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VIBRATION ANALYSIS OF PRE-TWISTED BEAMS USING THE SPLINE COLLOCATION METHOD

Ming-Hung Hsu*

Key words: natural frequency, pre-twisted beam, spline collocation method, tapered beam, boundary condition.

ABSTRACT

The variation in the accuracy of the calculated natural frequencies of pre-twisted beams solved with spline collocation method is investigated in this study. The spline collocation method is used to formulate the eigenvalue problems of pre-twisted beams. Three types of boundary conditions are considered. Numerical results indicate that the accuracy of the calculated natural frequencies is significantly dependent upon the pre-twisted angle of the non-uniform beam. Results that show that spline collocation method is very competitive for the vibration analysis of pre-twisted beams are presented.

I. INTRODUCTION

Speaking of many kinds of design, dynamic characteristics of pre-twisted beams absolutely play a vital role. In the field of turbo or compressor engineering, for simplicity, the beam is frequently approximated as a pre-twisted beam. At the design stage, accurate prediction of national frequencies of non-uniform pre-twisted beam is of considerable importance at the design stage. The coupled natural frequencies of a pre-twisted beam were also investigated using a number of different methods. Abrate [1] studied the vibration of a pre-twisted blade using the Rayleigh-Ritz method. Anderson [2] studied the flexural vibration of rotating bars. Dawson [4, 5] studied the vibration of a pre-twisted blade using the Rayleigh-Ritz method. Gupa and Rao [8] applied the finite element method for finding the variation of natural frequencies of doubly tapered and twisted Timoshenko beams. Hodges *et al.* [9] used the transfer matrices to compute the fundamental frequencies and corresponding modal displacements along the non-uniform rotating beams. They displayed that a blade has a complex geometry that makes an exact investigation of its characteristics somewhat complex. Kuang and Hsu [10, 11] presented that the blade is frequently approximated as a pre-twisted beam for simplicity in the field of turbo or compressor engineering. Lin *et al.* [12] presented the accurate modified transfer matrix method for studying the dynamic behavior of a non-uniform pre-twisted Timoshenko beam. Rao [15, 16] studied the natural frequency of pre-twisted beam to consider the complex shape of beam. They presented the vibration problems of wind blades and turbo blades are crucial parts of the design. Storti and Aboelnaga [19] studied the transverse deflections of a straight tapered symmetric beam attached to a rotating hub as a model for the bending vibration of blades in turbomachinery. Subrahmanyam *et al.* [20] showed coupled bending-bending vibrations of pre-twisted cantilever blading allowing for shear deflection and rotary inertia by the Reissner method. Subrahmanyam and Rao [21] presented coupled bending-bending vibrations of pre-twisted tapered cantilever beams treated by the Reissner method. Swaminathan and Rao [22] solved the vibrations of rotating, pre-twisted and tapered blades. Young [27] dealt with the dynamics of a pre-twisted beam using Rao's comparison functions. In this work, the spline collocation method is implemented to formulate the eigenvalue problem of a pre-twisted beam in the discrete form. The integrity and computational efficiency of spline collocation method in this problem will be demonstrated through a series of case studies.

II. SPLINE COLLOCATION METHOD

The solutions to numerous complex pre-twisted beam problems have been efficiently obtained as the use of fast computers and range of available numerical methods, including the Galerkin method, differential quadrature method, finite element technique, differential transform, boundary element method, and Rayleigh-Ritz method. In this study, spline collocation method is employed to formulate the discrete eigenvalue problems of various pre-twisted beams. Prenter *et al.* [7, 14, 18] investigated spline and variation methods. Bert and Sheu [3] presented static analysis of beams and plates using spline collocation method. El-Hawary *et al.* [6] discussed quartic spline collocation methods for solving linear elliptic partial differential equations. They derived optimal quartic spline approximations to generated high order perturbations of partial differential equations. Patlashenko and Weller [13] applied the spline collocation method to solve two-dimensional problems, and determined the postbucking

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behavior of laminated panels subjected to mechanical and heating induced loadings. Viswanathan and Navaneethakrishnan [23] studied the free vibration of circular cylindrical thin shells using point collocation method. Rao and Kumar [17] presented a B-spline collocation method of higher order for a class of self-adjoint singularly pertured boundary value problems. Wu *et al.* [24, 25, 26] displayed the application of the spline collocation method to analysis of rigid frame structures under various loading conditions. Their results from the spline collocation method are compared with those obtained from the differential quadrature method, the finite element method, and other available analytical methods. The spline function can be derived from backward or central finite difference. In this work, we consider the knots z_i as follows.

$$z_i = \overline{a} + ih \quad \text{for } i = -2, -1, 0, ..., N+1, N+2 \tag{1}$$

where z_0, z_1, z_2, …, z_{N-1}, z_N are the abscissas of knots and z_{-2}, z_{-1}, z_{N+1}, z_{N+2} are the abscissas of extended fictitious knots.

$$h = \frac{\overline{b} - \overline{a}}{N} \tag{2}$$

where distance h between two adjacent knots keep constant. Spline function is given as follows [3, 6, 13].

$$B_{ui}(z) = B_{vi}(z) = \begin{cases} \dfrac{(z-z_{i-3})^5}{h^5} & \text{if } z \in [z_{i-3}, z_{i-2}] \\ \dfrac{(z-z_{i-3})^5 - 6(z-z_{i-2})^5}{h^5} & \text{if } z \in [z_{i-2}, z_{i-1}] \\ \dfrac{(z-z_{i-3})^5 - 6(z-z_{i-2})^5 + 15(z-z_{i-1})^5}{h^5} & \text{if } z \in [z_{i-1}, z_i] \\ \dfrac{(z-z_{i-3})^5 - 6(z-z_{i-2})^5 + 15(z-z_{i-1})^5 - 20(z-z_i)^5}{h^5} & \text{if } z \in [z_i, z_{i+1}] \\ \dfrac{(z-z_{i-3})^5 - 6(z-z_{i-2})^5 + 15(z-z_{i-1})^5 - 20(z-z_i)^5 + 15(z-z_{i+1})^5}{h^5} & \text{if } z \in [z_{i+1}, z_{i+2}] \\ \dfrac{(z-z_{i-3})^5 - 6(z-z_{i-2})^5 + 15(z-z_{i-1})^5 - 20(z-z_i)^5 + 15(z-z_{i+1})^5 - 6(z-z_{i+2})^5}{h^5} & \text{if } z \in [z_{i+2}, z_{i+3}] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $B_{u,-2}(z)$, $B_{u,-1}(z)$, $B_{u,0}(z)$, …, $B_{u,N+1}(z)$, $B_{u,N+2}(z)$, and $B_{v,-2}(z)$, $B_{v,-1}(z)$, $B_{v,0}(z)$, …, $B_{v,N+1}(z)$, $B_{v,N+2}(z)$ form a basis for the function defined over the region $\overline{a} \le z \le \overline{b}$. The values at the knots are given by the following equations.

$$U(z) = \sum_{i=-2}^{N+2} a_{ui} B_{ui}(z) \tag{4}$$

$$V(z) = \sum_{i=-2}^{N+2} a_{vi} B_{vi}(z) \tag{5}$$

where a_{ui} and a_{vi} are coefficients to be determined. There are $N + 5$ collocation points in the domain. The spline functions should be at least one order higher than that of the governing differential equation so that accuracy and smoothness of the approximate solution can be guaranteed [24, 25, 26].

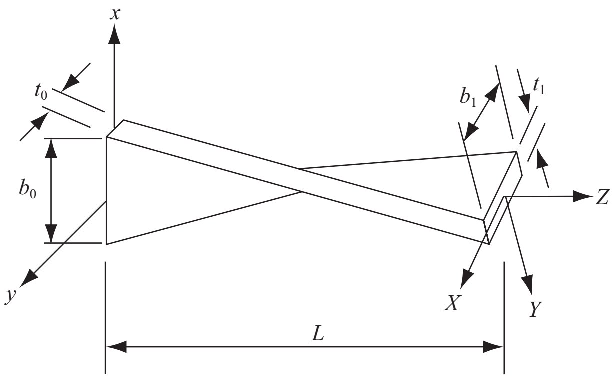

Fig. 1. Geometry of a pre-twisted beam.

III. FORMATION OF THE EIGENVALUE PROBLEM

The spline collocation method is employed to formulate the eigenvalue problems for pre-twisted beams. The pre-twisted beam is shown in Fig. 1.

The length of the pre-twisted beam is L. b_0 and t_0 denote the width and thickness of the pre-twisted beam at $z = 0$, respectively. b_1 and t_1 denote the width and thickness of the pre-twisted beam at $z = L$, respectively. The deflection components u and v are the transverse flexible deflections of the pre-twisted beam. The kinetic energy of the beam, due to the lateral bending vibration [2, 10, 11], is

$$T = \frac{1}{2}\int_0^L \rho A \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right) dz \tag{6}$$

Consider the cross sectional area of the pre-twisted beam material at position z to be

$$A(z) = b_0\, t_0 \left(1 + \alpha \frac{z}{L} \right)\left(1 + \beta \frac{z}{L} \right) \tag{7}$$

where ρ is the density of the pre-twisted beam, and the tapered angles of the pre-twisted beam are

$$\alpha = \frac{b_1 - b_0}{b_0} \tag{8}$$

$$\beta = \frac{t_1 - t_0}{t_0} \tag{9}$$

The strain energy of the pre-twisted beam is [2, 10, 11]

$$U = \frac{1}{2}\int_0^L E \left(I_{yy} \left(\frac{\partial^2 u}{\partial z^2} \right)^2 + 2 I_{xy} \left(\frac{\partial^2 u}{\partial z^2} \right)\left(\frac{\partial^2 v}{\partial z^2} \right) + I_{xx} \left(\frac{\partial^2 v}{\partial z^2} \right)^2 \right) dz \tag{10}$$

In this equation, I_{xx}, I_{yy} and I_{xy} are the area moments of inertia. Consider the tapered beam to be pre-twisted with a uniform pre-twisted angle θ_t and the moments of cross sectional area at the position z can be derived as

$$I_{xx}(z) = I_{XX}\cos^2\left(\frac{z}{L}\theta_t\right) + I_{YY}\sin^2\left(\frac{z}{L}\theta_t\right) \tag{11}$$

$$I_{yy}(z) = I_{XX}\sin^2\left(\frac{z}{L}\theta_t\right) + I_{YY}\cos^2\left(\frac{z}{L}\theta_t\right) \tag{12}$$

$$I_{xy}(z) = (I_{YY} - I_{XX})\sin\left(\frac{z}{L}\theta_t\right)\cos\left(\frac{z}{L}\theta_t\right) \tag{13}$$

where L is the length of the pre-twisted beam and the area moments of inertia with respect to the axes X and Y are

$$I_{XX}(z) = \frac{b_0 t_0^3}{12}\left(1+\alpha\frac{z}{L}\right)\left(1+\beta\frac{z}{L}\right)^3 \tag{14}$$

$$I_{YY}(z) = \frac{b_0^3 t_0}{12}\left(1+\alpha\frac{z}{L}\right)^3\left(1+\beta\frac{z}{L}\right) \tag{15}$$

Hamilton's principle of motion is

$$\int_{t_1}^{t_2}(\delta T - \delta U + \delta W)\,dt = 0 \tag{16}$$

where δW, δT and δU are the virtual work, the variation of kinetic energy and the variation of strain energy, respectively. By using Hamilton principle, the equations of motion of this pre-twisted beam can be derived as:

$$\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 EI_{yy}}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + 2\frac{\partial EI_{yy}}{\partial z}\frac{\partial^3 u}{\partial z^3} + EI_{yy}\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2 EI_{xy}}{\partial z^2}\frac{\partial^2 v}{\partial z^2} + 2\frac{\partial EI_{xy}}{\partial z}\frac{\partial^3 v}{\partial z^3} + EI_{xy}\frac{\partial^4 v}{\partial z^4} = 0 \tag{17}$$

$$\rho A\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 EI_{xx}}{\partial z^2}\frac{\partial^2 v}{\partial z^2} + 2\frac{\partial EI_{xx}}{\partial z}\frac{\partial^3 v}{\partial z^3} + EI_{xx}\frac{\partial^4 v}{\partial z^4} + \frac{\partial^2 EI_{xy}}{\partial z^2}\frac{\partial^2 u}{\partial z^2} + 2\frac{\partial EI_{xy}}{\partial z}\frac{\partial^3 u}{\partial z^3} + EI_{xy}\frac{\partial^4 u}{\partial z^4} = 0 \tag{18}$$

The corresponding boundary conditions of the clamped-free beam are

$$u = 0 \quad \text{at} \quad z = 0 \tag{19a}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{19b}$$

$$v = 0 \quad \text{at} \quad z = 0 \tag{19c}$$

$$\frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{19d}$$

$$EI_{yy}\frac{\partial^2 u}{\partial z^2} + EI_{xy}\frac{\partial^2 v}{\partial z^2} = 0 \quad \text{at} \quad z = L \tag{19e}$$

$$\frac{\partial}{\partial z}\left(EI_{yy}\frac{\partial^2 u}{\partial z^2} + EI_{xy}\frac{\partial^2 v}{\partial z^2}\right) = 0 \quad \text{at} \quad z = L \tag{19f}$$

$$EI_{xx}\frac{\partial^2 v}{\partial z^2} + EI_{xy}\frac{\partial^2 u}{\partial z^2} = 0 \quad \text{at} \quad z = L \tag{19g}$$

$$\frac{\partial}{\partial z}\left(EI_{xx}\frac{\partial^2 v}{\partial z^2} + EI_{xy}\frac{\partial^2 u}{\partial z^2}\right) = 0 \quad \text{at} \quad z = L \tag{19h}$$

The corresponding boundary conditions of the simply supported beam are

$$u = 0 \quad \text{at} \quad z = 0 \tag{20a}$$

$$EI_{yy}\frac{\partial^2 u}{\partial z^2} + EI_{xy}\frac{\partial^2 v}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \tag{20b}$$

$$v = 0 \quad \text{at} \quad z = 0 \tag{20c}$$

$$EI_{xx}\frac{\partial^2 v}{\partial z^2} + EI_{xy}\frac{\partial^2 u}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \tag{20d}$$

$$u = 0 \quad \text{at} \quad z = L \tag{20e}$$

$$EI_{yy}\frac{\partial^2 u}{\partial z^2} + EI_{xy}\frac{\partial^2 v}{\partial z^2} = 0 \quad \text{at} \quad z = L \tag{20f}$$

$$v = 0 \quad \text{at} \quad z = L \tag{20g}$$

$$EI_{xx}\frac{\partial^2 v}{\partial z^2} + EI_{xy}\frac{\partial^2 u}{\partial z^2} = 0 \quad \text{at} \quad z = L \tag{20h}$$

The corresponding boundary conditions of the clamped-clamped beam are

$$u = 0 \quad \text{at} \quad z = 0 \tag{21a}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{21b}$$

$$v = 0 \quad \text{at} \quad z = 0 \tag{21c}$$

$$\frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{21d}$$

$$u = 0 \quad \text{at} \quad z = L \tag{21e}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = L \tag{21f}$$

$$v = 0 \quad \text{at} \quad z = L \tag{21g}$$

$$\frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = L \tag{21h}$$

The system is composed of eight boundary conditions and two coupled governing equations. With the solution assumed to be of the form $u = U\exp(i\omega t)$ and $v = V\exp(i\omega t)$, Eqs. (17) and (18) can then be simplified to

$$\frac{d^2EI_{yy}}{dz^2}\frac{d^2U}{dz^2}+2\frac{dEI_{yy}}{dz}\frac{d^3U}{dz^3}+EI_{yy}\frac{d^4U}{dz^4}+\frac{d^2EI_{xy}}{dz^2}\frac{d^2V}{dz^2}+2\frac{dEI_{xy}}{dz}\frac{d^3V}{dz^3}+EI_{xy}\frac{d^4V}{dz^4}=\omega^2\rho AU \quad (22)$$

$$\frac{d^2EI_{xx}}{dz^2}\frac{d^2V}{dz^2}+2\frac{dEI_{xx}}{dz}\frac{d^3V}{dz^3}+EI_{xx}\frac{d^4V}{dz^4}+\frac{d^2EI_{xy}}{dz^2}\frac{d^2U}{dz^2}+2\frac{dEI_{xy}}{dz}\frac{d^3U}{dz^3}+EI_{xy}\frac{d^4U}{dz^4}=\omega^2\rho AV \quad (23)$$

where ω is the angular frequency of vibration. The corresponding boundary conditions of the clamped-free beam are:

$$U=0 \text{ for } z=0 \quad (24a)$$

$$\frac{dU}{dz}=0 \text{ for } z=0 \quad (24b)$$

$$V=0 \text{ for } z=0 \quad (24c)$$

$$\frac{dV}{dz}=0 \text{ for } z=0 \quad (24d)$$

$$EI_{yy}\frac{d^2U}{dz^2}+EI_{xy}\frac{d^2V}{dz^2}=0 \text{ for } z=L \quad (24e)$$

$$\frac{d}{dz}\left(EI_{yy}\frac{d^2U}{dz^2}+EI_{xy}\frac{d^2V}{dz^2}\right)=0 \text{ for } z=L \quad (24f)$$

$$EI_{xx}\frac{d^2V}{dz^2}+EI_{xy}\frac{d^2U}{dz^2}=0 \text{ for } z=L \quad (24g)$$

$$\frac{d}{dz}\left(EI_{xx}\frac{d^2V}{dz^2}+EI_{xy}\frac{d^2U}{dz^2}\right)=0 \text{ for } z=L \quad (24h)$$

The corresponding boundary conditions of the simple supported beam can be shown as

$$U=0 \text{ for } z=0 \quad (25a)$$

$$EI_{yy}\frac{d^2U}{dz^2}+EI_{xy}\frac{d^2V}{dz^2}=0 \text{ for } z=0 \quad (25b)$$

$$V=0 \text{ for } z=0 \quad (25c)$$

$$EI_{xx}\frac{d^2V}{dz^2}+EI_{xy}\frac{d^2U}{dz^2}=0 \text{ for } z=0 \quad (25d)$$

$$U=0 \text{ for } z=L \quad (25e)$$

$$EI_{yy}\frac{d^2U}{dz^2}+EI_{xy}\frac{d^2V}{dz^2}=0 \text{ for } z=L \quad (25f)$$

$$V=0 \text{ for } z=L \quad (25g)$$

$$EI_{xx}\frac{d^2V}{dz^2}+EI_{xy}\frac{d^2U}{dz^2}=0 \text{ for } z=L \quad (25h)$$

The corresponding boundary conditions of the clamped-clamped beam may be presented as stated below:

$$U=0 \text{ for } z=0 \quad (26a)$$

$$\frac{dU}{dz}=0 \text{ for } z=0 \quad (26b)$$

$$V=0 \text{ for } z=0 \quad (26c)$$

$$\frac{dV}{dz}=0 \text{ for } z=0 \quad (26d)$$

$$U=0 \text{ for } z=L \quad (26e)$$

$$\frac{dU}{dz}=0 \text{ for } z=L \quad (26f)$$

$$V=0 \text{ for } z=L \quad (26g)$$

$$\frac{dV}{dz}=0 \text{ for } z=L \quad (26h)$$

In seeking an efficient discretization technique to acquire an accurate numerical solution with very small number of knots, the spline collocation method is utilized to solve numerically these partial differential equations. By applying the spline collocation method, Eqs. (4) and (5) are substituted into (22) and (23). The equation of motion of a pre-twisted beam can be rearranged into the spline collocation method formula. This leads to

$$\left[\frac{d^2EI_{yy}(z_i)}{dz^2}\frac{d^2B_{u,-2}(z_i)}{dz^2} \quad \frac{d^2EI_{yy}(z_i)}{dz^2}\frac{d^2B_{u,-1}(z_i)}{dz^2} \quad \cdots \quad \frac{d^2EI_{yy}(z_i)}{dz^2}\frac{d^2B_{u,N+2}(z_i)}{dz^2}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \cdots \quad a_{u,N+2}\right]^T+$$

$$\left[2\frac{dEI_{yy}(z_i)}{dz}\frac{d^3B_{u,-2}(z_i)}{dz^3} \quad 2\frac{dEI_{yy}(z_i)}{dz}\frac{d^3B_{u,-1}(z_i)}{dz^3} \quad \cdots \quad 2\frac{dEI_{yy}(z_i)}{dz}\frac{d^3B_{u,N+2}(z_i)}{dz^3}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \cdots \quad a_{u,N+2}\right]^T+$$

$$\left[EI_{yy}(z_i)\frac{d^4B_{u,-2}(z_i)}{dz^4} \quad EI_{yy}(z_i)\frac{d^4B_{u,-1}(z_i)}{dz^4} \quad \cdots \quad EI_{yy}(z_i)\frac{d^4B_{u,N+2}(z_i)}{dz^4}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \cdots \quad a_{u,N+2}\right]^T+$$

$$\left[\frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{v,-2}(z_i)}{dz^2} \quad \frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{v,-1}(z_i)}{dz^2} \quad \cdots \quad \frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{v,N+2}(z_i)}{dz^2}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \cdots \quad a_{u,N+2}\right]^T+$$

$$\left[2\frac{dEI_{xy}(z_i)}{dz}\frac{d^3B_{v,-2}(z_i)}{dz^3} \quad 2\frac{dEI_{xy}(z_i)}{dz}\frac{d^3B_{v,-1}(z_i)}{dz^3} \quad \ldots \right.$$

$$\left. 2\frac{dEI_{xy}(z_i)}{dz}\frac{d^3B_{v,N+2}(z_i)}{dz^3}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T +$$

$$\left[EI_{xy}(z_i)\frac{d^4B_{v,-2}(z_i)}{dz^4} \quad EI_{xy}(z_i)\frac{d^4B_{v,-1}(z_i)}{dz^4} \quad \ldots \right.$$

$$\left. EI_{xy}(z_i)\frac{d^4B_{v,N+2}(z_i)}{dz^4}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T =$$

$$\left[\omega^2\rho A(z_i)B_{u,-2}(z_i) \quad \omega^2\rho A(z_i)B_{u,-1}(z_i) \quad \ldots \right.$$

$$\left. \omega^2\rho A(z_i)B_{u,N+2}(z_i)\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T \quad \text{for}$$

$$i = 0, 1, \ldots, N \tag{27}$$

$$\left[\frac{d^2EI_{xx}(z_i)}{dz^2}\frac{d^2B_{v,-2}(z_i)}{dz^2} \quad \frac{d^2EI_{xx}(z_i)}{dz^2}\frac{d^2B_{v,-1}(z_i)}{dz^2} \quad \ldots \right.$$

$$\left. \frac{d^2EI_{xx}(z_i)}{dz^2}\frac{d^2B_{v,N+2}(z_i)}{dz^2}\right]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T +$$

$$\left[2\frac{dEI_{xx}(z_i)}{dz}\frac{d^3B_{v,-2}(z_i)}{dz^3} \quad 2\frac{dEI_{xx}(z_i)}{dz}\frac{d^3B_{v,-1}(z_i)}{dz^3} \quad \ldots \right.$$

$$\left. 2\frac{dEI_{xx}(z_i)}{dz}\frac{d^3B_{v,N+2}(z_i)}{dz^3}\right]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T +$$

$$\left[EI_{xx}(z_i)\frac{d^4B_{v,-2}(z_i)}{dz^4} \quad EI_{xx}(z_i)\frac{d^4B_{v,-1}(z_i)}{dz^4} \quad \ldots \right.$$

$$\left. EI_{xx}(z_i)\frac{d^4B_{v,N+2}(z_i)}{dz^4}\right]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T$$

$$+\left[\frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{u,-2}(z_i)}{dz^2} \quad \frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{u,-1}(z_i)}{dz^2} \quad \ldots \right.$$

$$\left. \frac{d^2EI_{xy}(z_i)}{dz^2}\frac{d^2B_{u,N+2}(z_i)}{dz^2}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T +$$

$$\left[2\frac{dEI_{xy}(z_i)}{dz}\frac{d^3B_{u,-2}(z_i)}{dz^3} \quad 2\frac{dEI_{xy}(z_i)}{dz}\frac{d^3B_{u,-1}(z_i)}{dz^3} \quad \ldots \right.$$

$$\left. 2\frac{dEI_{xx}(z_i)}{dz}\frac{d^3B_{v,N+2}(z_i)}{dz^3}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T$$

$$+\left[EI_{xy}(z_i)\frac{d^4B_{u,-2}(z_i)}{dz^4} \quad EI_{xy}(z_i)\frac{d^4B_{u,-1}(z_i)}{dz^4} \quad \ldots \right.$$

$$\left. EI_{xy}(z_i)\frac{d^4B_{u,N+2}(z_i)}{dz^4}\right]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T =$$

$$\left[\omega^2\rho A(x_i)B_{v,-2}(x_i) \quad \omega^2\rho A(x_i)B_{v,-1}(x_i) \quad \ldots \right.$$

$$\left. \omega^2\rho A(x_i)B_{v,N+2}(x_i)\right]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T \quad \text{for}$$

$$i = 0, 1, \ldots, N \tag{28}$$

Using the spline collocation method, the boundary conditions of the clamped-free beam can be rearranged into the matrix forms as

$$\left[B_{u,-2}(z_0) \quad B_{u,-1}(z_0) \quad \ldots \quad B_{u,N+2}(z_0)\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{29a}$$

$$\left[\frac{dB_{u,-2}(z_0)}{dz} \quad \frac{dB_{u,-1}(z_0)}{dz} \quad \ldots \quad \frac{dB_{u,N+2}(z_0)}{dz}\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{29b}$$

$$\left[B_{v,-2}(z_0) \quad B_{v,-1}(z_0) \quad \ldots \quad B_{v,N+2}(z_0)\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{29c}$$

$$\left[\frac{dB_{v,-2}(z_0)}{dz} \quad \frac{dB_{v,-1}(z_0)}{dz} \quad \ldots \quad \frac{dB_{v,N+2}(z_0)}{dz}\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{29d}$$

$$\left[EI_{yy}(z_N)\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad EI_{yy}(z_N)\frac{d^2B_{u,-1}(z_N)}{dz^2} \quad \ldots. \right.$$

$$\left. EI_{yy}(z_N)\frac{d^2B_{u,N+2}(z_N)}{dz^2}\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T +$$

$$\left[EI_{xy}(z_N)\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad EI_{xy}(z_N)\frac{d^2B_{v,-1}(z_N)}{dz^2} \quad \ldots. \right.$$

$$\left. EI_{xy}(z_N)\frac{d^2B_{v,N+2}(z_N)}{dz^2}\right]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0]$$

$$\tag{29e}$$

$$\Big[EI_{xx}(z_N)\frac{d^3B_{v,-2}(z_N)}{dz^3} \quad EI_{xx}(z_N)\frac{d^3B_{v,-1}(z_N)}{dz^3} \quad .... \quad EI_{xx}(z_N)\frac{d^3B_{v,N+2}(z_N)}{dz^3}\Big]\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T +$$
$$\Big[\frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad \frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,-1}(z_N)}{dz^2} \quad .... \quad \frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,N+2}(z_N)}{dz^2}\Big]\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T +$$
$$\Big[EI_{xy}(z_N)\frac{d^3B_{u,-2}(z_N)}{dz^3} \quad EI_{xy}(z_N)\frac{d^3B_{u,-1}(z_N)}{dz^3} \quad .... \quad EI_{xy}(z_N)\frac{d^3B_{u,N+2}(z_N)}{dz^3}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T +$$
$$\Big[\frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad \frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{u,-1}(z_N)}{dz^2} \quad .... \quad \frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{N+2}(z_N)}{dz^2}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T = [0] \tag{29f}$$

$$\Big[EI_{xx}(z_N)\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad EI_{xx}(z_N)\frac{d^2B_{v,-1}(z_N)}{dz^2} \quad .... \quad EI_{xx}(z_N)\frac{d^2B_{v,N+2}(z_N)}{dz^2}\Big]$$
$$\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T +$$
$$\Big[EI_{xy}(z_N)\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad EI_{xy}(z_N)\frac{d^2B_{u,-1}(z_N)}{dz^2} \quad .... \quad EI_{xy}(z_N)\frac{d^2B_{u,N+2}(z_N)}{dz^2}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T = [0] \tag{29g}$$

$$\Big[EI_{xx}(z_N)\frac{d^3B_{v,-2}(z_N)}{dz^3} \quad EI_{xx}(z_N)\frac{d^3B_{v,-1}(z_N)}{dz^3} \quad .... \quad EI_{xx}(z_N)\frac{d^3B_{v,N+2}(z_N)}{dz^3}\Big]\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T +$$
$$\Big[\frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad \frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,-1}(z_N)}{dz^2} \quad .... \quad \frac{dEI_{xx}(z_N)}{dz}\frac{d^2B_{v,N+2}(z_N)}{dz^2}\Big]\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T +$$
$$\Big[EI_{xy}(z_N)\frac{d^3B_{u,-2}(z_N)}{dz^3} \quad EI_{xy}(z_N)\frac{d^3B_{u,-1}(z_N)}{dz^3} \quad .... \quad EI_{xy}(z_N)\frac{d^3B_{u,N+2}(z_N)}{dz^3}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T +$$
$$\Big[\frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad \frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{u,-1}(z_N)}{dz^2} \quad .... \quad \frac{dEI_{xy}(z_N)}{dz}\frac{d^2B_{u,N+2}(z_N)}{dz^2}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T = [0] \tag{29h}$$

Using the spline collocation method, the boundary conditions of the simple supported beam can be rearranged into the matrix forms as

$$\Big[B_{u,-2}(z_0) \quad B_{u,-1}(z_0) \quad \dots \quad B_{u,N+2}(z_0)\Big]$$
$$\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T = [0] \tag{30a}$$

$$\Big[EI_{yy}(z_0)\frac{d^2B_{u,-2}(z_0)}{dz^2} \quad EI_{yy}(z_0)\frac{d^2B_{u,-1}(z_0)}{dz^2} \quad \dots \quad EI_{yy}(z_0)\frac{d^2B_{u,N+2}(z_0)}{dz^2}\Big]\Big[a_{u,-2} \quad a_{u,-1} \quad \dots \quad a_{u,N+2}\Big]^T +$$
$$\Big[EI_{xy}(z_0)\frac{d^2B_{v,-2}(z_0)}{dz^2} \quad EI_{xy}(z_0)\frac{d^2B_{v,-1}(z_0)}{dz^2} \quad \dots \quad EI_{xy}(z_0)\frac{d^2B_{v,N+2}(z_0)}{dz^2}\Big]\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T = [0] \tag{30b}$$

$$\Big[B_{v,-2}(z_0) \quad B_{v,-1}(z_0) \quad \dots \quad B_{v,N+2}(z_0)\Big]$$
$$\Big[a_{v,-2} \quad a_{v,-1} \quad \dots \quad a_{v,N+2}\Big]^T = [0] \tag{30c}$$

$$\Big[EI_{xx}(z_0)\frac{d^2B_{v,-2}(z_0)}{dz^2} \quad EI_{xx}(z_0)\frac{d^2B_{v,-1}(z_0)}{dz^2}$$

$$\ldots \quad EI_{xx}(z_0)\frac{d^2B_{v,N+2}(z_0)}{dz^2}\Bigg]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T +$$

$$\Bigg[EI_{xy}(z_0)\frac{d^2B_{u,-2}(z_0)}{dz^2} \quad EI_{xy}(z_0)\frac{d^2B_{u,-1}(z_0)}{dz^2}$$

$$\ldots \quad EI_{xy}(z_0)\frac{d^2B_{u,N+2}(z_0)}{dz^2}\Bigg]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{30d}$$

$$\left[B_{u,-2}(z_N) \quad B_{u,-1}(z_N) \quad \ldots \quad B_{u,N+2}(z_N)\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{30e}$$

$$\Bigg[EI_{yy}(z_N)\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad EI_{yy}(z_N)\frac{d^2B_{u,-1}(z_N)}{dz^2}$$

$$\ldots \quad EI_{yy}(z_N)\frac{d^2B_{u,N+2}(z_N)}{dz^2}\Bigg]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T +$$

$$\Bigg[EI_{xy}(z_N)\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad EI_{xy}(z_N)\frac{d^2B_{v,-1}(z_N)}{dz^2}$$

$$\ldots \quad EI_{xy}(z_N)\frac{d^2B_{v,N+2}(z_N)}{dz^2}\Bigg]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{30f}$$

$$\left[B_{v,-2}(z_N) \quad B_{v,-1}(z_N) \quad \ldots \quad B_{v,N+2}(z_N)\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{30g}$$

$$\Bigg[EI_{xx}(z_N)\frac{d^2B_{v,-2}(z_N)}{dz^2} \quad EI_{xx}(z_N)\frac{d^2B_{v,-1}(z_N)}{dz^2}$$

$$\ldots \quad EI_{xx}(z_N)\frac{d^2B_{v,N+2}(z_N)}{dz^2}\Bigg]\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T +$$

$$\Bigg[EI_{xy}(z_N)\frac{d^2B_{u,-2}(z_N)}{dz^2} \quad EI_{xy}(z_N)\frac{d^2B_{u,-1}(z_N)}{dz^2}$$

$$\ldots \quad EI_{xy}(z_N)\frac{d^2B_{u,N+2}(z_N)}{dz^2}\Bigg]\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{30h}$$

Using the spline collocation method, the boundary conditions of the clamped-clamped beam can be rearranged into the matrix forms as

$$\left[B_{u,-2}(z_0) \quad B_{u,-1}(z_0) \quad \ldots \quad B_{u,N+2}(z_0)\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{31a}$$

$$\left[\frac{dB_{u,-2}(z_0)}{dz} \quad \frac{dB_{u,-1}(z_0)}{dz} \quad \ldots \quad \frac{dB_{u,N+2}(z_0)}{dz}\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{31b}$$

$$\left[B_{v,-2}(z_0) \quad B_{v,-1}(z_0) \quad \ldots \quad B_{v,N+2}(z_0)\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{31c}$$

$$\left[\frac{dB_{v,-2}(z_0)}{dz} \quad \frac{dB_{v,-1}(z_0)}{dz} \quad \ldots \quad \frac{dB_{v,N+2}(z_0)}{dz}\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{31d}$$

$$\left[B_{u,-2}(z_N) \quad B_{u,-1}(z_N) \quad \ldots \quad B_{u,N+2}(z_N)\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{31e}$$

$$\left[\frac{dB_{u,-2}(z_N)}{dz} \quad \frac{dB_{u,-1}(z_N)}{dz} \quad \ldots \quad \frac{dB_{u,N+2}(z_N)}{dz}\right]$$

$$\left[a_{u,-2} \quad a_{u,-1} \quad \ldots \quad a_{u,N+2}\right]^T = [0] \tag{31f}$$

$$\left[B_{v,-2}(z_N) \quad B_{v,-1}(z_N) \quad \ldots \quad B_{v,N+2}(z_N)\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{31g}$$

$$\left[\frac{dB_{v,-2}(z_N)}{dz} \quad \frac{dB_{v,-1}(z_N)}{dz} \quad \ldots \quad \frac{dB_{v,N+2}(z_N)}{dz}\right]$$

$$\left[a_{v,-2} \quad a_{v,-1} \quad \ldots \quad a_{v,N+2}\right]^T = [0] \tag{31h}$$

The eigenvalues of the resultant algebraic equation system provide the natural frequencies of the pre-twisted beam problem.

IV. NUMERICAL RESULTS AND DISCUSSION

Figure 2 shows the calculated natural frequencies of clamped-free beams with different pre-twisted angles. The data for this pre-twisted beam are [4]: $b_0/t_0 = 4$. The non-dimensional natural frequencies of the pre-twisted beam are defined as $\bar{\omega}_i = \omega_i\sqrt{12\rho AL^4/Eb_0t_0^3}$. Numerical results

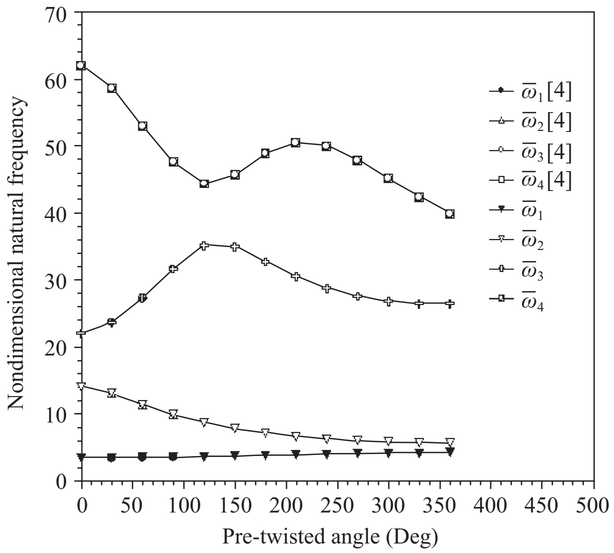

Fig. 2. The calculated natural frequencies of the clamped-free beams with different pre-twisted angles.

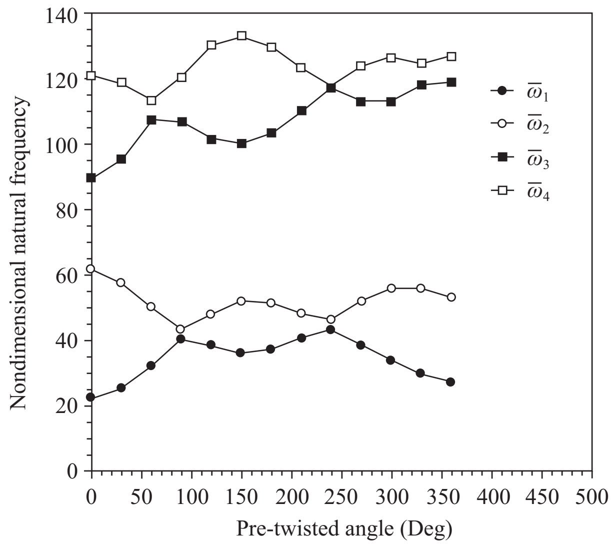

Fig. 4. The natural frequencies of the clamped-clamped beams with various pre-twisted angles.

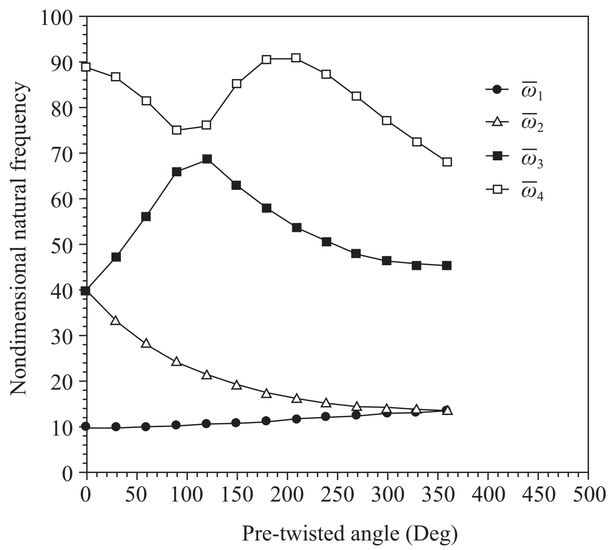

Fig. 3. The natural frequencies of the simple-supported beams with various pre-twisted angles.

indicate that the natural frequencies of a pre-twisted beam calculated using the spline collocation method are shown to be in favorable agreement with the numerical results solved using the Rayleigh-Ritz method [4]. No significant error is found for the results calculated using the spline collocation method. Results indicated that a higher first natural frequency is calculated for the beam with a higher total pre-twisted angle for $\theta_t < 360°$. Numerical results also indicated that the calculated second natural frequency is decreased while the pre-twisted angle increasing for $\theta_t < 360°$.

Figure 3 shows the natural frequencies of simple-supported beams with various pre-twisted angles. Numerical results indicated that a higher first natural frequency is calculated for the beam with a higher pre-twisted angle for $\theta_t < 360°$. Numerical results also indicated that the calculated second natural frequency is decreased while the pre-twisted angle increasing for $\theta_t < 360°$. The pre-twisted angles deeply affect the third and fourth natural frequencies.

Figure 4 shows the natural frequencies of the clamped-clamped beams with various pre-twisted angles. Numerical results indicated that a higher first natural frequency is calculated for the beam with a higher pre-twisted angle for $\theta_t < 90°$. Numerical results also indicated that the calculated second natural frequency is decreased while the pre-twisted angle increasing for $\theta_t < 90°$. Numerical results in this example show that the pre-twisted angle can significantly affect the natural frequencies of the pre-twisted beams.

Figure 5 describes the natural frequencies of the clamped-free beams with various tapered angles β. The fundamental frequencies are almost constant. Numerical results also indicated that the calculated second, third and fourth natural frequencies are increased while the tapered angles increasing.

Figure 6 shows the natural frequencies of the simple-supported beams with various tapered angles β. Numerical results in this example show that the tapered angle can significantly affect the first frequency. The first, second, third and fourth natural frequencies increase with tapered angles β, almost linearly.

Figure 7 shows the natural frequencies of the clamped-clamped beams with various tapered angles β. Numerical results also indicated that the calculated natural frequencies are increased while the tapered angles increasing in general. Numerical results indicate that the tapered angle is a very sensitive parameter for the vibration of the tapered beam. The clamped-clamped boundary conditions give rise to higher natural frequencies of the beams in comparison with the simply supported boundary conditions.

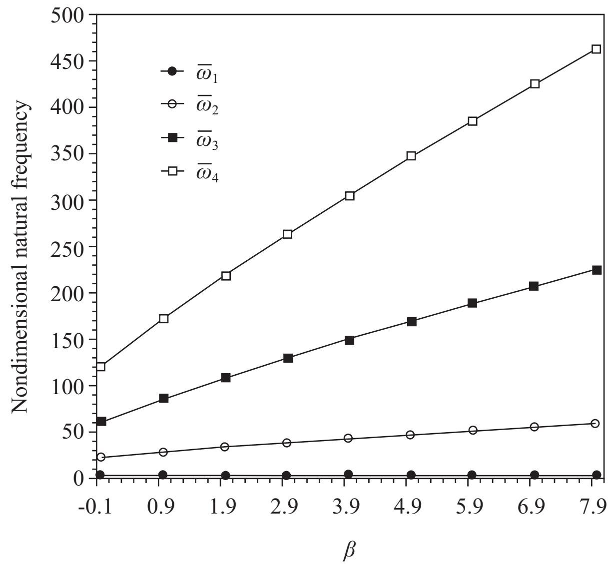

Fig. 5. The natural frequencies of the clamped-free beams with various tapered angles β.

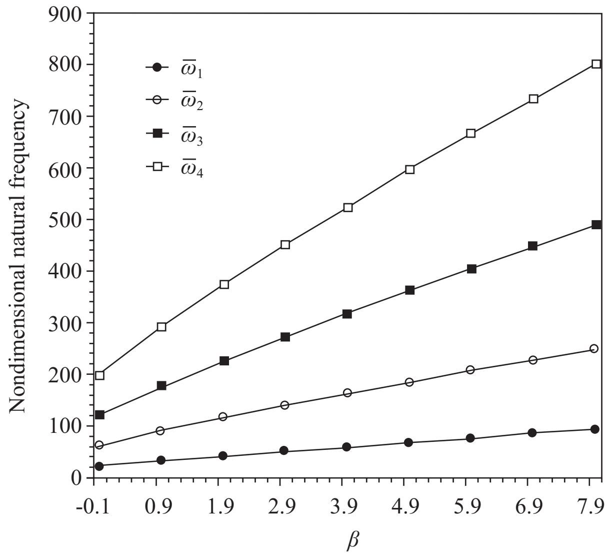

Fig. 7. The natural frequencies of the clamped-clamped beams with various tapered angles β.

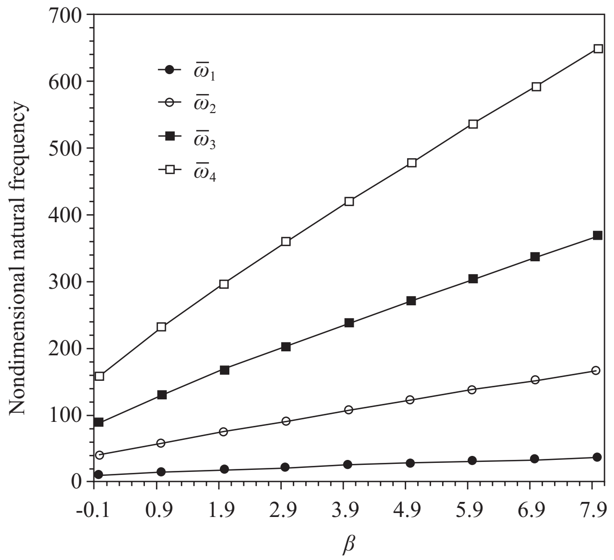

Fig. 6. The natural frequencies of the simple-supported beams for various tapered angles β.

V. CONCLUDING REMARKS

The variation in calculated natural frequencies for the pre-twisted beams using the spline collocation method is investigated. The solution of the governing fourth-order differential equation is approximated by the spline function with polynomial. The efficiency and accuracy of the proposed method is ascertained by comparison with existing solutions. Numerical results in different cases validated the applicability of the proposed method for solving such an engineering problem. The pre-twisted angles influence the natural frequencies of the beams. The demonstrated accuracy and simplicity of the proposed method makes it a good candidate for modeling more complicated pre-twisted beam problems.

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