A DELAY-DEPENDENT APPROACH TO ROBUST STABILITY FOR UNCERTAIN HYBRID BIDIRECTIONAL ASSOCIATIVE MEMORY NEURAL NETWORKS WITH TIME-VARYING DELAYS

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ABSTRACT

This paper performs a global robust stability analysis of a particular class of hybrid bidirectional associative memory time-varying delayed neural network with norm-bounded time-varying parameter uncertainties. The activation functions are assumed to be globally Lipschitz continuous. Globally delay-dependent robust stability criteria are derived in the form of linear matrix inequalities by introducing relaxation matrices which, when chosen properly, produce a less conservative result. Two numerical examples are given to illustrate the significant improvement obtained in the conservativeness of the delay bound.

I. INTRODUCTION

Bidirectional associative memory (BAM) neural networks were originally introduced in [18-20] and have been successfully deployed in pattern recognition and artificial intelligence applications [9, 14, 21, 23, 25]. The principal feature of BAM neural networks is the generalization of the single-layer autoassociative Hebbian correlator as a two-layered pattern-matched heteroassociative circuit. Accordingly, BAM neural networks have received significant attention in the literature [1, 10, 17]. It is well known that time delays are frequently encountered in the electronic implementation of artificial neural networks due to the finite switching speeds of the amplifiers and the inherent time required for communication will introduce time delays in the interaction between neurons [22]. Time delays introduce oscillation and instability in BAM neural networks. Therefore, the stability of time-delay BAM neural networks has been the subject of intensive theoretical and practical investigation [1, 7, 8, 12, 15, 16, 24, 27].

The studies cited above provide only pure-delay models. However, hybrid BAM neural networks with both instantaneous and delayed signaling have also been investigated. For example, the convergence dynamics of hybrid BAM neural networks with distributed delays were considered in [22], and a sufficient condition for their solvability was presented. Adopting a similar approach, [2] presented a sufficient condition for the existence, uniqueness and global asymptotic stability of the equilibrium point for hybrid BAM neural networks with distributed time delays. However, in both [22] and [2] the results are delay-independent, that is, they do not include any information relating to the magnitude of the distributed or discrete delays. It is known that delay-dependent conditions are generally less conservative than delay-independent conditions; particularly when the magnitude of the delay is small. Although delay-dependent results for BAM neural networks were developed in [15] and [24] using pure-delay models, the literature does not report on globally delay-dependent robust stability criteria for hybrid BAM neural networks with time-varying delays and parameter uncertainties. The objective of the present study is to address this perceived gap in the literature.

Accordingly, this paper addresses the problem of attaining globally robust stability in uncertain hybrid BAM neural networks with time-varying delays. In the present analysis, the parameter uncertainties are assumed to be time-varying but norm-bounded. Combining the Lyapunov-Krasovskii functional with linear matrix inequality (LMI) techniques, globally

delay-dependent robust stability conditions for uncertain hybrid BAM neural networks with time-varying delays, expressed in terms of quadratic forms of state and LMI, are derived. This study also introduce relaxation matrices which, when chosen properly, produce a less conservative result [13, 26]. The maximum bound for the time delays which ensure that the hybrid uncertain BAM neural network with time-varying delays achieves a globally robust stability is obtained by solving a quasi-convex optimization problem. Two numerical examples are provided to demonstrate the effectiveness of the method.

Throughout this paper, the notation $X \geq Y$ ($X > Y$) for symmetric matrices X and Y indicates that the matrix $X - Y$ is positive and semi-definite (respectively, positive definite), Z^T represents the transpose of matrix Z, and the vector norm $\|\cdot\|$ indicates the Euclidean vector norm, that is $\|W\| = \lambda_M^{1/2}(W^T W)$, where $\lambda_M(W)$ (respectively $\lambda_m(W)$) denotes the operation of taking the maximum (respectively, minimum) eigenvalue of W.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the following hybrid BAM neural network with a time-varying delay described by non-linear differential equations of the form [2]

$$\dot{u}_i(t) = -(a_i + \Delta a_i(t))u_i(t) + \sum_{j=1}^{m}(w_{ij} + \Delta w_{ij}(t))g_j(z_j(t))$$

$$+ \sum_{j=1}^{m}(w_{ij}^v + \Delta w_{ij}^v(t))g_j(z_j(t - h(t))) + I_i,$$

$$i = 1, 2, \cdots, n, \qquad (1)$$

and

$$\dot{z}_j(t) = -(b_j + \Delta b_j(t))z_j(t) + \sum_{i=1}^{n}(c_{ji} + \Delta c_{ji}(t))g_i(u_i(t))$$

$$+ \sum_{j=1}^{n}(c_{ji}^v + \Delta c_{ji}^v(t))g_i(u_i(t - \tau(t))) + J_j,$$

$$j = 1, 2, \cdots, m, \qquad (2)$$

with

$$u_i(t) = \phi_i(t), \quad z_j(t) = \varphi_j(t), \quad t \in \left[-\max\{h, \tau\}, 0\right],$$

$$i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m. \qquad (3)$$

In the above, $u_i(t)$ and $z_j(t)$ denote the state of the ith and jth neurons at time t, respectively; a_i and b_j are the rate at which the ith and jth units reset their potential to the resting state in isolation, i.e. when disconnected from the network and the external inputs at time t, respectively; and w_{ij}, w_{ij}^v, c_{ji} and c_{ji}^v

are synaptic connection strengths at the time t, $\Delta a_i(t)$, $\Delta b_j(t)$, $\Delta w_{ij}(t)$, $\Delta w_{ij}^v(t)$, $\Delta c_{ji}(t)$ and $\Delta c_{ji}^v(t)$ denote unknown constants representing time-varying parameter uncertainties. Furthermore, g_i and g_j are the activation functions of the ith and jth neurons, respectively; and I_i and J_j are the external biases on the ith and jth neurons at time t. Finally, in (3), $\phi_i(t)$ and $\varphi_j(t)$ denote the initial conditions of the ith and jth neurons, respectively; $h(t)$ and $\tau(t)$ represent transmission delays which satisfy

$$\begin{cases} 0 \leq \tau(t) \leq \tau, & \dot{\tau}(t) \leq d_\tau \\ 0 \leq h(t) \leq h, & \dot{h}(t) \leq d_h. \end{cases} \qquad (4)$$

Denote

$$u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T, z(t) = [z_1(t), z_2(t), \cdots, z_m(t)]^T,$$

$$g(u(t)) = [g_1(u_1(t), g_2(u_2(t)), \cdots, g_n(u_n(t))]^T,$$

$$g(z(t)) = [g_1(z_1(t), g_2(z_2(t)), \cdots, g_m(z_m(t))]^T,$$

$$I = [I_1, I_2, \cdots, I_n]^T, J = [J_1, J_2, \cdots, J_m]^T,$$

$$\phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_n(t)]^T, \varphi(t) = [\varphi_1(t), \varphi_2(t), \cdots, \varphi_m(t)]^T.$$

Therefore, Eqs. (1) and (2) can be expressed as

$$\dot{u}(t) = -(A + \Delta A(t))u(t) + (W + \Delta W(t))g(z(t))$$

$$+ (W^v + \Delta W^v(t))g(z(t - h(t))) + I, \qquad (5)$$

$$\dot{z}(t) = -(B + \Delta B(t))z(t) + (C + \Delta C(t))g(u(t))$$

$$+ (C^v + \Delta C^v(t))g(u(t - \tau(t))) + J, \qquad (6)$$

with

$$u(t) = \phi(t), z(t) = \varphi(t), t \in \left[-\max\{h, \tau\}, 0\right], \qquad (7)$$

where

$$A = diag\ (a_1, a_2, \ldots, a_n), B = diag\ (b_1, b_2, \ldots, b_m), W = [w_{ij}]_{n \times m},$$
$$W^v = [w_{ij}^v]_{n \times m}, C = [c_{ji}]_{m \times n}, \text{ and } C^v = [c_{ji}^v]_{m \times n}.$$

Assume that $\Delta A(t)$, $\Delta W(t)$, $\Delta W^v(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta C^v(t)$ are of the form

$$[\Delta A(t)\ \Delta W(t)\ \Delta W^v(t)\ \Delta B(t)\ \Delta C(t)\ \Delta C^v(t)]$$
$$= DF(t)[E_1\ E_2\ E_3\ E_4\ E_5\ E_6], \qquad (8)$$

where D, E_1, E_2, E_3, E_4, E_5 and E_6 are known real constant matrices and $F(t)$ is an unknown real-valued time-varying matrix satisfying

$$F^T(t)F(t) \le I, \forall t. \tag{9}$$

Assume further that all the elements of $F(t)$ are Lebesgue measurable. $\Delta A(t)$, $\Delta W(t)$, $\Delta W^v(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta C^v(t)$ are said to be admissible if both (5) and (6) hold.

Throughout this paper it is assumed that the activation functions satisfy the following assumption.

Assumption 1 (Lipschitz condition) [2].
(A1) There exist positive constants α_i and β_j, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$ such that

$$\left| g_i(\zeta_1) - g_i(\zeta_2) \right| \le \alpha_i \left| \zeta_1 - \zeta_2 \right|, \zeta_1 \ne \zeta_2, i = 1, \cdots, n, \tag{10}$$

$$\left| g_j(\xi_1) - g_i(\xi_2) \right| \le \beta_j \left| \xi_1 - \xi_2 \right|, \xi_1 \ne \xi_2, j = 1, \cdots, m, \tag{11}$$

for all ζ_1, ζ_2, ξ_1 and $\xi_2 \in R$.

(A2) There exist positive constants M_i, $i = 1, \ldots, n$ and $L_i, j = 1, \ldots, m$ such that $|g_i(u)| \le M_i$ and $|g_j(z)| \le L_j$ for all u, $z \in R$. Note that this assumption requires that the functions be bounded. Under assumption (A2), the hybrid BAM neural network defined by (5) and (6) always has an equilibrium point which can be found from various fixed point theorems, e.g. Brouwer's fixed point theorem, Schauder's fixed point theorem, or contraction mapping. In the following, this study analyzes the global asymptotic stability of the equilibrium point, which in turn implies the uniqueness of the equilibrium point.

To simplify the asymptotic stability analysis of (5) and (6), it is assumed that $u^* = [u_1^*, u_2^*, \cdots, u_n^*]^T$ and $z^* = [z_1^*, z_2^*, \cdots, z_m^*]^T$ are the equilibrium points of the hybrid BAM neural network. The equilibrium points u^* and z^* are then shifted to the origin by letting $x(t) = u(t) - u^*$, $y(t) = z(t) - z^*$, $f(x(t)) = g(u(t)) - g(u^*)$ and $f(y(t)) = g(z(t)) - g(z^*)$. Consequently, the system given in (5) and (6) can be expressed as

$$\dot{x}(t) = -(A + \Delta A(t))x(t) + (W + \Delta W(t))f(y(t)) + (W^v + \Delta W^v(t))f(y(t - h(t))), \tag{12}$$

$$\dot{y}(t) = -(B + \Delta B(t))y(t) + (C + \Delta C(t))f(x(t)) + (C^v + \Delta C^v(t))f(x(t - \tau(t))), \tag{13}$$

with

$$u(t) = \phi(t), z(t) = \varphi(t), t \in \left[-\max\{h, \tau\}, 0 \right], \tag{14}$$

where

$$x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T, y(t) = [y_1(t), y_2(t), \cdots, y_m(t)]^T,$$

$$f(x(t)) = [f_1(x_1(t), f_2(x_2(t)), \cdots, f_n(x_n(t))]^T,$$

$$f(y(t)) = [f_1(y_1(t), f_2(y_2(t)), \cdots, f_m(y_m(t))]^T.$$

Hence, it is easily shown that

$$\left| f_i(x_i) \right| \le \alpha_i \left| x_i \right|, \text{ and } f_i(0) = 0, i = 1, \cdots, n.$$

$$\left| f_j(y_j) \right| \le \beta_j \left| y_j \right|, \text{ and } f_j(0) = 0, j = 1, \cdots, m.$$

Definition 1 [5, 11]. The equilibrium point (trivial solution) 0 is said to be globally asymptotically stable if it is locally stable in the Lyapunov sense and is globally attractive, where global attractivity indicates that every trajectory tends to the equilibrium point as $t \to \infty$.

The objective of the present analysis is to establish the delay-dependent conditions which cause the hybrid uncertain BAM neural network with time-varying delays given in (12) and (13) to attain globally asymptotically robust stability. More specifically, for given scalars $\tau > 0$ and $h > 0$, the main aim of this study is to determine whether the system in (12) and (13) has globally asymptotically robust stability for all delays in the range $0 \le \tau(t) \le \tau$ and $0 \le h(t) \le h$.

III. MATHEMATICAL FORMULATION OF THE PROPOSED APPROACH

This section explores the globally robust stability for system given in (12) and (13). The analysis commences by using the LMI approach to develop some results which are essential to introduce the following Lemma 1 for the development of our main theorem.

Lemma 1 [3]. Let A, D, S, F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^T(t)F(t) \le I$. Then the following statements hold
(a) For any $\varepsilon > 0$ and vectors $x, y \in R^n$

$$2x^T DFSy \le \varepsilon^{-1} x^T DD^T x + \varepsilon y^T S^T S y, \tag{15}$$

(b) For vectors $x, y \in R^n$

$$2x^T DSy \le x^T DPD^T x + y^T S^T P^{-1} S y. \tag{16}$$

For any matrices Z_i, S_i, N_i and $M_i (i = 1, 2, 3, 4, 5)$ of appropriate dimensions, it follows from null equations that [13, 26]

$$[x^T(t)Z_1 + x^T(t-\tau(t))Z_2 + f^T(y(t))Z_3 + f^T(y(t-h(t)))Z_4$$
$$+ \dot{x}^T(t)Z_5] \times [x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds] = 0,$$
(17)

$$[y^T(t)S_1 + y^T(t-h(t))S_2 + f^T(x(t))S_3 + f^T(x(t-\tau(t)))S_4$$
$$+ \dot{y}^T(t)S_5] \times [y(t) - y(t-h(t)) - \int_{t-h(t)}^{t} \dot{y}(s)ds] = 0,$$
(18)

$$[x^T(t)M_1 + x^T(t-\tau(t))M_2 + f^T(y(t))M_3 + f^T(y(t-h(t)))M_4$$
$$+ \dot{x}^T(t)M_5] \times \{\dot{x}(t) + (A + \Delta A(t))x(t) - (W + \Delta W(t))f(y(t))$$
$$- (W_1 + \Delta W_1(t))f(y(t-h(t)))\} = 0,$$
(19)

$$[y^T(t)N_1 + y^T(t-h(t))N_2 + f^T(x(t))N_3 + f^T(x(t-\tau(t)))N_4$$
$$+ \dot{y}^T(t)N_5] \times \{\dot{y}(t) + (B + \Delta B(t))x(t) - (C + \Delta C(t))f(x(t))$$
$$- (C_1 + \Delta C_1(t))f(x(t-\tau(t)))\} = 0.$$
(20)

To study the globally robust stable for the hybrid uncertain BAM neural network with time-varying delays, the following theorem reveals that such conditions can be expressed in terms of LMIs.

Theorem 1. Under the assumption 1 (A1), for any given $\tau > 0$, $h > 0$, satisfying $0 \le \tau(t) \le \tau$, $0 \le h(t) \le h$, $\dot{\tau}(t) \le d_\tau$ and $\dot{h}(t) \le d_h$, (12) and (13) are globally robust stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$, $T_1 > 0$, $T_2 > 0$ and matrices Z_i, S_i, N_i and $M_i (i = 1, 2, 3, 4, 5)$ of appropriate dimensions and positive scalars ε_1 and ε_2 such that the following linear matrix inequalities hold

$$\begin{bmatrix} \Omega & MD & \varepsilon_1 E^T & \tau Z \\ D^T M^T & -\varepsilon_1 & 0 & 0 \\ \varepsilon_1 E & 0 & -\varepsilon_1 & 0 \\ \tau Z^T & 0 & 0 & -\tau T_1 \end{bmatrix} < 0,$$
(21)

$$\begin{bmatrix} \Pi & ND & \varepsilon_2 \bar{E}^T & hS \\ D^T N^T & -\varepsilon_2 & 0 & 0 \\ \varepsilon_2 \bar{E} & 0 & -\varepsilon_2 & 0 \\ hS^T & 0 & 0 & -hT_2 \end{bmatrix} < 0,$$
(22)

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ \Omega_{14}^T & \Omega_{24}^T & \Omega_{34}^T & \Omega_{44} & \Omega_{45} \\ \Omega_{15}^T & \Omega_{25}^T & \Omega_{35}^T & \Omega_{45}^T & \Omega_{55} \end{bmatrix},$$

$$\Omega_{11} = M_1 A + A^T M_1^T + Q_1 + Z_1 + Z_1^T,$$

$$\Omega_{12} = Z_2 - Z_1 + A^T M_2^T, \quad \Omega_{13} = -M_1 W + A^T M_3^T + Z_3,$$

$$\Omega_{14} = Z_4 - M_1 W_1 + A^T M_4^T, \quad \Omega_{15} = P_1 + M_1 + A^T M_5^T + Z_5,$$

$$\Omega_{22} = -(1-d_\tau)Q_1 - Z_2 - Z_2^T, \quad \Omega_{23} = -M_2 W - Z_3,$$

$$\Omega_{24} = -M_2 W_1 - Z_4, \quad \Omega_{25} = M_2 - Z_5, \quad \Omega_{33} = R_2 - M_3 W - W^T M_3^T,$$

$$\Omega_{34} = -M_3 W_1 - W^T M_4^T, \quad \Omega_{35} = M_3 - W^T M_5^T,$$

$$\Omega_{44} = -(1-d_h)R_2 - M_4 W_1 - W_1^T M_4^T, \quad \Omega_{45} = M_4 - W_1^T M_5^T,$$

$$\Omega_{55} = M_5 + M_5^T + \tau T_1,$$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ \Pi_{12}^T & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & \Pi_{34} & \Pi_{35} \\ \Pi_{14}^T & \Pi_{24}^T & \Pi_{34}^T & \Pi_{44} & \Pi_{45} \\ \Pi_{15}^T & \Pi_{25}^T & \Pi_{35}^T & \Pi_{45}^T & \Pi_{55} \end{bmatrix},$$

$$\Pi_{11} = S_1 + S_1^T + Q_2 + N_1 B + B^T N_1^T, \quad \Pi_{12} = S_2 - S_1 + B^T N_2^T,$$

$$\Pi_{13} = -N_1 C + B^T N_3^T + S_3, \quad \Pi_{14} = S_4 - N_1 C_1 + B^T N_4^T,$$

$$\Pi_{15} = P_2 + N_1 + B^T N_5^T + S_5, \quad \Pi_{22} = -(1-d_h)Q_2 - S_2 - S_2^T,$$

$$\Pi_{23} = -N_2 C - S_3, \quad \Pi_{24} = -N_2 C_1 - S_4, \quad \Pi_{25} = N_2 - S_5,$$

$$\Pi_{33} = R_1 - N_3 C - C^T N_3^T, \quad \Pi_{34} = -N_3 C_1 - C^T N_4^T,$$

$$\Pi_{35} = N_3 - C^T N_5^T, \quad \Pi_{44} = -(1-d_\tau)R_1 - N_4 C_1 - C_1^T N_4^T,$$

$$\Pi_{45} = N_4 - C_1^T N_5^T, \quad \Pi_{55} = N_5 + N_5^T + hT_2,$$

$$M = [M_1^T \quad M_2^T \quad M_3^T \quad M_4^T \quad M_5^T]^T,$$

$$Z = \begin{bmatrix} Z_1^T & Z_2^T & Z_3^T & Z_4^T & Z_5^T \end{bmatrix}^T,$$

$E = [E_1 \quad 0 \quad -E_2 \quad -E_3 \quad 0], \bar{E} = [E_4 \quad 0 \quad -E_5 \quad -E_6 \quad 0],$

$N = [N_1^T \quad N_2^T \quad N_3^T \quad N_4^T \quad N_5^T]^T,$

$S = [S_1^T \quad S_2^T \quad S_3^T \quad S_4^T \quad S_5^T]^T.$

Proof. Consider the following Lyapunov-Krasovskii functional for the system in (12) and (13)

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t) + V_7(t) + V_8(t), \tag{21}$$

where

$$V_1(t) = x^T(t)P_1 x(t), \tag{22}$$

$$V_2(t) = y^T(t)P_2 y(t), \tag{23}$$

$$V_3(t) = \int_{t-\tau(t)}^{t} x^T(\alpha)Q_1 x(\alpha)d\alpha, \tag{24}$$

$$V_4(t) = \int_{t-h(t)}^{t} y^T(\alpha)Q_2 y(\alpha)d\alpha, \tag{25}$$

$$V_5(t) = \int_{t-\tau(t)}^{t} f^T(x(\alpha))R_1 f(x(\alpha))d\alpha, \tag{26}$$

$$V_6(t) = \int_{t-h(t)}^{t} f^T(y(\alpha))R_2 f(y(\alpha))d\alpha, \tag{27}$$

$$V_7(t) = \int_{t-\tau}^{t}\int_{\alpha}^{t} \dot{x}^T(\theta)T_1\dot{x}(\theta)d\theta d\alpha, \tag{28}$$

$$V_8(t) = \int_{t-h}^{t}\int_{\alpha}^{t} \dot{y}^T(\theta)T_2\dot{y}(\theta)d\theta d\alpha. \tag{29}$$

Taking the derivative of $V(t)$ with respect to t along the trajectories of (12) and (13) yields

$\dot{V}(t) = 2x^T(t)P_1\dot{x}(t) + 2y^T(t)P_2\dot{y}(t) + x^T(t)Q_1 x(t)$

$\quad -(1-\dot{\tau}(t))x^T(t-\tau(t))Q_1 x(t-\tau(t)) + y^T(t)Q_2 y(t)$

$\quad -(1-\dot{h}(t))y^T(t-\tau(t))Q_2 y(t-\tau(t)) + f^T(x(t))R_1 f(x(t))$

$\quad -(1-\dot{\tau}(t))f^T(x(t-\tau(t)))R_1 f(x(t-\tau(t)))$

$\quad +f^T(y(t))R_2 f(y(t)) - (1-\dot{h}(t))f^T(y(t-h(t)))R_2 f(y(t-h(t)))$

$\quad +\tau \dot{x}^T(t)T_1\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(\alpha)T_1\dot{x}(\alpha)d\alpha$

$$\quad +h\dot{y}^T(t)T_2\dot{y}(t) - \int_{t-h}^{t} \dot{y}^T(\alpha)T_2\dot{y}(\alpha)d\alpha. \tag{30}$$

Substituting (17)-(20) into (30) yields

$\dot{V}(t) \le 2x^T(t)P_1\dot{x}(t) + 2y^T(t)P_2\dot{y}(t) + x^T(t)Q_1 x(t)$

$\quad -(1-d_x)x^T(t-\tau(t))Q_1 x(t-\tau(t)) + y^T(t)Q_2 y(t)$

$\quad -(1-d_h(t))y^T(t-\tau(t))Q_2 y(t-\tau(t)) + f^T(x(t))R_1 f(x(t))$

$\quad -(1-d_x(t))f^T(x(t-\tau(t)))R_1 f(x(t-\tau(t))) + f^T(y(t))R_2 f(y(t))$

$\quad -(1-d_h(t))f^T(y(t-h(t)))R_2 f(y(t-h(t)))$

$\quad +\tau \dot{x}^T(t)T_1\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}^T(\alpha)T_1\dot{x}(\alpha)d\alpha$

$\quad +h\dot{y}^T(t)T_2\dot{y}(t) - \int_{t-h}^{t} \dot{y}^T(\alpha)T_2\dot{y}(\alpha)d\alpha$

$\quad +2[x^T(t)Z_1 + x^T(t-\tau(t))Z_2 + f^T(y(t))X_3 + f^T(y(t-h(t)))Z_4$

$\quad +\dot{x}^T(t)Z_5]\times[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(\alpha)d\alpha]$

$\quad +2[y^T(t)S_1 + y^T(t-h(t))S_2 + f^T(x(t))S_3 + f^T(x(t-\tau(t)))S_4$

$\quad +\dot{y}^T(t)S_5]\times[y(t) - y(t-h(t)) - \int_{t-h(t)}^{t} \dot{y}(\alpha)d\alpha]$

$\quad +2[x^T(t)M_1 + x^T(t-\tau(t))M_2 + f^T(x(t))M_3 + f^T(x(t-\tau(t)))M_4$

$\quad +\dot{x}^T(t)M_5]\times\{\dot{x}(t) + (A+\Delta A(t))x(t) - (W+\Delta W(t))f(y(t))$

$\quad -(W_1+\Delta W_1(t))f(y(t-h(t)))\}$

$\quad +2[y^T(t)N_1 + y^T(t-h(t))N_2 + f^T(x(t))N_3 + f^T(x(t-\tau(t)))N_4$

$\quad +\dot{y}^T(t)N_5]\times\{\dot{y}(t) + (B+\Delta B(t))y(t) - (C+\Delta C(t))f(x(t))$

$$\quad -(C_1+\Delta C_1(t))f(x(t-\tau(t)))\}. \tag{31}$$

From Lemma 1, it can be shown that

$$-2[x^T(t)Z_1 + x^T(t-\tau(t))Z_2 + f^T(y(t))Z_3$$
$$+ f^T(y(t-h(t)))Z_4 + \dot{x}^T(t)Z_5] \int_{t-\tau(t)}^{t} \dot{x}(\alpha)d\alpha$$

$$\leq \tau q_1^T(t)ZT_1^{-1}Z^T q_1(t) + \int_{t-\tau(t)}^{t} \dot{x}^T(\alpha)T_1\dot{x}(\alpha)d\alpha, \quad (32)$$

$$-2[y^T(t)S_1 + y^T(t-h(t))S_2 + f^T(x(t))S_3$$
$$+ f^T(x(t-\tau(t)))S_4 + \dot{y}^T(t)S_5] \int_{t-h(t)}^{t} \dot{y}(\alpha)d\alpha$$

$$\leq h q_2^T(t)ST_2^{-1}S^T q_2(t) + \int_{t-h(t)}^{t} \dot{y}^T(\alpha)T_2\dot{y}(\alpha)d\alpha, \quad (33)$$

$$2[x^T(t)M_1 + x^T(t-\tau(t))M_2 + f^T(x(t))M_3 + f^T(x(t-\tau(t)))M_4$$
$$+ \dot{x}^T(t)M_5] \times \{\dot{x}(t) + (A+\Delta A(t))x(t) - (W+\Delta W(t))f(y(t))$$
$$- (W_1 + \Delta W_1(t))f(y(t-h(t)))\} \leq 2q_1^T(t)M(\bar{A}+DF(t)E)$$
$$\leq 2q_1^T(t)M\bar{A}q_1(t) + q_1^T(t)\varepsilon_1^{-1}MDD^TM^T q_1(t)$$
$$+ q_1^T(t)\varepsilon_1 E^T E q_1(t), \quad (34)$$

$$2[y^T(t)N_1 + y^T(t-h(t))N_2 + f^T(x(t))N_3 + f^T(x(t-\tau(t)))N_4$$
$$+ \dot{y}^T(t)N_5] \times \{\dot{y}(t) + (B+\Delta B(t))y(t) - (C+\Delta C(t))f(x(t))$$
$$- (C_1 + \Delta C_1(t))f(x(t-\tau(t)))\} \leq 2q_2^T(t)N(\bar{B}+DF(t)\bar{E})$$
$$\leq 2q_2^T(t)N\bar{B}q_2(t) + q_2^T(t)\varepsilon_2^{-1}NDD^TN^T q_2(t)$$
$$+ q_2^T(t)\varepsilon_2\bar{E}^T\bar{E}q_2(t), \quad (35)$$

where

$$q_1(t) = [x^T(t) \quad x^T(t-\tau(t)) \quad f^T(y(t)) \quad f^T(y(t-h(t))) \quad \dot{x}^T(t)]^T,$$

$$q_2(t) = [y^T(t) \quad y^T(t-h(t)) \quad f^T(x(t)) \quad f^T(x(t-\tau(t))) \quad \dot{y}^T(t)]^T,$$

$$Z = [Z_1^T \quad Z_2^T \quad Z_3^T \quad Z_4^T \quad Z_5^T]^T, S = [S_1^T \quad S_2^T \quad S_3^T \quad S_4^T \quad S_5^T]^T,$$

$$M = [M_1^T \quad M_2^T \quad M_3^T \quad M_4^T \quad M_5^T]^T,$$

$$N = [N_1^T \quad N_2^T \quad N_3^T \quad N_4^T \quad N_5^T]^T,$$

$$\bar{A} = [A \quad 0 \quad -W \quad -W_1 \quad I], \bar{B} = [B \quad 0 \quad -C \quad -C_1 \quad I],$$

$$E = [E_1 \quad 0 \quad -E_2 \quad -E_3 \quad 0], \bar{E} = [E_4 \quad 0 \quad -E_5 \quad -E_6 \quad 0].$$

Combining Eqs. (31)-(35) yields

$$\dot{V}(t) \leq q_1^T(t)\Omega_1 q_1(t) + q_2^T(t)\Pi_1 q_2(t), \quad (36)$$

where

$$\Omega_1 = \Omega + \tau ZT_1^{-1}Z^T + \varepsilon_1^{-1}MDD^TM^T + \varepsilon_1 E^T E,$$

$$\Pi_1 = \Pi + hST_2^{-1}S^T + \varepsilon_2^{-1}NDD^TN^T + \varepsilon_2\bar{E}^T\bar{E}.$$

From (18) and (19) and using the well-known Schur complement [5], it can be shown that $\dot{V}(t) < 0$. Therefore, it can be concluded from the Lyapunov-Krasovskii functional that the hybrid uncertain BAM neural network expressed in (12)-(13) attains globally asymptotically robust stability. This completes the proof of Theorem 1.

Remark 1. Theorem 1 provides a sufficient condition for the globally asymptotically robust stability for system given in (12)-(13) and proposes a delay-dependent stability criterion. Recently the results have been studied in [2, 22] and are delay-independent stability criteria. Since, it is less conservative than a delay-independent condition in [2, 22]. Moreover, parameter uncertainties have also not been fully investigated in [2, 22]. Also, the present results in [15] ignore some useful terms. This may bring conservativeness. In Theorem 1, all terms are reserved, which can reduce conservativeness to keep the stability of systems. In [15], the system is pure-delay model with constant time delay. Therefore, the results obtained in Theorem 1 for uncertain hybrid BAM neural networks with time-varying delays are more general than that presented in the literature [15].

Remark 2: In many cases, the derivative of time-varying delay $h(t)$ and $\tau(t)$ is known and may be small. Relaxation matrices are introduced into Theorem 1, so, d_h and d_τ can be any value or unknown due to $\Pi_{22} = -(1-d_h)Q_2 - S_2 - S_2^T$, $\Omega_{22} = -(1-d_\tau)Q_1 - Z_2 - Z_2^T$, where Z_2, S_2 are relaxation matrices with appropriate dimensions. The traditional assumption that derivatives of the delays are less than 1 is no longer required in our analysis. Therefore, Theorem 1 is applicable to more general and practical than the one investigated in [2] and [22].

Two numerical examples are now presented to demonstrate the usefulness of the proposed approach.

IV. NUMERICAL EXAMPLES

Example 1: Consider the following hybrid uncertain BAM delayed neural network

$$\dot{u}(t) = -(A + \Delta A(t))u(t) + (W + \Delta W(t))g(z(t))$$
$$+ (W^v + \Delta W^v(t))g(z(t - h(t))) + I, \quad (37)$$

$$\dot{z}(t) = -(B + \Delta B(t))z(t) + (C + \Delta C(t))g(u(t))$$
$$+ (C^v + \Delta C^v(t))g(u(t - \tau(t))) + J, \quad (38)$$

where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0.9 \end{bmatrix}, W = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix}, W_1 = \begin{bmatrix} 0.9 & -1.2 \\ 0.05 & -0.9 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.4 \end{bmatrix}, C_1 = \begin{bmatrix} 0.7 & 0.7 \\ 0.5 & 1.3 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.02 & 0.02 \\ -0.1 & 0.2 \end{bmatrix}, E_2 = \begin{bmatrix} -0.07 & 0.3 \\ -0.01 & 0.2 \end{bmatrix}, E_3 = \begin{bmatrix} 0.02 & -0.02 \\ 0.1 & -0.1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 0.01 & 0.02 \\ -0.1 & 0.3 \end{bmatrix}, E_5 = \begin{bmatrix} -0.04 & 0.2 \\ -0.01 & 0.1 \end{bmatrix}, E_6 = \begin{bmatrix} 0.01 & -0.02 \\ 0.2 & -0.1 \end{bmatrix}.$$

Utilizing Theorem 1 and $F^T(t)F(t) \leq I$, it is found that LMI equations (21) and (22) are feasible for any arbitrarily large τ and h for different d_τ and d_h (provided that the numerical computation is reliable). Therefore, in view of Theorem 1, the current hybrid uncertain BAM neural network attains globally robust stability in the sense of Definition 1.

Example 2: Consider the following uncertain BAM delayed neural network

$$\dot{u}(t) = -(A + \Delta A(t))u(t) + (W^v + \Delta W^v(t))g(z(t - h(t))) + I,$$
$$(39)$$

$$\dot{z}(t) = -(B + \Delta B(t))z(t) + (C^v + \Delta C^v(t))g(u(t - \tau(t))) + J,$$
$$(40)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, W^v = \begin{bmatrix} 0.05 & 0.25 & 0.05 \\ 0.10 & 0.05 & 0.15 \\ 0.15 & 0.15 & 0.05 \end{bmatrix},$$

$$C^v = \begin{bmatrix} 0.75 & 0.75 & 0.95 \\ 0 & 0.50 & 0.75 \\ 0.15 & 0.95 & 0.95 \end{bmatrix}, D = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.3 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.02 & 0.02 & 0 \\ -0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix}, E_3 = \begin{bmatrix} 0.02 & -0.02 & 0 \\ 0.1 & -0.1 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 0.01 & 0.02 & 0 \\ -0.1 & 0.3 & 0.1 \\ 0.1 & 0 & 0.2 \end{bmatrix}, E_6 = \begin{bmatrix} 0.01 & -0.02 & 0 \\ 0.2 & -0.1 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}.$$

All uncertain terms are zero in (39) and (40), it can be verified that Theorem 1 in [24] is feasible for all delays $0 \leq \sigma \leq 0.5$ and $0 \leq \tau \leq 0.78$, Corollary 2 in [13] is feasible for all delays $0 \leq \tau \leq 1$ and $0 \leq \sigma \leq 0.5$. Also, Theorem 3.1 in [8] does not satisfy globally delay-dependent conditions for all delays τ and δ when N and M are zero. Theorem 1 in this paper is feasible for all delays $0 \leq \tau(t) \leq 3.78$ and $0 \leq h(t) \leq 3.78$. Therefore, Theorem 1 in this paper is less conservative than those in [8, 15, 24] for delayed BAM neural network. Moreover, it can be checked that the delay-dependent robust stability conditions in [1, 2, 22] is not satisfied. Therefore, they fail to conclude that uncertain hybrid BAM neural network with time-varying delays is globally delay-dependent robust stability. By resorting to Theorem 1 in this paper, it is found that LMIs in (21) and (22) are feasible for all delays satisfying $0 \leq \tau(t) \leq 10.58$ and $0 \leq h(t) \leq 10.58$. The solutions of Theorem 1 are as follows

$$P_1 = \begin{bmatrix} 2.3061 & -0.2657 & -0.4443 \\ -0.2657 & 2.5420 & -0.3180 \\ -0.4443 & -0.3180 & 2.4346 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.4861 & -0.4376 & -0.9482 \\ -0.4376 & 2.6898 & -1.0997 \\ -0.9482 & -1.0997 & 1.9763 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.5769 & -0.1885 & -0.3018 \\ -0.1885 & 1.7086 & -0.2005 \\ -0.3018 & -0.2005 & 1.6656 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 1.6066 & -0.4482 & -0.7091 \\ -0.4482 & 1.9974 & -0.4595 \\ -0.7091 & -0.4595 & 1.6406 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 2.6751 & -0.0527 & 0.0321 \\ -0.0527 & 2.4971 & 0.5400 \\ 0.0321 & 0.5400 & 2.3424 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 2.5036 & 0.3361 & 0.4186 \\ 0.3361 & 2.9811 & 0.2920 \\ 0.4186 & 0.2920 & 2.3636 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.1745 & -0.0219 & -0.0403 \\ -0.0219 & 0.1960 & -0.0334 \\ -0.0403 & -0.0334 & 0.1843 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0.0261 & -0.0055 & -0.0203 \\ -0.0055 & 0.0659 & -0.0341 \\ -0.0203 & -0.0341 & 0.0453 \end{bmatrix},$$

$\varepsilon_1 = 1.5575, \varepsilon_2 = 1.4104.$

Thus, by Theorem 1, Eqs. (39) and (40) attains globally delay-dependent robust stability. This shows that the condition given in Theorem 1 is less conservative than those in [1, 2, 22] when checking the globally delay-dependent robust stability for a given uncertain hybrid BAM neural network with time-varying delays.

V. CONCLUSIONS

This study has investigated the problem of globally delay-dependent robust stability for a particular class of uncertain hybrid BAM neural network with time-varying delays. A sufficient condition for the solvability of this problem, which depends on the magnitude of the time delay, has been derived using the Lyapunov-Krasovskii functional and the LMI approach. It has been shown that the hybrid uncertain BAM with time-varying delays attains globally delay-dependent robust stability can be obtained by solving a quasi-convex optimization problem. Two numerical examples have been presented to demonstrate the effectiveness of the proposed approach.

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