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INVERSE TIME TRANSLATIONAL SYMMETRY OF THE WAVE EQUATION FOR BLATZ-KO CYLINDERS

Hin-Chi Lei* and Sheng-Wei Chen*

Key words: Blatz-Ko material, symmetries, nonlinear elasticity.

ABSTRACT

The symmetries for the wave equation governing the radial deformations of circular cylinders composed of Blatz-Ko materials are studied. It is found that the wave equation possesses a special symmetry --- the inverse time translational symmetry (ITTS) named by us. We find that this special symmetry is not possessed by the wave equations for cylinders composed of other two compressible elastic materials. However, it appears again when we study the invariant properties of the wave equation governing radially deformed Blatz-Ko spheres and the one governing Blatz-Ko blocks in uniaxial tensile motion. We therefore infer that ITTS is a special property inherited by many dynamical problems associated with Blatz-Ko materials. It is also found that this special symmetry can help us to construct correspondence between different initial-boundary value problems of the wave equation for Blatz-Ko cylinders. Making use of this correspondence we can obtain non-trivial solutions from known solutions. Also, if we need to perform an experiment for a Blatz-Ko cylinder for a long time period, we may use this correspondence to design a substitute experiment that only runs in a short time period.

INTRODUCTION

As void growth is an important subject in the studies of the behaviors of solids [9, 16, 21, 22, 28], many researches about the radially symmetric deformations of elastic solids had been done after the publication of a fundamental paper by John Ball in 1982 [3]. An extensive review of these researches was given by Horgan and Polignone [15]. And in this article we try to carry out some investigations on the invariant properties of the wave equation governing the dynamic radial deformations of a typical cross section of a circular cylinder composed of Blatz-Ko material [5]. We adopt the Blatz-Ko material model because it was derived from experimental data of real materials. It had been found to be important in modeling the fuel of jet engine and thus many researches about it had been done [1, 4, 14, 16, 17, 19]. In recent years the material model was also applied to describe vascular behavior [8, 30].

The Lie groups of the wave equation contain some coordinate transformations and the form of the wave equation will be kept invariant when these coordinate transformations are applied to it [6, 23, 24]. Previously, the invariant properties of many governing equations for *incompressible* nonlinear elastic materials had been investigated [12, 18, 20]. And here we focus on the invariant properties of the wave equation for Blatz-Ko cylinders which are elastic and *compressible*. It is found that the wave equation possesses a special symmetry, the inverse time translational symmetry (ITTS) named by us. The form of ITTS is so special that we want to examine it more deeply.

We want to clarify three things. First, we want to know whether if ITTS frequently appears in the invariant properties of the wave equations for cylinders composed of other compressible elastic materials. We thus analyze two cylinders composed of the Shang-Cheng material [27] and the generalized Varga material [7, 13] respectively and find that none of their governing equations possesses ITTS. This suggests that ITTS does not frequently occur in the invariant properties of the wave equations for cylinders composed of materials other than the Blatz-Ko material. The second thing we want to clarify is whether if the governing equations for other objects composed of Blatz-Ko materials also possess ITTS. We thus investigate the wave equation governing radially deformed Blatz-Ko spheres and the one governing uniaxial tensile blocks composed of Blatz-Ko materials. And it is found that both of them possess ITTS. This suggests that ITTS might appear in many dynamical problems for Blatz-Ko materials. The third thing we want to study is the usefulness of ITTS. It is found that ITTS can be applied to establish the correspondence between different initial-boundary value problems of the wave equation for Blatz-Ko cylinders. The correspondence is useful since it can help us to obtain non-trivial solutions from known solutions. Also, if we need to perform an experiment for a Blatz-Ko

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cylinder for a long time period, we may use the above correspondence to design a substitute experiment that only runs in a short time period. It should be pointed out that, using other groups such as scaling groups or time translation groups we can also obtain new solutions from known solutions. However, the new solutions obtained by using these groups are trivial since they are different with the old solutions only by simple time translation or coordinate scaling. It should also be noted that correspondence principles linking boundary value problems have been the subject of much literature in linear elasticity [10, 11, 25, 26, 29]. In nonlinear elastostatics there is the famous Adkins's duality principle connecting boundary value problems [2]. And the correspondence between initial-boundary value problems reported in this paper is applied to connect nonlinear elastody-namic problems.

WAVE EQUATION FOR BLATZ-KO CYLINDERS

We consider a cylinder with circular cross sections. Assuming that the cylinder is in plane-strain condition we only investigate the deformations of one of its cross sections. If radially symmetric deformations are considered the Eulerian coordinates of a material point in the cross section can be denoted as $r = r(R, t)$ and $\theta = \Theta$. Here R and Θ are the Lagrangian coordinates of the material point. The deformation gradient can be expressed as

$$\mathbf{F} = \begin{pmatrix} F_{rR} & F_{r\Theta} \\ F_{\theta R} & F_{\theta\Theta} \end{pmatrix} = \begin{pmatrix} dr/dR & 0 \\ 0 & r/R \end{pmatrix}. \tag{1}$$

We can denote $\lambda_1 = F_{rR} = dr/dR$ and $\lambda_2 = F_{\theta\Theta} = r/R$ as the principal stretches. If $W = W(\lambda_1, \lambda_2)$ is the strain energy density function, the first Piola-Kirchhoff stress tensor can be computed by

$$\mathbf{S} = \begin{pmatrix} S_{rR} & S_{r\Theta} \\ S_{\theta R} & S_{\theta\Theta} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}, \tag{2}$$

where $W_1 = \partial W/\partial\lambda_1$ and $W_2 = \partial W/\partial\lambda_2$. The Cauchy stress tensor $\mathbf{T}$ is linked to $\mathbf{S}$ by $\mathbf{S} = J_F\mathbf{T}\mathbf{F}^{-T}$ with $J_F = \mathbf{det\ (F)}$ and it can be computed by

$$\mathbf{T} = \begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} W_1/\lambda_2 & 0 \\ 0 & W_2/\lambda_1 \end{pmatrix}. \tag{3}$$

The stress tensor $\mathbf{S}$ satisfies the momentum-balance equation

$$\mathbf{Div\,S} = \rho_0 \frac{\partial^2 r}{\partial t^2}, \tag{4}$$

where ρ_0 is the constant mass density, **Div** denotes the divergence operator in the Lagrangian coordinates (R, Θ) and t is the time variable. Due to the radial symmetry the above equation can be written as

$$\frac{\partial S_{rR}}{\partial R} + \frac{S_{rR} - S_{\theta\Theta}}{R} = \rho_0 \frac{\partial^2 r}{\partial t^2}. \tag{5}$$

Combining (5) with (2) yields

$$\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial \lambda_1}\right) + \frac{1}{R}\left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right) = \rho_0 \frac{\partial^2 r}{\partial t^2}. \tag{6}$$

If the cylinder is composed of Blatz-Ko material [8] the strain energy density function will be

$$W = \frac{\mu}{2}(\lambda_1^{-2} + \lambda_2^{-2} - 2) + \mu(\lambda_1\lambda_2 - 1) \tag{7}$$

and its Equation of motion will take the form

$$3Rr^3 r_{RR} + r_R^4 R^3 - r_R r^3 = \rho R r_R^4 r^3 r_{tt} \tag{8}$$

according to (6). Here μ denotes the shear modulus at infinitesimal deformations and ρ is defined by $\rho = \rho_0/\mu$.

LIE GROUPS OF THE WAVE EQUATION

We consider a Lie group $\mathbf{G}$ of coordinate transformations:

$$\bar{R} = F(R, t, r; s),$$

$$\bar{t} = G(R, t, r; s),$$

$$\bar{r} = H(R, t, r; s), \tag{9}$$

where s is a parameter. We want to determine the expressions of the transformation functions F, G, H such that the form of the governing Eq. (8) will be kept invariant when it is transformed by (9). In order to obtain the expressions of F, G, H we consider the infinitesimal coordinate transformations

$$\bar{R} = R + s\xi(R, t, r) + O(s^2),$$

$$\bar{t} = t + s\tau(R, t, r) + O(s^2),$$

$$\bar{r} = r + s\eta(R, t, r) + O(s^2). \tag{10}$$

The function ξ, τ and η can be determined by a routine but tedious procedure [6, 23, 24]. After a lengthy calculation it is found that

$$\xi = A_1 R,$$

$$\tau = A_4 t^2 + (2A_2 - A_1)\, t + A_3,$$

$$\eta = (A_4 t + A_2)\, r. \tag{11}$$

Here A_i (i = 1, 2, 3, 4) are arbitrary constants. The finite coordinate transformations are obtained by integrating the following ODEs [6, 23, 24]:

$$\frac{d\bar{R}}{ds} = A_1 \bar{R},$$

$$\frac{d\bar{t}}{ds} = A_4 \bar{t}^2 + (2A_2 - A_1)\, \bar{t} + A_3,$$

$$\frac{d\bar{r}}{ds} = (A_4 \bar{t} + A_2)\, \bar{r}. \tag{12}$$

The initial conditions for the above ODEs are

$$\bar{R} = R,\ \bar{t} = t,\ \bar{r} = r \text{ at } s = 0. \tag{13}$$

We define $\mathbf{G}_i$ (i = 1, 2, 3, 4) as the one-parameter group of coordinate transformations corresponding to the choice of arbitrary constants $A_i = 1$ and $A_j = 0$ ($j \neq i$, j = 1, 2, 3, 4). We can integrate (12) with (13), $A_1 = 1$ and $A_2 = A_3 = A_4 = 0$ to get

$$\mathbf{G}_1\text{: } \bar{R} = Re^s,\ \bar{t} = te^{-s},\ \bar{r} = r. \tag{14}$$

In the similar way we can get

$$\mathbf{G}_2\text{: } \bar{R} = R,\ \bar{t} = te^{2s},\ \bar{r} = re^s; \tag{15}$$

$$\mathbf{G}_3\text{: } \bar{R} = R,\ \bar{t} = t + s,\ \bar{r} = r. \tag{16}$$

$$\mathbf{G}_4\text{: } \bar{R} = R,\ \bar{t} = \frac{t}{1 - st},\ \bar{r} = \frac{r}{1 - st}; \tag{17}$$

We see that $\mathbf{G}_1$ and $\mathbf{G}_2$ are the groups of scaling transformations, and $\mathbf{G}_3$ is the group of translations of time. The group $\mathbf{G}_4$ contains the translations of the "inverse time". In fact the second relation in (17) implies $\frac{1}{\bar{t}} = \frac{1}{t} - s$. The relation $\bar{t} = \frac{t}{1 - st}$ is plotted in Figure 1 for different value of s. One can see that for any positive value of s the t in $[0, \frac{1}{s})$ is mapped to the $\bar{t}$ in $[0, \infty)$ by the group operation. Further, for any negative value of s the t in $[0, \infty)$ is mapped to the $\bar{t}$ in $[0, \frac{-1}{s})$. It is clear that $\mathbf{G}_4$ is a local Lie group [16-18]. The inverse operation of $\mathbf{G}_4$ takes the form

$$R = \bar{R},\ t = \frac{\bar{t}}{1 + s\bar{t}},\ r = \frac{\bar{r}}{1 + s\bar{t}}.$$

The relation $t = \frac{\bar{t}}{1 + s\bar{t}}$ is plotted in Figure 2. We see that for positive s the $\bar{t}$ in $[0, \infty)$ is mapped to the t in $[0, \frac{1}{s})$ while for negative s the $\bar{t}$ in $[0, \frac{-1}{s})$ is mapped to the t in $[0, \infty)$. The physical interpretation of this special

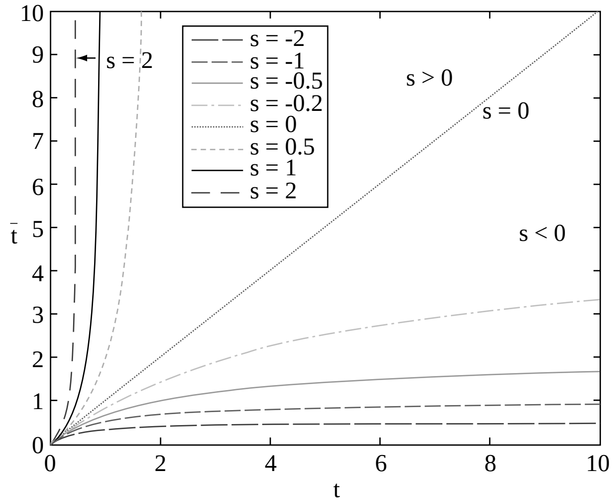

Fig. 1. The relation $\bar{t} = \frac{t}{1 - st}$.

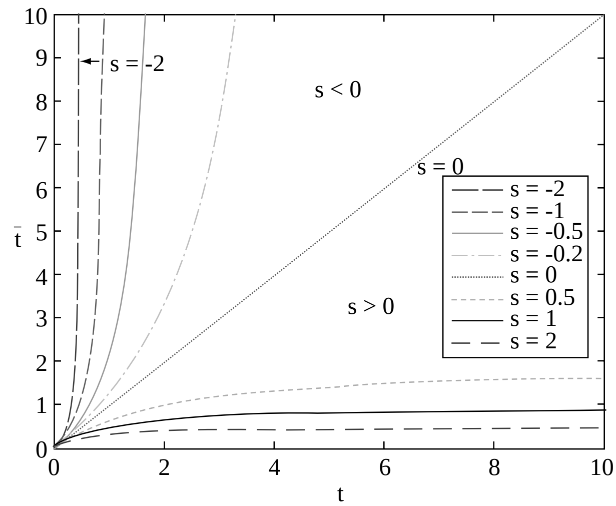

Fig. 2. The relation $t = \frac{\bar{t}}{1 + s\bar{t}}$.

group action will be given in Section 6.

When the above coordinate transformations are acting on Eq. (8) the form of the equation will not be changed. For example, the coordinate transformations in (14) imply that $r_R = e^s \bar{r}_{\bar{R}}$, $r_{RR} = e^{2s} \bar{r}_{\bar{R}\bar{R}}$, $r_{tt} = e^{-2s} \bar{r}_{\bar{t}\bar{t}}$. Therefore, when (14) is applied to (8) we shall have

$$3e^s \bar{R}\bar{r}^3 \bar{r}_{\bar{R}\bar{R}} + e^s \bar{r}_{\bar{R}}^4 \bar{R}^3 - e^s \bar{r}_{\bar{R}} \bar{r}^3 = e^s \rho \bar{R} \bar{r}_{\bar{R}}^4 \bar{r}^3 \bar{r}_{\bar{t}\bar{t}},$$

which is in the same form of (8) since the term e^s is just a nonzero constant and can be cancelled out. It should be noted that scaling symmetries as well as time translational symmetries appear in the invariant properties of many linear and nonlinear wave equations. This is not the case for the symmetry associated with $\mathbf{G}_4$, the inverse time translational symmetry (ITTS) named by us. In fact, many *linear wave* equations possess ITTS, but few *nonlinear wave* equations do [16-18]. Therefore we would like to investigate it more deeply. There are three things we want to clarify about ITTS. First, we want to see whether if ITTS frequently appears in the invariant properties of the wave equations for cylinders composed of other compressible elastic materials. Second, we want to see whether if the governing equations for other objects composed of Blatz-Ko materials also possess ITTS. The third thing we want to explore is the usefulness of ITTS. We shall deal with these three issues in the following sections.

CYLINDERS COMPOSED OF OTHER COMPRESSIBLE ELASTIC MATERIALS

In this section we want to study the symmetries of the wave equations for the cylinders composed of other compressible elastic materials.

We first study the cylinders composed of the material whose model was proposed by Shang and Cheng in [27]. We call it the Shang-Cheng material. This material model was proved to be effective in helping one to construct analytical solutions for void growth problems as can be seen in [27]. The strain energy density function for Shang-Cheng material is

$$W = c_1 (\lambda_1 + \lambda_2 - 2) + c_2 \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - 2\right) + c_3 (\lambda_1\lambda_2 - 1), \tag{18}$$

in which

$$c_1 = \mu\frac{1-3\nu}{1-2\nu},\ c_2 = \mu\frac{1-\nu}{1-2\nu},\ c_3 = \mu\frac{2\nu}{1-2\nu}.$$

Here μ and ν are respectively the shear modulus and the Poisson ratio at infinitesimal deformations. Putting (18) into (6) we get the wave equation for Shang-Cheng cylinders:

$$2Rr^2 r_{RR} - r^2 r_R + R^2 r_R^3 = \rho^* R r^2 r_R^3 r_{tt},\ \rho^* = \frac{\rho_0}{c_2}. \tag{19}$$

After some calculations [6, 23, 24] it is found that the above wave equation admits the following groups of coordinate transformations:

$$\mathbf{G}_5:\ \bar{R} = Re^s,\ \bar{t} = te^{-\frac{1}{2}s},\ \bar{r} = r. \tag{20}$$

$$\mathbf{G}_6:\ \bar{R} = R,\ \bar{t} = te^{\frac{3}{2}s},\ \bar{r} = re^s. \tag{21}$$

$$\mathbf{G}_7:\ \bar{R} = R,\ \bar{t} = t + s,\ \bar{r} = r; \tag{22}$$

It is clear that the wave equation for Shang-Cheng cylinders does not possess ITTS.

In order to get more information for our understanding we consider one more compressible elastic material, namely the generalized Varga material. This material model was also proved to be useful in helping one to construct analytical solutions for void growth problems as can be seen in [13]. The strain energy density function for generalized Varga material is

$$W = c_1^* (\lambda_1 + \lambda_2 - 2) + c_2^* (\lambda_1\lambda_2 - 1) + g(J),\quad J = \lambda_1\lambda_2. \tag{23}$$

Here c_1^* and c_2^* are some constants related to μ and υ [23, 24]. To simplify the calculations we consider the case where $g(J)$ takes the form

$$g(J) = b_0 + b_1 J + b_2 J^2 \tag{24}$$

with b_0, b_1 and b_2 being some constants related to μ and υ [7, 13]. Substituting (23) together with (24) into (6) gives us the wave equation for generalized Varga cylinders:

$$Rr^2 r_{RR} - r^2 r_R + Rrr_R^2 = \rho^{**} R^3 r_{tt},\ \rho^{**} = \frac{\rho_0}{2b_2}. \tag{25}$$

And it is found that the above wave equation admits the following groups of coordinate transformations:

$$\mathbf{G}_8:\ \bar{R} = Re^s,\ \bar{t} = te^{2s},\ \bar{r} = r; \tag{26}$$

$$\mathbf{G}_9: \bar{R} = Re^s,\ \bar{t} = t,\ \bar{r} = re^{2s}; \tag{27}$$

$$\mathbf{G}_{10}: \bar{R} = R,\ \bar{t} = t + s,\ \bar{r} = r; \tag{28}$$

$$\mathbf{G}_{11}: \bar{R}^2 = R^2 + 2s,\ \bar{t} = t,\ \bar{r} = r. \tag{29}$$

Once again ITTS is missing in the symmetries associated with the above Lie groups. According to our studies so far we infer that ITTS does not frequently occur in the invariant properties of the wave equations for cylinders composed of materials other than the Blatz-Ko material.

OTHER OBJECTS COMPOSED OF BLATZ-KO MATERIALS

In this section we study the symmetries of the wave equations for other objects composed of Blatz-Ko materials. We want to see whether if we shall find ITTS again.

We first study the dynamic radial deformations of a sphere composed of the Blatz-Ko material. Let

$$D_0 = \{(R, \Theta, \Phi) \mid a < R < b,\ 0 < \Theta \le 2\pi,\ 0 \le \Phi \le \pi\}$$

denote a hollow sphere in its undeformed configuration. A radial deformation takes a point in D_0 with spherical polar coordinates (R, Θ, Φ) to the point (r, θ, ϕ) in the deformed region D. Since the deformation is radially symmetric we have

$$r = r(R, t) > 0,\ \theta = \Theta \text{ and } \phi = \Phi. \tag{30}$$

The deformation gradient tensor associated with (30) is given by

$$\mathbf{F} = \begin{pmatrix} F_{rR} & F_{r\Theta} & F_{r\Phi} \\ F_{\theta R} & F_{\theta\Theta} & F_{\theta\Phi} \\ F_{\phi R} & F_{\phi\Theta} & F_{\phi\Phi} \end{pmatrix} = \begin{pmatrix} dr/dR & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & r/R \end{pmatrix}. \tag{31}$$

We denote $\lambda_1 = F_{rR} = dr/dR$, $\lambda_2 = F_{\theta\Theta} = r/R$ and $\lambda_3 = F_{\phi\Phi} = r/R$ as the principal stretches. It is obvious that $\lambda_2 = \lambda_3$. If $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain energy density function, the first Piola-Kirchhoff stress tensor can be computed by

$$\mathbf{S} = \begin{pmatrix} S_{rR} & S_{r\Theta} & S_{r\Phi} \\ S_{\theta R} & S_{\theta\Theta} & S_{\theta\Phi} \\ S_{\phi R} & S_{\phi\Theta} & S_{\phi\Phi} \end{pmatrix} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & W_2 & 0 \\ 0 & 0 & W_3 \end{pmatrix}, \tag{32}$$

where $W_1 = \partial W/\partial\lambda_1$, $W_2 = \partial W/\partial\lambda_2$ and $W_3 = \partial W/\partial\lambda_3$. The stress tensor $\mathbf{S}$ satisfies the momentum-balance Eq.

$$\frac{\partial S_{rR}}{\partial R} + \frac{2(S_{rR} - S_{\theta\Theta})}{R} = \rho_0 \frac{\partial^2 r}{\partial t^2}. \tag{33}$$

In this case the strain energy density function for the Blatz-Ko material is

$$W = \frac{\mu}{2}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5). \tag{34}$$

Combining (32), (33) and (34) we get the wave equation for Blatz-Ko spheres:

$$3Rr^3 r_{RR} + 2R^3 r_R^4 - 2r^3 r_R = \rho R r^3 r_R^4 r_{tt}. \tag{35}$$

In the above equation we had denoted $\rho = \rho_0/\mu$. The Lie groups of the above wave equation are found to be

$$\mathbf{G}_{12}: \bar{R} = Re^s,\ \bar{t} = te^{-s},\ \bar{r} = r; \tag{36}$$

$$\mathbf{G}_{13}: \bar{R} = R,\ \bar{t} = te^{2s},\ \bar{r} = re^s; \tag{37}$$

$$\mathbf{G}_{14}: \bar{R} = R,\ \bar{t} = \frac{t}{1 - st},\ \bar{r} = \frac{r}{1 - st}; \tag{38}$$

$$\mathbf{G}_{15}: \bar{R} = R,\ \bar{t} = t + s,\ \bar{r} = r. \tag{39}$$

Obviously the symmetry associated with (38) is ITTS. In fact, we see that the Lie groups of the two wave Eqs., (8) and (35), for Blatz-Ko cylinders and Blatz-Ko spheres are the same.

In order to obtain more information for our understanding we consider one more object --- an uniaxial tensile block composed of Blatz-Ko material. The strain energy density function for Blatz-Ko material in this case is just

$$W = \frac{\mu}{2}(\lambda_1^{-2} + 2\lambda_1 - 3), \tag{40}$$

which can be obtained by setting $\lambda_2 = \lambda_3 = 1$ in (34). Here the principal stretch is computed by $\frac{dy}{dx}$ with y and x denoting the deformed and reference positions correspondingly. It is easy to find that the governing equation for $y(x, t)$ is just

$$y_{xx} = \rho y_x^4 y_{tt} \tag{41}$$

with ρ defined by $\rho = \rho_0/3\mu$. It is easy to check that the above equation admits the group

$$\mathbf{G}_{16}: \bar{R} = R,\ \bar{t} = \frac{t}{1 - st},\ \bar{r} = \frac{r}{1 - st}, \tag{42}$$

and thus the wave Eq. (41) also possesses ITTS. The Lie groups of all the wave Equations studied in this paper are summarized in Table 1. According to the investigations we have done so far we can infer that ITTS might appear in many dynamical problems for Blatz-Ko materials. Of course, more researches must be done in the future before we can obtain a clearer picture about this inference. In this article, we just try to shed some light on this interesting phenomenon and draw the attention of those people who are interested in it. In the next sections we shall investigate the usefulness of ITTS.

CONNECTING INITIAL-BOUNDARY VALUE PROBLEMS FOR (8) BY G_4

In this section we look at the applications of ITTS. In fact, the group G_4 can be used to connect different initial-boundary value problems for the governing Eq. (8). We consider the radial deformations of a cylinder with its cross section bounded by $R = A$ and $R = B$ ($B > A$). A solution of the governing (8),

$$r = f(R, t), \tag{43}$$

is considered which satisfies the following initial-boundary conditions:

$$r = \theta(R) \qquad \text{when } t = 0, \tag{44}$$

$$\frac{\partial r}{\partial t}(R, 0) = \phi(R) \qquad \text{when } t = 0, \tag{45}$$

$$r = p(t) \qquad \text{at } R = A, \tag{46}$$

$$r = q(t) \qquad \text{at } R = B. \tag{47}$$

When the group G_4 acts on the solution (43) it becomes

Table 1. Lie groups of all the wave equations

	Lie groups	**Does the wave equation process ITTS?**
Wave equation for Blatz-Ko cylinders	$\mathbf{G}_1$: $\bar{R} = Re^s$, $\bar{t} = te^{-s}$, $\bar{r} = r$ $\mathbf{G}_2$: $\bar{R} = R$, $\bar{t} = te^{2s}$, $\bar{r} = re^s$ $\mathbf{G}_3$: $\bar{R} = R$, $\bar{t} = t + s$, $\bar{r} = r$ $\mathbf{G}_4$: $\bar{R} = R$, $\bar{t} = \frac{t}{1-st}$, $\bar{r} = \frac{r}{1-st}$	Yes
Wave equation for Shang-Chang cylinders	$\mathbf{G}_5$: $\bar{R} = Re^s$, $\bar{t} = te^{-\frac{1}{2}s}$, $\bar{r} = r$ $\mathbf{G}_6$: $\bar{R} = R$, $\bar{t} = te^{\frac{3}{2}s}$, $\bar{r} = re^s$ $\mathbf{G}_7$: $\bar{R} = R$, $\bar{t} = t + s$, $\bar{r} = r$	No
Wave equation for generalized Varga cylinders	$\mathbf{G}_8$: $\bar{R} = Re^s$, $\bar{t} = te^{2s}$, $\bar{r} = r$ $\mathbf{G}_9$: $\bar{R} = Re^s$, $\bar{t} = t$, $\bar{r} = re^{2s}$ $\mathbf{G}_{10}$: $\bar{R} = R$, $\bar{t} = t + s$, $\bar{r} = r$ $\mathbf{G}_{11}$: $\bar{R}^2 = R^2 + 2s$, $\bar{t} = t$, $\bar{r} = r$	No
Wave equation for Blatz-Ko spheres	$\mathbf{G}_{12}$: $\bar{R} = Re^s$, $\bar{t} = te^{-s}$, $\bar{r} = r$ $\mathbf{G}_{13}$: $\bar{R} = R$, $\bar{t} = te^{2s}$, $\bar{r} = re^s$ $\mathbf{G}_{14}$: $\bar{R} = R$, $\bar{t} = \frac{t}{1-st}$, $\bar{r} = \frac{r}{1-st}$ $\mathbf{G}_{15}$: $\bar{R} = R$, $\bar{t} = t + s$, $\bar{r} = r$	Yes
Wave equation for Blatz-Ko blocks	$\mathbf{G}_{16}$: $\bar{R} = R$, $\bar{t} = \frac{t}{1-st}$, $\bar{r} = \frac{r}{1-st}$	Yes

Note: Not all the groups of this wave equation are presented here. One can consult [6, 23, 24] for more details.

$$\bar{r} = (1 + s\bar{t}) \cdot f(\bar{R}, \frac{\bar{t}}{1 + s\bar{t}}) \tag{48}$$

On the other hand, when G_4 acts on Eq. (8) it will be turned into the same equation written in terms of the transformed coordinates ($\bar{R}$, $\bar{t}$, $\bar{r}$). Therefore, (48) is a solution of $3\bar{R}\bar{r}^3\bar{r}_{\bar{R}\bar{R}} + \bar{r}_{\bar{R}}^4\bar{R}^3 - \bar{r}_{\bar{R}}\bar{r}^3 = \rho\bar{R}\bar{r}_{\bar{R}}^4\bar{r}^3\bar{r}_{\bar{t}\bar{t}}$. If the overhead bars are omitted then it is clear that

$$r = (1 + st) \cdot f(R, \frac{t}{1 + st}) \tag{49}$$

is also a solution of (8) and it satisfies the following conditions:

$$r = \theta(R) \qquad \textit{when } t = 0, \tag{50}$$

$$\frac{\partial r}{\partial t} = s\theta(R) + \phi(R) \qquad \textit{when } t = 0, \tag{51}$$

$$r = (1 + st) \cdot p(\frac{t}{1 + st}) \qquad \textit{at } R = A, \tag{52}$$

$$r = (1 + st) \cdot q(\frac{t}{1 + st}) \qquad \textit{at } R = B. \tag{53}$$

The meaning of the above connection of initial-boundary value problems can be clarified by an example. Let us consider the case where s equals to -10. In this case, the term $t/1 + st = t/1 - 10t$ will tend to infinity as t tends to $\frac{1}{10}$. We see that the solution (49) and its boundary conditions are valid for $t \in [0, \frac{1}{10})$ while the solution (43) and its boundary conditions are valid for $t \in [0, \infty)$. So, if we need to perform an experiment for a Blatz-Ko cylinder for an infinite long time period, we may use the above correspondence to design a substitute experiment that only runs in a finite time domain. In order to make this concept more precise we consider a cross section of a Blatz-Ko cylinder with inner and outer boundaries described by $R = 1$ and $R = 2$ respectively. The initial-boundary conditions for the cylinder are assumed to be

$$r = \theta(R) = 2R \qquad \textit{when } t = 0, \tag{54}$$

$$\frac{\partial r}{\partial t}(R, 0) = \phi(R) = 0 \qquad \textit{when } t = 0, \tag{55}$$

$$r = p(t) = 1 + e^{-3t}\cos(2t) \qquad \textit{at } R = 1, \tag{56}$$

$$r = q(t) = 3 + e^{-t}\cos(3t) \qquad \textit{at } R = 2. \tag{57}$$

The above conditions are assigned in accordance with (44)-(47) and the solution satisfying these conditions is still represented by $r = f(R, t)$ just as that in (43). Note that the above boundary conditions (56) and (57) are valid for $t \in [0, \infty)$. Using the transformations in G_4 with $s = -10$, the solution (43) is transformed into

$$r = (1 - 10t) f(R, \frac{t}{1 - 10t}). \tag{58}$$

At the same time the initial-boundary conditions (54)-(57) are transformed into

$$r = \theta(R) = 2R \qquad \textit{when } t = 0, \tag{59}$$

$$\frac{\partial r}{\partial t} = 10\theta(R) + \phi(R) = 20R \qquad \textit{when } t = 0, \tag{60}$$

$$\begin{aligned} r &= (1 - 10t)\, p(\frac{t}{1 - 10t}) \\ &= (1 - 10t)(1 + e^{\frac{-3t}{1-10t}} \cos(\frac{2t}{1 - 10t})) \qquad \textit{at } R = 1, \end{aligned} \tag{61}$$

$$\begin{aligned} r &= (1 - 10t)\, q(\frac{t}{1 - 10t}) \\ &= (1 - 10t)(3 + e^{\frac{-t}{1-10t}} \cos(\frac{3t}{1 - 10t})) \qquad \textit{at } R = 2. \end{aligned} \tag{62}$$

Note that the boundary conditions (61) and (62) are applied only from $t = 0$ to $t = \frac{1}{10} - \varepsilon$ with ε being a positive infinitesimal number. For $t = \frac{1}{10} + \varepsilon$ the boundary conditions (61) and (62) will no longer be valid because the deformed positions of the inner and outer boundaries will became negative at $t = \frac{1}{10} + \varepsilon$. Moreover, at $t = \frac{1}{10} - \varepsilon$ the deformed positions of the inner and outer boundaries will both be shrunk to zero, and this means that the cylinder will be squeezed into a thin rod with infinitesimal volume. This is not a physically acceptable situation, however, in reality, there is no test needs to be performed for infinitely long time, so the above connection might be applied to replace a long term test by a short term one and the above physically unacceptable situation will not occur.

We should remark that, using other groups, G_i (i = 1, 2, 3), we can also obtain new solutions from known solutions of (8). However, the new solutions obtained by using these groups are trivial since they are different with the old solutions only by simple time translations or coordinate scalings. Therefore is useful since it can generate non-trivial solutions from known solutions of (8).

An example is presented to illustrate the above results. Consider

$$r = g\ (Rt) = g\ (\xi),\ \xi = Rt. \tag{63}$$

Putting (63) into (8) we get the governing Eq. of g:

$$3\xi g^3 g'' + \xi^3 g'^4 - g^3 g' - \rho \xi^3 g^3 g'^4 g'' = 0, \tag{64}$$

which is an ordinary differential equation. This solution is of theoretical interest since we only needs to solve ordinary differential equation instead of partial differential equation in order to understand its behavior. According to the argument for taking (49) as a solution of (8) it is evident that

$$r = (1 + st)\ g\left(\frac{Rt}{1 + st}\right) \tag{65}$$

is also a solution of (8) and its behavior is essentially different from that of (63). In other words, it is a solution generated non-trivially from (63) by G_4.

CONCLUSION

In this paper we have obtained the following results.

1. We have found that the wave equation for Blatz-Ko cylinders possesses a special symmetry – the inverse time translational symmetry. It is also found that this special symmetry is not possessed by the wave equations for cylinders composed of other two materials, namely, the Shang-Chang material and the generalized Varga material.
2. All of the three wave equations governing respectively the Blatz-Ko cylinders, the Blatz-Ko spheres and the Blatz-Ko blocks possess the inverse time translational symmetry.
3. The inverse time translational symmetry can helps us to construct correspondence between different initial-boundary value problems. Making use of this correspondence we can obtain non-trivial solutions from known solutions. Also, if we need to perform an experiment for a Blatz-Ko cylinder for a long time period, we may use the above correspondence to design a substitute experiment that only runs in a short time period.

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