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HAMILTON DECOMPOSITION OF COMPLETE BIPARTITE 3-UNIFORM HYPERGRAPHS

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Key words: hypergraph, bipartite hypergraph, Hamilton cycle.

ABSTRACT

The problem of finding a Hamilton decomposition of the complete 3-uniform hypergraph K_n^3 has been solved for $n \equiv 2(\text{mod } 3)$, $n \equiv 4(\text{mod } 6)$ [2], $n \equiv 1(\text{mod } 6)$, $n \equiv 0(\text{mod } 3)$ (for $K_n^3 - I$, the complete 3-uniform hypergraph minus a 1-factor) [5]. In this paper, we give the concept of the bipartite hypergraphs and find a Hamilton decomposition of the complete bipartite hypergraph $K_{m,m}^3$ for m be prime.

I. INTRODUCTION

A k-uniform hypergraph H is a pair (V, ε), where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of n vertices and ε is a family of k-subset of V called hyperedges. If ε consists of all k-subset of V, then H is a complete k-uniform hypergraph on n vertices and is denoted by K_n^k. At the same time, we may refer to a vertex $v_i \in V$ as v_{i+n}. A cycle of length l of H is a sequence of the form $(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, e_1)$, where $v_1, v_2, \ldots, v_n$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are k-edges of H, satisfying:

(i) $v_i, v_{i+1} \in e_i$ $1 \le i \le l$, where addition on the subscripts is modulo n, and
(ii) $e_i \neq e_j$ for $i \neq j$

This cycle is known as a Berge cycle, having been introduced by Berge in his book [1]. A Hamilton cycle of a hypergraph H on n vertices is a cycle of length n, and a Hamilton decomposition of H is a partition of the hyperedges of H into Hamilton cycles.

Definition 1. Let H be a hypergraph on V. H is called bipartite if V can be partitioned into two subsets V_1 and V_2 such that $e \cap V_1 \neq \phi$ and $e \cap V_2 \neq \phi$ for any $e \in \varepsilon$. Furthermore, if $|e| = r$ for any $e \in \varepsilon$ then we call H a bipartite r-uniform hypergraph, written $H^r(V_1, V_2)$. H is called the complete bipartite r-uniform hypergraph with vertex-set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$ if $\varepsilon = \{e: e \subseteq V, |e| = r \text{ and } e \cap V_i \neq \phi, \text{ for } i = 1, 2\}$ and denoted it by $K^r(V_1, V_2)$ or $K^r{}_{n,m}$ when $|V_1| = n$, $|V_2| = m$.

A set of Hamilton cycles of $K^r{}_{n,m}$, say $C_1, C_2, \ldots, C_m$ is called a Hamilton decomposition if $\bigcup_{i=1}^m \varepsilon(C_i) = \varepsilon(k_{n,m}^r)$ and $\varepsilon(C_i) \cap \varepsilon(C_j) = \phi$ for $i \neq j$.

In this paper, we give a Hamilton decomposition of complete bipartite hypergraph $K_{m,m}^3$ for m being prime.

II. RESULTS

Let m be a positive integer and let D denote the set of all pairs (k, r) of odd positive integers such that $k < r$.

Given a $(k, r) \in D$ and an integer j, define an edge $e_j(k, r)$ as follows:

(1) if $r \neq m$,
$e_j(k, r) = \{jr, jr + k, (j + 1)r\}$ (mod $2m$);
(2) if $r = m$ and k is odd,
$e_j(k, m) = \{jk, jk + m, (j + 1)k + m\}$ (mod $2m$); In both cases, define

$$C(k, r) = \left\{e_j(k, r) : j = 0, 1, 2, \cdots, 2m-1\right\} \text{ (mod } 2m) \quad (1)$$

Lemma 1 Let $m > 3$ be a prime. Then, for any $(k, r) \in D$, $e_j(k, r) = e_{j'}(k, r)$ if and only if $j \equiv j'$ (mod $2m$).

Proof. By definition it is easily seen that $e_{j+2m}(k, r) = e_j(k, r)$.

Suppose $e_j(k, r) = e_{j'}(k, r)$ with $0 \le j, j' \le 2m–1$. Set $t = j' - j$ and we consider two cases.

Case 1: $r \neq m$. We have that $\{jr, jr + k, (j + 1)r\} \equiv \{j'r, j'r + k, (j' + 1)r\}$ (mod $2m$), which implies that $\{0, k, r\} \equiv \{tr, tr + k_1, (t + 1)r\}$ (mod $2m$). If $tr \neq 0$ (mod $2m$) (equivalently, $tr + k \neq k$ (mod $2m$) and $(t + 1)r \neq r$ (mod $2m$)), then,

(i) $tr \equiv k$ (mod $2m$), $tr + k \equiv r$ (mod $2m$) and $(t + 1)r \equiv 0$ (mod $2m$); or

(ii) $tr \equiv r$ (mod $2m$), $tr + k \equiv 0$ (mod $2m$) and $(t + 1)r \equiv k$ (mod $2m$). Both cases imply that $3k \equiv 0$ (mod $2m$), a contradiction. It shows that $tr \equiv 0$ (mod $2m$). Recall that r is odd and $r \neq m$, in other words, r and $2m$ are coprime, which implies that $j \equiv j'$ (mod $2m$).

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Case 2: $r = m$. In this case we see that m is a prime. We have that $\{jk, jk + m, (j + 1)k + m\} \equiv \{j'k, j'k + m, (j' + 1)k + m\}$ (mod $2m$), which implies that $\{0, m, k + m\} \equiv \{tk, tk + m, (t + 1)k + m\}$ (mod $2m$). If $tk \neq 0$ (mod $2m$) (equivalently, $tk + m \neq m$ (mod $2m$) and $(t + 1)k + m \neq k + m$, (mod $2m$) then, (i) $tk \equiv m$ (mod $2m$) and $(t + 1)k + m \equiv 0$ (mod $2m$)); or (ii) $tk \equiv k + m$ (mod $2m$) and $tk + m \equiv 0$ (mod $2m$). Both cases imply that $k \equiv 0$ (mod $2m$), a contradiction. It shows that $tk \equiv 0$ (mod $2m$). Since m is an odd prime and $k < m$ is an odd integer, we have that k and $2m$ are coprime, which implies that $j \equiv j'$ (mod $2m$).

Lemma 2 Let $m > 3$ be a prime and let $V_1 = \{0, 2, \ldots, 2m–2\}$, $V_2 = \{1, 3, \ldots, 2m–1\}$, and $K^3_{m,m} = K^3(V_1, V_2)$. Then the edge squence $C(k, r)$ defined in (1) and (2) is a Hamilton cycle of $K^3_{m,m}$.

Proof. By the definition of e_j, we see that, for every edge $e_j(k, r)$ of $C(k, r)$, exactly one of the following statements holds:

(1) $|e_j \cap V_1| = 2$ and $|e_j \cap V_2| = 1$, or

(2) $|e_j \cap V_1| = 1$ and $|e_j \cap V_2| = 2$.

From Lemma 1 it follows that $|C(k, r)| = 2m$, for each $(k, r) \in D$. Note that if $r \neq m$, then $(j + 2)r \neq jr$ (mod $2m$) for any integer j, and if $r = m$, then m is prime and $(j + 2)k + m \neq jk + m$ (mod $2m$). From this it is easy to see that

$$e_j(k, r) \cap e_{j+1}(k, r) = \begin{cases} (j+1)r & \text{if } r \neq m, \\ (j+1)k + m & \text{if } r = m. \end{cases}$$

This proves that for each $(k, r) \in D$, $C(k, r)$ is a Hamilton cycle.

Lemma 3 Let (k, r) and (k', r') be two distinct elements of D. Then $C(k, r) \cap C(k', r') = \phi$.

Proof. Let us put the reduced residues modulo $2m$ equidistantly and clockwise on a circle. Take three of them, say, a, b and c. Then $\{a, b, c\} \in C(k, r)$ for some $(k, r) \in D$ if and only if the spaces among the three elements are in turn k, $r–k$ and $2m–r$. Therefore, if $e_j(k, r) = e_{j'}(k', r')$, then the cycle permutations $(k, r–k, 2m–r)$ and $(k', r'–k', 2m–r')$ are identical. Note that there are only $r–k$ and $r'–k'$ are even. We therefore obtain that $k = k'$ and $r–k = r'–k'$, which yields that $(k, r) = (k', r')$.

Theorem 4 Let $m > 3$, m be prime. Then $K^3_{m,m} = \bigcup_{(k,r)\in D} C(k, r)$ is a Hamilton decomposition.

Proof. Let $V_1 = \{0, 2, \ldots, 2m–2\}$, $V_2 = \{1, 3, \ldots, 2m–1\}$, and $K^3_{m,m} = K^3(V_1, V_2)$. By Lemma 2, for any $(k, r) \in D$, $C(k, r)$ is a Hamilton cycle of $K^3(V_1, V_2)$. Therefore, in order to complete the proof it suffices to show that for each 3-element set $\{a, b, c\} \subseteq \{0, 1, \ldots, 2m–1\}$ with $\{a, b, c\} \cap V_1 \neq \phi$ and $\{a, b, c\} \cap V_2 \neq \phi$ there is a $(k, d) \in D$ and an integer j such that $\{a, b, c\} \equiv e_j(k, r)$ (mod $2m$).

Without loss of generality we assume that $a < b < c$. Consider $b – a$, $c – b$, and $2m – c + a$. Since not all of them are even, while their sum is even, there are two among them are odd and one even. We label $b – a$, $c – b$, and $2m – c + a$ as k_1, k_2 and k_3 such that $k_1 \leq k_3$ are odd and k_2 is even. We now complete the proof by six cases.

Case 1: $(k_1, k_2, k_3) = (b – a, c – b, 2m – c + a)$. In this case, put $k = k_1 = b – a$ and $r = k_1 + k_2 = c + a$. If $r \neq m$, then $(r, 2m) = 1$, there is a j such that $a \equiv jr$ (mod $2m$), hence $b \equiv jr + k$ and $c \equiv (j + 1)r$ (mod $2m$), that is, $\{a, b, c\} \equiv e_j(k, r)$ (mod $2m$). If $r = m$, then $(k, 2m) = 1$, there is a j such that $a \equiv jk + m$ (mod $2m$), hence $b \equiv (j + 1)k + m$ and $c \equiv jk$ (mod $2m$), that is, $\{a, b, c\} \equiv e_j(k, m)$ (mod $2m$).

Case 2: $(k_1, k_2, k_3) = (b – a, 2m – c + a, c – b)$. In this case, put $k = k_3 = c – b$ and $r = k_2 + k_3 = 2m – c + a + c – b \equiv a – b$. If $r \neq m$, then $(r, 2m) =1$, there is a j such that $b \equiv jr$ (mod $2m$), hence $c \equiv jr + k$ and $a \equiv (j + 1)r$ (mod $2m$), that is, $\{a, b, c\} \equiv e_j(k, r)$ (mod $2m$). If $r = m$, then $(k, 2m) = 1$, there is a j such that $b \equiv jk + m$ (mod $2m$), hence $c \equiv (j + 1)k + m$ and $a \equiv jk$ (mod $2m$), that is, $\{a, b, c\} \equiv e_j(k, m)$ (mod $2m$).

Case 3: $(k_1, k_2, k_3) = (c – b, b – a, 2m – c + a)$. In this case, put $k = k_3 = 2m – c + a$ and $r = k_2 + k_3 = 2m – c + a + b – a = b – c$. The remainder is similar to Case 2.

Case 4: $(k_1, k_2, k_3) = (c – b, 2m – c + a, b – a)$. In this case, put $k = k_1 = c – b$ and $r = k_1 + k_2 = c – b + 2m – c + a \equiv a – b$ (mod $2m$). The remainder is similar to Case 1.

Case 5: $(k_1, k_2, k_3) = (2m – c + a, b – a, c – b)$. In this case, put $k = k_1 = 2m – c + a$ and $r = k_1 + k_2 = 2m – c + a + b – a = a – b$ (mod $2m$). The remainder is similar to Case 1.

Case 6: $(k_1, k_2, k_3) = (2m – c + a, c – b, b – a)$. In this case, put $k = k_3 = b – a$ and $r = k_2 + k_3 = c – b + b – a = c – a$. The remainder is similar to Case 2.

The proof is completed.

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