = \frac{1}{2}\pi\upsilon_2 - C.P.V. \int_B L^i(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) \tag{74}$$

$$\int_B M^e(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) = -\frac{1}{2}\pi\upsilon_2 + H.P.V. \int_B M^e(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) = -\frac{1}{2}\pi\upsilon_2 - H.P.V. \int_B M^i(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) \tag{75}$$

Substituting Eqs. (72) and (73) into Eq. (61), and Eqs. (74) and (75) into Eq. (63), we have

$$0 = -\pi u_2(x) - C.P.V. \int_B V^i(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) - R.P.V. \int_B U^i(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) \tag{76}$$

$$0 = -\pi\upsilon_2(x) + H.P.V. \int_B M^i(\hat{x}, x)\, u_2(\hat{x})\, dB(\hat{x}) - C.P.V. \int_B L^i(\hat{x}, x)\, \upsilon_2(\hat{x})\, dB(\hat{x}) \tag{77}$$

Adding Eq. (60) into Eq. (76), we have

$$2\pi u_1 = \pi(u_1 - u_2) + C.P.V. \int_B V^i(\hat{x}, x)\, (u_1 - u_2)\, dB(\hat{x}) - R.P.V. \int_B U^i(\hat{x}, x)\, (\upsilon_1 + \upsilon_2)\, dB(\hat{x}) = \pi\psi + C.P.V. \int_B V^i(\hat{x}, x)\, \psi(\hat{x})\, dB(\hat{x}) - R.P.V. \int_B U^i(\hat{x}, x)\, \phi(\hat{x})\, dB(\hat{x}) \tag{78}$$

where $\phi = \upsilon_1 + \upsilon_2$ and $\psi = u_1 - u_2$ [24]. By subtracting Eq. (62) from Eq. (77), we have

$$2\pi\upsilon_1 = \pi(\upsilon_1 + \upsilon_2) + H.P.V. \int_B M^i(\hat{x}, x)\, (u_1 - u_2)\, dB(\hat{x}) - C.P.V. \int_B L^i(\hat{x}, x)\, (\upsilon_1 + \upsilon_2)\, dB(\hat{x}) = \pi\phi + H.P.V. \int_B M^i(\hat{x}, x)\, \psi(\hat{x})\, dB(\hat{x}) - C.P.V. \int_B L^i(\hat{x}, x)\, \phi(\hat{x})\, dB(\hat{x}) \tag{79}$$

Let us make $u_1 = u_2$ on the boundary, that is $\psi = 0$ on the boundary. The tangential velocities on the two sides of the boundary are then continous, whereas the normal velocities are discontinuous, so that

$$2\pi u_1 = -R.P.V. \int_B U^i(\hat{x}, x)\, \phi(\hat{x})\, dB(\hat{x}) \tag{80}$$

$$2\pi\upsilon_1 = \pi\phi - C.P.V. \int_B L^i(\hat{x}, x)\, \phi(\hat{x})\, dB(\hat{x}) \tag{81}$$

Eqs. (80) and (81) shows the boundary integral equations using the single layer density, $-\phi(\hat{x})$. Secondly, we may suppose that $\phi = 0$ along the boundary. This gives continuous normal velocity, while discontinuous tangential velocity, along the original boundary. We have

$$2\pi u_1 = \pi\psi + C.P.V. \int_B V^i(\hat{x}, x)\, \psi(\hat{x})\, dB(\hat{x}) \tag{82}$$

$$2\pi\upsilon_1 = H.P.V. \int_B M^i(\hat{x}, x)\, \psi(\hat{x})\, dB(\hat{x}) \tag{83}$$

Eqs. (82) and (83) are the boundary integral equations obtained from the double layer density, $\psi(\hat{x})$. The final results are the same to those deriving to the limiting process, where the free term is descended from V kernel ($\pi\psi$) and L kernel ($\pi\phi$) only. But from the deriving process, the free terms are different from those of the direct method. Since ϕ and ψ are defined on boundary, so the contour integration surrounding singularity method can not be employed. The main reason is that the boundary densities ϕ and ψ can not be expanded on the surrounding path.

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利用繞道奇異點的方法探討荷姆茲方程在角點的對偶邊界積分方程

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摘 要

本文探討經由推到邊界及繞道奇異點的方法導出在角點荷姆茲方程的對偶積分表示式。結果發現，利用環繞邊界法它的跳躍項是由L及M核函數經積分各貢獻一半，這與經由極限過程所得自由項完全由L核函數貢獻有所不同。在超強奇異積分方程中阿達馬主值的觀念在此從雙層勢能的法向微分推廣到切向微分以便於與傳統的定義對照。同時對於荷姆茲方程對偶邊界積分方程式中的六個核函數在角點的自由項也予以檢驗。荷姆茲方程的核函數與拉普拉斯方程的核函數完全不同，但是它們的自由項卻相同。值得一提的是拉普拉斯方程僅爲荷姆茲方程當波數k趨近於零時的一個特例而已。