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Jong-Dar Yau

Assistant Professor, Department of Architecture & Bldg. Technology, Tamkang University, Taipei, Taiwan, R.O.C.

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# RESONANCE OF CONTINUOUS BRIDGES DUE TO HIGH SPEED TRAINS

Jong-Dar Yau\*

Keywords: Impact response, continuous bridges, high-speed train, resonance.

## ABSTRACT

In this paper, the vibration of continuous bridges subjected to the passage of high-speed trains is studied. Only the bridges with uniform spans are considered and the train moving over a bridge is modeled as a series of moving loads. According to the resonant formula of simple beams to moving loads, the effect of multiple resonant peaks for continuous bridges subjected to high-speed trains can be determined by analogy method. Based on the finite element method, the effect of the number of spans on the impact response of the continuous beams is presented. The results show that the more the number of spans of the continuous beam, the smaller the impact response is for the displacement.

## INTRODUCTION

The vibration behaviors of rail bridges under the passage of moving vehicles have been studied by many researchers. Recently, the author and Yang have presented a series of papers on the impact response of bridges to high-speed trains and on the interaction of vehicle/bridge systems [1-6]. By an analytical approach, the resonance phenomenon of simply supported railway bridges subjected to a series of moving loads has been investigated, while vehicle-bridge interaction (VBI) elements of various complexities were developed for simulating the interaction behavior between the bridge and the vehicles moving over it, aimed particularly at high-speed railway systems.

A review of the literature indicates that most of the previous works have been concentrated on the dynamic response of simply supported bridges. In comparison, relatively few studies have been conducted on the dynamic response of continuous bridges to moving loads. As far as the continuous bridges are concerned, the impact response of highway bridges has been studied by

Yang *et al.* [2] using the dynamic condensation method, and the dynamic interaction between the bridge and moving vehicles for high-speed railways by Yau *et al.* [5] using the VBI elements. Cheung *et al.* [8] used the modified beam vibration functions to investigate the response of multi-span non-uniform bridges under moving vehicles and trains.

The objective of this article is to study the impact response of multi-span continuous bridges to high-speed trains. By the finite element method, a continuous bridge is modeled as a series of beam elements and a train as a sequence of moving loads. The results show that the increase in the number of spans of the continuous beams will result in the increase of resonant peaks and reduction of the impact response.

## RESONANCE ANALYSIS

As shown in Fig. 1, by modeling a bridge as a Bernoulli-Euler beam, a simple beam with length  $L$  and uniform cross section is considered. The train moving over the beam at speed  $v$  is modeled as a sequence of equidistant moving loads. The interval between two adjacent moving loads is  $d$  and the weight of each moving load is  $p$ . By neglecting the damping effect of beam structures, the equation of motion for the beam traveled by the moving loads can be written as [1, 6]

$$m\ddot{u} + EIU'''' = p \sum_{k=1}^N \delta[x - v(t - t_k)] \times [H(t - t_k) - H(t - t_k - L/v)] \quad (1)$$

where a prime denotes derivative with respect to the coordinate  $x$ , an over-dot denotes derivative with respect to time  $t$ ,  $m$  = the mass per unit length of the beam,  $u(x, t)$  = vertical displacement,  $E$  = elastic modulus,  $I$  = moment of inertia of the beam,  $\delta$  = Dirac's delta function,  $H(t)$  = unit step function,  $N$  = total number of moving loads, and  $t_k = (k - 1)d/v$  = arriving time of the  $k$ th load at the beam. The closed form solution of the dynamic response  $u(x, t)$  of the beam traveled by the moving loads with equidistant is expressed as [1, 6]:

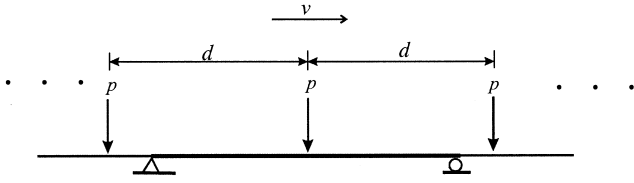


Fig. 1. Model of simple beams and moving loads.

$$u(x, t) = \Delta_{st} \sin\left(\frac{\pi x}{L}\right) \times [Q_1(v, t)H(t - t_N) + Q_2(v, t)H(t - t_{N-1} - L/v)] \quad (2)$$

where  $\Delta_{st} = 2pL^3/(\pi^4 EI) \approx pL^3/(48EI)$  = the maximum static deflection of the corresponding simple beam,

$$Q_1(v, t) = \frac{\sin \Omega(t - t_N) - S \sin \omega_0(t - t_N)}{1 - S^2}$$

$$Q_2(v, t) = -\frac{2S \cos\left(\frac{\pi}{2S}\right)}{1 - S^2} \times \left[ \sin \omega_0\left(t - \frac{L}{2v}\right) + \sin \omega_0\left(t - \frac{t_N + L/v}{2}\right) \times \frac{\sin \omega_0[(t_N - d/v)/2]}{\sin(\omega_0 d/2v)} \right]$$

$$S = \frac{\Omega}{\omega_0} = \frac{\pi v/L}{\omega_0}$$

$$\omega_0 = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{m}} \quad (3a-d)$$

$S$  = the speed parameter, which represents the ratio of the driving frequency  $\Omega$  to the fundamental frequency  $\omega_0$  of the beam. From equation (2), it can be seen that the term  $Q_1(v, t)$  is associated with  $H(t - t_N)$ , which represents the dynamic response induced by the  $N$ th moving load acting on the beam, and the term  $Q_2(v, t)$  is associated with  $H(t - t_{N-1} - L/v)$ , which represents the residual response induced by the  $N - 1$  moving loads that have passed the beam.

From equations (3b), it can be seen that the response reaches a maximum when the denominator  $\sin(\omega d/2v)$  equals zero, or  $S = dl/(2nL)|_{n=1, 2, 3, \dots}$ . This is exactly the condition for *resonance* to occur. Correspondingly, when  $n$  is set to be 1, the resonance is called the main resonant condition and the resonant speed is denoted as  $v_{res}$ . By the L'Hospital rule, the dynamic response factor for resonance  $Q_2(v_{res}, t)$  in equation (3b) can be manipulated to yield [1, 6]:

$$Q_2(v_{res}, t) = 2(N - 1) \left[ -S_{res} \frac{\cos(\pi/2S_{res})}{1 - S_{res}^2} \right] \times \sin \omega \left( t - \frac{L}{2v_{res}} \right) \quad (4)$$

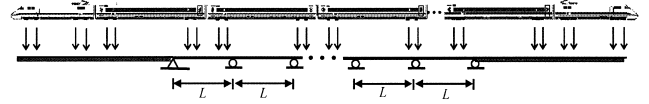


Fig. 2. Models of Continuous Bridges and High Speed Trains.

where the subscript *res* means resonance. The preceding equation indicates that under the condition of resonance, larger response will be induced on the beam when there are more loads passing the beam, as implied by  $(N - 1)$ .

From above derivation, it is concluded that the main resonance speed  $v_{res}$  of vehicles occurs if the excitation circular-frequency  $v/d$  of the vehicle meets the fundamental circular-frequency  $\omega_0/2\pi$  of the beam, that is,  $v_{res} = (\omega_0 d)/(2\pi)$ . Further, when the excitation circular-frequency caused by the running speed of the train is approaching to one of the circular-frequency  $\omega/2\pi$  of the bridge, the resonance will be also excited, that is,  $v_{res} = (\omega d)/(2\pi)$ . It must be emphasized that the resonant condition for bridges subjected to the vehicle moving at high speeds has been verified in many related papers [9-13].

## METHOD OF ANALYSIS

Multi-span continuous bridges with constant span lengths are one type of structures commonly used in practice, because they can be constructed in a rather systematic way. As shown in Fig. 2, a multi-span continuous bridge with uniform spans and simple supports is considered in this study. By modeling a continuous bridge as a beam-like structure, and a train with constant speed  $v$  as a sequence of moving loads, the dynamic response of the beam to moving loads is studied using the finite element method. In the present study, the most commonly used beam element with 6 degrees of freedom at each node, as shown in Fig. 3, is used to represent each segment of the beam, for which the elastic stiffness matrix  $[k_b]$  and the mass matrix  $[m]$  are constructed using methods available in textbooks of structural dynamics. In particular, the concept of consistent mass, rather than lumped mass, is used to construct the mass matrix  $[m]$  for its better accuracy, as required in the analysis for multi-span continuous beams, and the mass matrix  $[m]$  is expressed as shown in appendix A.

As shown in Fig. 4, a continuous beam with uniform cross sections is divided into a series of beam elements of equal length. Based on the eigen-value analysis, the natural frequencies  $\omega$  solved for the continuous beams with different numbers of spans have to

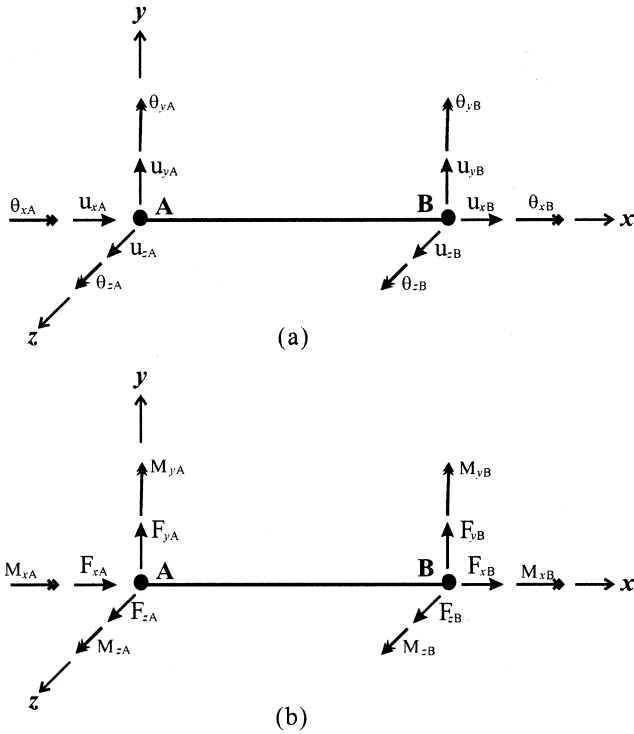


Fig. 3. Beam Element.



Fig. 4. Finite Element Model of Continuous Beams.

been plotted in Fig. 5, with respect to the frequency  $\omega_0$  of the corresponding simple beam that has a length equal to the span length of the continuous beams. As can be seen from Fig. 5, the *fundamental* frequencies of the uniformly multi-span continuous beams remain the same regardless of the variation in the number of spans. However, the increase in the number of spans does make the distribution of frequencies much denser. Such a phenomenon implies that the excitation energy (through the wheel contact points) to one span of a continuous beam with more spans can be transferred more quickly the neighboring spans.

**IMPACT FACTOR AND SPEED PARAMETER**

In design practice, the impact factor  $I$  is used to account for the amplification effect of the bridge due to the passage of moving vehicles through increase of the design forces and stresses. The impact factor is defined as follows [7]:

$$I = \frac{R_d(x) - R_s(x)}{R_s(x)} \tag{6}$$

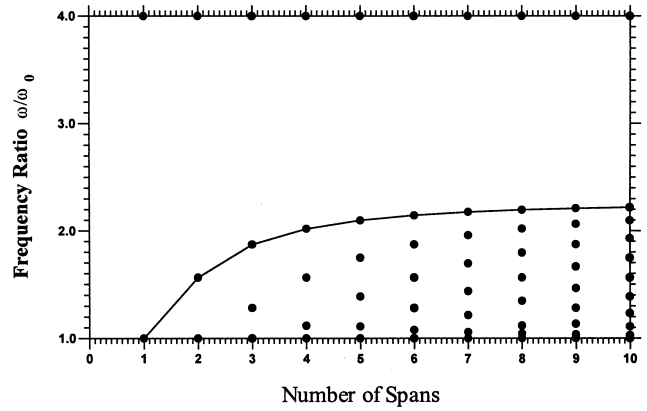


Fig. 5. Frequency Distribution of Multi-span Continuous Beams.

**Table 1. Frequency Distribution of Uniform Continuous Beams**

	Frequencies: $\omega$ (rad/s)					
1-span	<b>19.2</b>	<b>76.9</b>				
3-span	<b>19.2</b>	24.6	36.0	<b>76.9</b>		
5-span	<b>19.2</b>	21.3	26.7	33.6	40.3	<b>76.9</b>
7-span	<b>19.2</b>	20.3	23.3	27.6	37.6	41.8 <b>76.9</b>

where  $R_d(x)$  and  $R_s(x)$ , respectively, denote the maximum dynamic and static responses of the beam at position  $x$  due to the action of the moving loads. It has been indicated that for simple and continuous beams subjected to the moving action of a single truck, the impact factor may be linearly related to a dimensionless speed parameter  $S$ , defined as the ratio of the excitation frequency of the moving vehicle, i.e.,  $\pi v/L_c$ , with  $v$  denoting the vehicle speed and  $L_c$  the characteristic length of the bridge, to the fundamental frequency  $\omega_0$  of the bridge, that is,  $S = (\pi v)/(\omega_0 L_c)$ . Here, the characteristic length, as shown in Fig. 6, is defined as the length of two inflection points of the fundamental mode of continuous bridges [7]. Obviously, one feature with the present definitions for both the impact factor  $I$  and speed parameter  $S$  is that both of them are dimensionless.

**IMPACT RESPONSE ANALYSIS**

To investigate the impact response of continuous beams to moving loads at high speeds, four types of beams with different spans, i.e., 1, 3, 5, and 7-span, are considered, which are made of pre-stressed concrete with flexural rigidity  $EI = 8.92 \times 10^3$  KN-m<sup>2</sup>, mass per unit length  $m = 37.6$ t, and length  $L = 50$ m. A damping ratio of 2.5% is assumed for these continuous beams. The corresponding dynamic properties of these beams are listed in Table 1. It is found that the increase in the

**Table 2. Train-Loadings of SKS-Model**

	Power-Cars	Trailer-Cars							
Distance of Wheel Loads (m)	0.0 2.50	25.00	27.50	42.50	45.00	50.00	52.50	67.50	70.00
	17.50 20.00	75.00	77.50	92.50	95.00	100.0	102.5	117.5	120.0
		125.0	127.5	142.5	145.0	150.0	152.5	167.5	170.0
		175.0	177.5	192.5	195.0	200.0	202.5	217.5	220.0
		225.0	227.5	242.5	245.0	250.0	252.5	267.5	270.0
		275.0	277.5	292.5	295.0	300.0	302.5	317.5	320.0
		325.0	327.5	342.5	345.0	350.0	352.5	367.5	370.0
		375.0 377.5							
	392.5 395.0								
Weight (kN)	150	138							

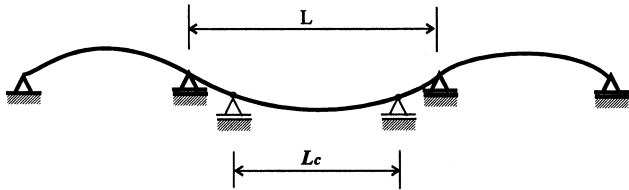


Fig. 6. Characteristic Length of Multi-span Continuous Beams.

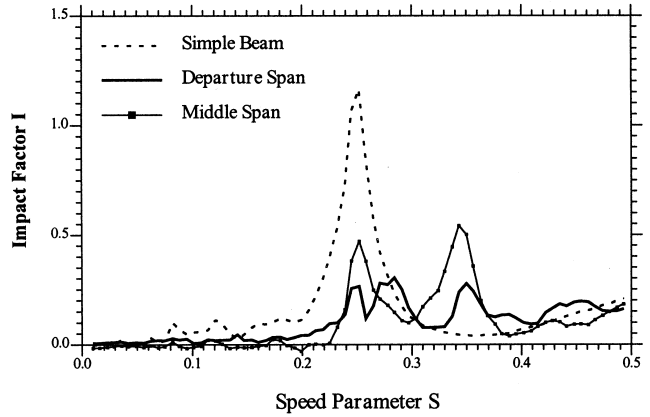


Fig. 8. Impact Response of 5-span Continuous Beams.

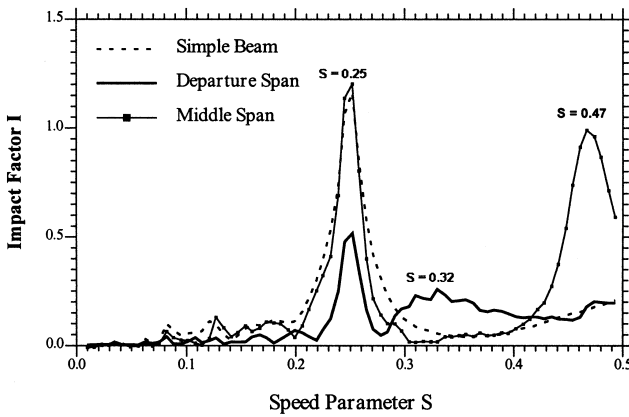


Fig. 7. Impact Response of 3-span Continuous Beams.

**Multiple Resonant Peaks of Continuous Beams**

For uniformly multiple-span continuous bridges, the impact response of the middle span is generally larger than that of the other two side spans [5]. In the present study, only the impact responses of the middle span and the departure span of continuous beams will be considered. The impact factor  $I$  solved for the midpoint of the middle span and the departure span of the continuous beams have been plotted against the speed parameter  $S$  (referred as  $I$ - $S$  plot) in Figs. 7-9. As can be seen, there exist multiple resonant peaks for the impact response of the departure span and the middle span of the continuous beam. This is mainly due to coincidence of some of the excitation frequencies implied by the moving wheel loads at different speeds with the fundamental or higher frequencies of the continuous beam. According to the analytical formulas as shown in Eqs. (2), (3), and (4), the resonant response for a simple beam subjected to a series of wheel loads will occur at  $S_{res} =$

number of spans does make the distribution of frequencies much denser. Because the continuous beams considered here are of uniform span lengths, the characteristic length  $L_c$  of the beam is equal to the span length  $L$ . In addition, the Shinkansen (SKS) train-loading model, as shown in Table 2, used by T. Y. Lin Taiwan [14] is adopted in the present study, which has an average car length  $d$  of 25 m.

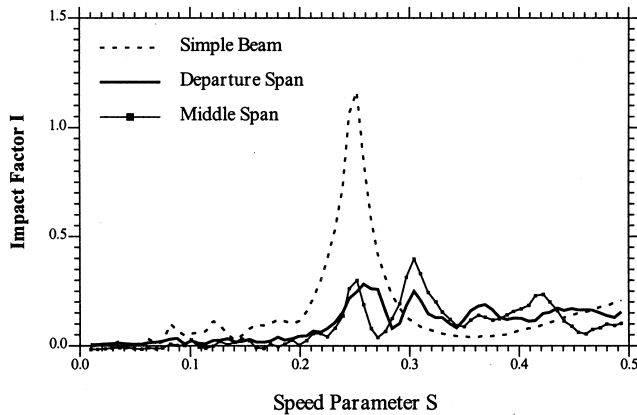


Fig. 9. Impact Response of 7-span Continuous Beams.

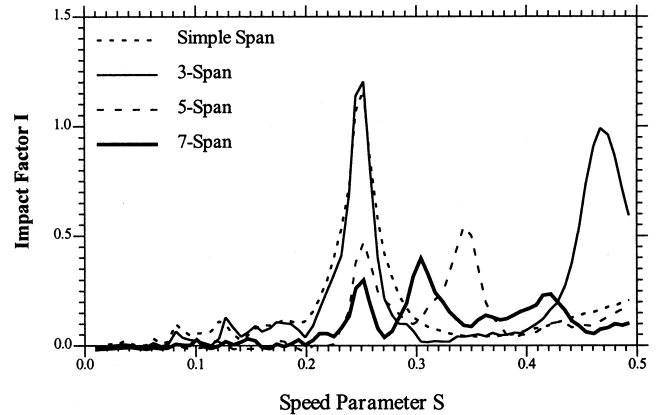


Fig. 10. Comparison of Impact Response of Mid-Span for Continuous Beams.

**Table 3. Resonant Speeds of Continuous Beams**

	Speed: $v_{res}$ (km/h) [ $S_{res} \leq 0.5$ ]			
1-span	<b>275</b> [0.25]			
3-span	<b>275</b> [0.25]	352 [0.32]	517 [0.47]	
5-span	<b>275</b> [0.25]	305 [0.28]	382 [0.35]	481 [0.44]
7-span	<b>275</b> [0.25]	291 [0.26]	334 [0.30]	395 [0.48]

$d/2L$ , where  $d$  = the wheel load interval. For the present case, the first resonance speed parameter for all of these continuous bridges is  $S_{res,1} = 25/(2 \times 50) = 0.25$ , or equivalently  $v = 275$  km/h. The second resonance speed parameter for 3-span continuous beams is  $S_{res,2} = S_{res,1} \omega_2/\omega_1 = 0.25 \times 1.28 = 0.32$ , or equivalently  $v = 352$  km/h, and the third resonance speed parameter is  $S_{res,3} = S_{res,1} \omega_3/\omega_1 = 0.25 \times 1.88 = 0.47$ , or equivalently  $v = 517$  km/h. It is noted that the second resonant peak at  $S_{res,2} = 0.32$  does not occur for the mid-span of the 3-span continuous beams, as shown in Fig. 7, the reason is that the inflection point of the second vibration mode is located at the midpoint of this span. Therefore, there is no contribution of the second mode of the 3-span continuous beams to the impact response of the mid-span of the beams. Evidently, all of the resonance speeds for continuous beams with different spans can be identified from the analytical formulas as shown in Table 3.

### Effect of Number of Spans

The impact factor  $I$  solved for the midpoint of mid-span for each continuous beam has been plotted against the speed parameter  $S$  in Fig. 10. As can be seen, the

impact response of the 7-span continuous beams is smaller than that of the other continuous beams. This is partly due to the fact that increase in the number of spans for the continuous beams allows the vibration energy excited by the moving loads to be transmitted more easily to the neighboring spans. The other reason is that for continuous beams with more spans, the restraining effect of the supports on the displacement appears to be higher. From Fig. 9, it can be observed as well that for continuous beams of the same characteristic length  $L_c$ , the speed parameter  $S$  (and therefore the vehicle speed  $v$ ) for the first resonance to occur is the same, regardless of the number of spans. Furthermore, the main resonant peaks of the impact response of continuous beams will become denser as well as the increase of the number of spans of the beams.

### Effect of Car/Span Length Ratio

For the present purposes, 5-span continuous beams of different span lengths made of pre-stressed concrete are considered. By letting each of the beams traveled by the moving loads, the impact factor  $I$  for the midpoint displacement of the arrival-span, mid-span, and departure-span of the beams has been plotted against the speed parameter  $S$  and the car/span length ratio  $d/L$  as shown in Figs. 11-13. It is observed that the impact factor at mid-span is larger than the other two cases. Meanwhile, when the ratio  $d/L$  approaches the value of  $2/3$ , the impact response becomes smaller than other length ratios, largely due to the fact that the cancellation condition for the impact response of the beams is met. Such an observation is consistent with that observed for simple beams subjected to high-speed trains in previous studies [1, 6].

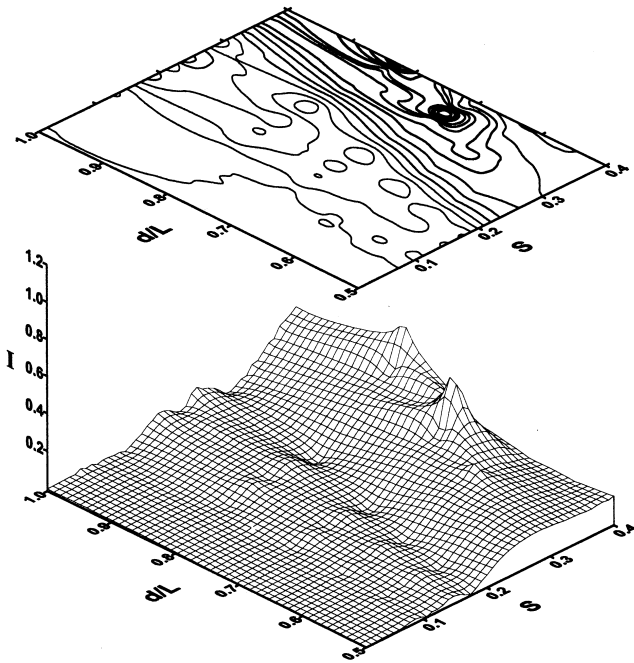


Fig. 11. Impact Response of the Arrival-Span.

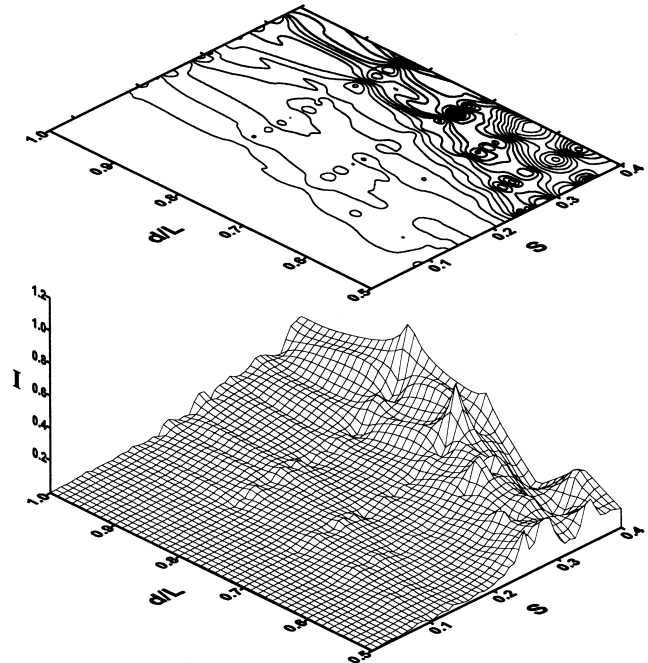


Fig. 13. Impact Response of the Departure-Span.

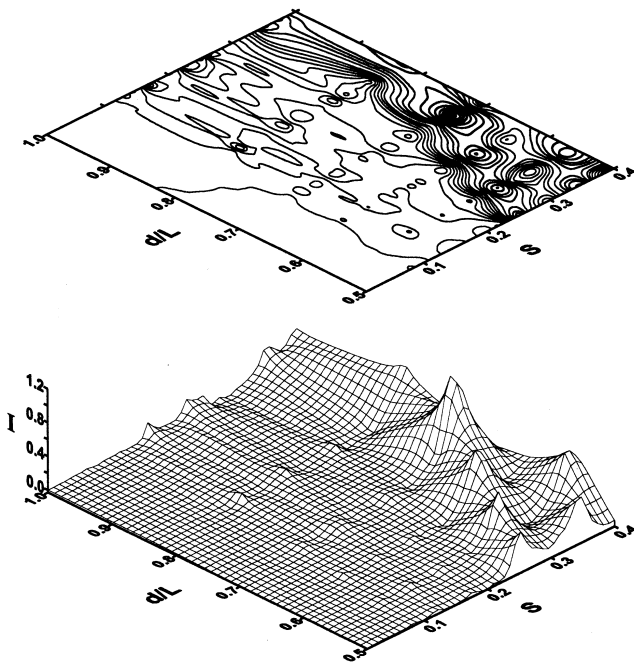


Fig. 12. Impact Response of the Mid-Span.

### CONCLUDING REMARKS

By modeling a train as a sequence of moving loads and the bridges as a series of beams of constant spans, the frequency distribution and impact response of the

beams are investigated. Based on the example studied, the following remarks can be made: 1) The frequency distribution of the continuous beams is related to the number of spans, that is, the more the number of spans for the beams, the denser the frequency distribution is. 2) By increasing the number of spans of the continuous beams, the main resonant peaks of continuous beams will increase and the impact response of the displacement of the beams is reduced. 3) When the car/span length ratio  $d/L$  is near the cancellation condition, i.e.,  $2/3$ , the impact response of the beams becomes minimal.

### ACKNOWLEDGEMENTS

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