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FREE VIBRATION ANALYSIS OF A HYBRID BEAM COMPOSED OF MULTIPLE ELASTIC BEAM SEGMENTS AND ELASTIC-SUPPORTED RIGID BODIES

Hsien-Yuan Lin
Department of Mechanical Engineering, Cheng Shiu University, Kaohsiung, Taiwan, R.O.C., lin.syg@msa.hinet.net

Chin-Yu Wang
Department of Mechanical Engineering, Cheng Shiu University, Kaohsiung, Taiwan, R.O.C.

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Hsien-Yuan Lin and Chin-Yu Wang

Key words: hybrid beam, rigid body, natural frequency, mode shape, exact solution.

ABSTRACT

This paper aims at presenting a method to determine the "exact" natural frequencies and mode shapes of a hybrid beam composed of multiple elastic beam segments and multiple rigid bodies with each rigid body connected with two adjacent elastic beam segments. Furthermore, each rigid body has its own mass and rotary inertia, and is supported by a translational spring and/or a rotational spring. First, based on the equations of the continuity of deformations and the equilibrium of moments and forces for each of the intermediate rigid bodies and boundary conditions, the coefficient matrices of the entire hybrid beam are derived. The overall coefficient matrix for the entire hybrid beam is obtained using the numerical assembly technique. The exact natural frequencies are determined by equating the determinant of the last overall coefficient matrix to zero. With respect to each of the natural frequencies, one may obtain the associated integration constants from the simultaneous equations. Finally, substituting these integration constants into the displacement functions for all the elastic beam segments and replacing the space occupied by each of the rigid bodies by a straight line, one determines each of the corresponding mode shapes of the hybrid beam. Finally, the influence of materials for the elastic beam segments on natural frequencies and mode shapes of the hybrid beam is studied.

I. INTRODUCTION

For the free vibration analysis of an elastic beam carrying a heavy tip body, the authors of Refs. [1, 4, 9, 17] are the pioneers in this aspect. Later, Liu and Huang [14] examined the vibrations of constrained beam carrying a heavy tip body with an elastically restrained condition and effects of the tip mass center. Zhou [20] studied the exact frequencies and mode shapes of a cantilever beam carrying a heavy tip mass with translational and rotational elastic supports. Kopmaz and Telli [7, 8] presented the eigenfrequencies of a two-part beam-mass system consisted of two beam segments carrying a mass. Naguleswaran [16], Banerjee and Sobey [2], and Ilanko [5] presented a set of amended equations of Ref. [7]. Ilanko [6] used the transcendental dynamic stability functions to determine the natural frequencies of a beam connected to a rigid body supported by elastic restraints. In the foregoing literature, by considering influence of "dimension (size)" of the rigid body, the free vibration characteristics for a single beam or a two-part beam carrying "one" rigid body are studied.

Recently, Maiz *et al*. [15] presented the exact natural frequencies of Bernoulli-Euler beams carrying point masses with rotary inertias. Wu and Chen [18, 19] studied the free vibration problem of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements by lumpedmass and continuous-mass transfer matrix methods, respectively. Lin [10-12] presented the exact natural frequencies and mode shapes of a beam carrying a number of concentrated elements. Since each mass and its rotary inertia are located at a "point", it is evident that the influence of dimension (size) for each concentrated element is not considered in the last literature [10-12, 15, 18, 19]. In Ref. [13], Lin presented the "exact" natural frequencies and mode shapes of a single beam carrying a number of elastic-supported rigid bars fixed on the beam. This paper is a continuation of Ref. [13] to present a method for investigating the "exact" natural frequencies and mode shapes of a hybrid beam composed of "multiple" elastic beam segments and "multiple" rigid bodies with effect of "dimension (size)" of each rigid body considered.

Paper submitted 10/29/10; revised 03/12/11; accepted 05/06/11. Author for correspondence: Hsien-Yuan Lin (e-mail: lin.syg@msa.hinet.net).
Department of Mechanical Engineering, Cheng Shiu University, Kaohsiung, Taiwan, R.O.C.

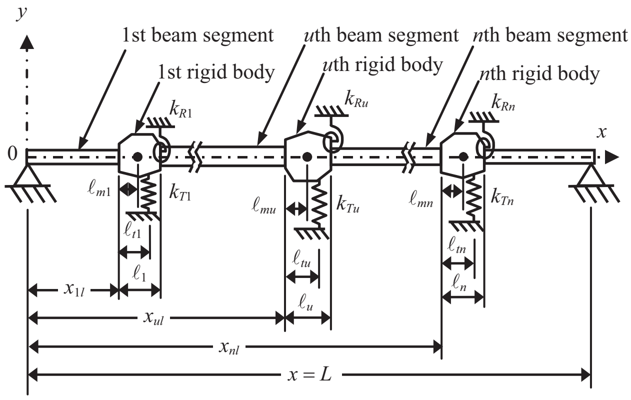

Fig. 1. Sketch for a hybrid beam composed of arbitrary elastic beam segments and elastic-supported rigid bodies in the pinned-pinned (support) condition.

II. FORMULATION OF THE PROBLEM

Fig. 1 shows the sketch of a hybrid beam composed of arbitrary elastic beam segments and elastic-supported rigid bodies with each rigid body connected with two adjacent elastic beam segments, furthermore, each rigid body has its own mass M and rotary inertia J and is supported by a translational spring k_T and a rotational spring k_R. The cross-section of each elastic beam segment is arbitrary (e.g., rectangular, square or circular). Each rigid body has two joints (for connecting with its left and right adjacent elastic beam segments), the position of its left joint is defined by x_{ul} with the subscripts u (u = 1, 2, 3, …) and l denoting the numbering and left joint of the uth rigid body, respectively. For the uth rigid body, its length is denoted by ℓ_u, the distance between its left joint and its center of gravity (represented by the symbol •) is denoted by ℓ_{mu}, and the distance between its left joint and the attaching point of the supporting translational spring is denoted by ℓ_{tu}. It is evident that total length of the entire hybrid beam is denoted by L as one may see from Fig. 1.

Based on the Euler-Bernoulli beam theory, the equation of motion for free vibration of the ith uniform elastic beam segment is given by [3]

$$E_i I_i \frac{\partial^4 y_i(x,t)}{\partial x^4} + \bar{m}_i \frac{\partial^2 y_i(x,t)}{\partial^2 t} = 0 \qquad i = 1, 2, 3, ... \tag{1}$$

where E_i, I_i and $\bar{m}_i$ are Young's modulus, moment of inertia of cross-sectional area and mass per unit length of the ith beam segment, respectively, while $y_i(x, t)$ is the transverse displacement at position x and time t of the ith beam segment.

For free vibrations, one has

$$y_i(x,t) = Y_i(x) e^{j\omega t} \tag{2}$$

where $Y_i(x)$ is amplitude of $y_i(x, t)$, ω is natural frequency of the vibrating system and $j = \sqrt{-1}$.

The substitution of Eq. (2) into Eq. (1) gives

$$Y_i''''(x) - \beta_{v,i}^4 Y_i(x) = 0 \tag{3}$$

where $\beta_{v,i}$ is the dimensional frequency parameter for the ith beam segment corresponding to the vth vibration mode defined by

$$\beta_{v,i}^4 = \frac{\omega_v^2 \bar{m}_i}{E_i I_i} \tag{4a}$$

or

$$\omega_v = \left(\beta_{v,i} L\right)^2 \left(\frac{E_i I_i}{\bar{m}_i L^4}\right)^{1/2} = \Omega_{v,i}^2 \left(\frac{E_i I_i}{\bar{m}_i L^4}\right)^{1/2} \tag{4b}$$

with

$$\Omega_{v,i} = \beta_{v,i} L \tag{4c}$$

It is evident that $\Omega_{v,i}$ is the non-dimensional frequency parameter for the ith beam segment corresponding to the vth vibration mode.

The general solution of Eq. (3) takes the form:

$$\begin{aligned} Y_i(x_i) = {} & C_{i,1} \sin \beta_{v,i} x_i + C_{i,2} \cos \beta_{v,i} x_i + C_{i,3} \sinh \beta_{v,i} x_i \\ & + C_{i,4} \cosh \beta_{v,i} x_i \end{aligned} \tag{5}$$

which is the displacement function for the ith beam segment located at left side of the ith rigid body. It is noted that $i \equiv u$ as one may see from Fig. 1.

1. Coefficient Matrix [B_u] for an Intermediate Rigid Body

If the numbering of an intermediate rigid body is u, then the continuity of deformations and the equilibrium of moments and forces (cf. Fig. 1) at the uth rigid body require that

$$Y_u(\xi_{ul}) + \frac{\ell_u}{L} Y_u'(\xi_{ul}) = Y_{u+1}(\xi_{ur}) \tag{6a}$$

$$Y_u'(\xi_{ul}) = Y_{u+1}'(\xi_{ur}) \tag{6b}$$

$$\begin{aligned} & E_u I_u \frac{1}{L^2} Y_u''(\xi_{ul}) - \left[J_u \omega^2 - k_{Ru} - k_{Tu} \left(\ell_{tu} - \ell_{mu}\right)^2 \right] \frac{1}{L} Y_u'(\xi_{ul}) \\ & + k_{Tu} \left(\ell_{tu} - \ell_{mu}\right) Y_u(\xi_{ul}) + E_u I_u \frac{\ell_{mu}}{L^3} Y_u'''(\xi_{ul}) \\ & = E_{u+1} I_{u+1} \frac{1}{L^2} Y_{u+1}''(\xi_{ur}) - E_{u+1} I_{u+1} \frac{\left(\ell_u - \ell_{mu}\right)}{L^3} Y_{u+1}'''(\xi_{ur}) \end{aligned} \tag{6c}$$

$$\begin{aligned} & E_u I_u \frac{1}{L^3} Y_u'''(\xi_{ul}) + \left[M_u \ell_{mu} \omega^2 - k_{Tu} \left(\ell_{tu} - \ell_{mu}\right) \right] \frac{1}{L} Y_u'(\xi_{ul}) \\ & + \left(M_u \omega^2 - k_{Tu}\right) Y_u(\xi_{ul}) = E_{u+1} I_{u+1} \frac{1}{L^3} Y_{u+1}'''(\xi_{ur}) \end{aligned} \tag{6d}$$

where $\xi_{ul} = x_{ul}/L$ and $\xi_{ur} = (x_{ul} + \ell_u)/L$ are the non- dimensional coordinates of left joint and right joint of the uth rigid body, respectively. In Eqs. (6a)-(6d), the primes refer to differentiations with the respect to the non-dimensional coordinate $\xi_u = x_u/L$.

From Eqs. (5) and (6a)-(6d) one obtains

$$\begin{pmatrix} C_{u,1}\sin\Omega_{v,u}\xi_{ul} + C_{u,2}\cos\Omega_{v,u}\xi_{ul} \\ +C_{u,3}\sinh\Omega_{v,u}\xi_{ul} + C_{u,4}\cosh\Omega_{v,u}\xi_{ul} \end{pmatrix} + \Omega_{v,u}\ell^*_u \begin{pmatrix} C_{u,1}\cos\Omega_{v,u}\xi_{ul} - C_{u,2}\sin\Omega_{v,u}\xi_{ul} \\ +C_{u,3}\cosh\Omega_{v,u}\xi_{ul} + C_{u,4}\sinh\Omega_{v,u}\xi_{ul} \end{pmatrix} - \begin{pmatrix} C_{u+1,1}\sin\Omega_{v,u+1}\xi_{ur} + C_{u+1,2}\cos\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,3}\sinh\Omega_{v,u+1}\xi_{ur} + C_{u+1,4}\cosh\Omega_{v,u+1}\xi_{ur} \end{pmatrix} = 0 \tag{7a}$$

$$\Omega_{v,u}\begin{pmatrix} C_{u,1}\cos\Omega_{v,u}\xi_{ul} - C_{u,2}\sin\Omega_{v,u}\xi_{ul} \\ +C_{u,3}\cosh\Omega_{v,u}\xi_{ul} + C_{u,4}\sinh\Omega_{v,u}\xi_{ul} \end{pmatrix} - \Omega_{v,u+1}\begin{pmatrix} C_{u+1,1}\cos\Omega_{v,u+1}\xi_{ur} - C_{u+1,2}\sin\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,3}\cosh\Omega_{v,u+1}\xi_{ur} + C_{u+1,4}\sinh\Omega_{v,u+1}\xi_{ur} \end{pmatrix} = 0 \tag{7b}$$

$$\begin{aligned}
&\left[-\Omega^2_{v,u} + k^*_{Tu}\left(\frac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)\right]\begin{pmatrix} C_{u,1}\sin\Omega_{v,u}\xi_{ul} \\ +C_{u,2}\cos\Omega_{v,u}\xi_{ul} \end{pmatrix} + \left[\Omega^2_{v,u} + k^*_{Tu}\left(\frac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)\right]\begin{pmatrix} C_{u,3}\sinh\Omega_{v,u}\xi_{ul} \\ +C_{u,4}\cosh\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&- \begin{bmatrix} J^*_u\left(\dfrac{\bar{m}_1}{\bar{m}_u}\right)\Omega^5_{v,u} - k^*_{Ru}\left(\dfrac{E_1I_1}{E_uI_u}\right)\Omega_{v,u} \\ -k^*_{Tu}\left(\dfrac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)^2\Omega_{v,u} + \ell^*_{mu}\Omega^3_{v,u} \end{bmatrix}\begin{pmatrix} C_{u,1}\cos\Omega_{v,u}\xi_{ul} \\ -C_{u,2}\sin\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&- \begin{bmatrix} J^*_u\left(\dfrac{\bar{m}_1}{\bar{m}_u}\right)\Omega^5_{v,u} - k^*_{Ru}\left(\dfrac{E_1I_1}{E_uI_u}\right)\Omega_{v,u} \\ -k^*_{Tu}\left(\dfrac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)^2\Omega_{v,u} - \ell^*_{mu}\Omega^3_{v,u} \end{bmatrix}\begin{pmatrix} C_{u,3}\cosh\Omega_{v,u}\xi_{ul} \\ +C_{u,4}\sinh\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&+\sigma_u\varepsilon_u\left(\ell^*_u - \ell^*_{mu}\right)\Omega^3_{v,u+1}\begin{pmatrix} -C_{u+1,1}\cos\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,2}\sin\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,3}\cosh\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,4}\sinh\Omega_{v,u+1}\xi_{ur} \end{pmatrix} \\
&-\sigma_u\varepsilon_u\Omega^2_{v,u+1}\begin{pmatrix} -C_{u+1,1}\sin\Omega_{v,u+1}\xi_{ur} - C_{u+1,2}\cos\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,3}\sinh\Omega_{v,u+1}\xi_{ur} + C_{u+1,4}\cosh\Omega_{v,u+1}\xi_{ur} \end{pmatrix} = 0
\end{aligned} \tag{7c}$$

$$\begin{aligned}
&\begin{bmatrix} M^*_u\left(\dfrac{\bar{m}_1}{\bar{m}_u}\right)\Omega^4_{v,u} \\ -k^*_{Tu}\left(\dfrac{E_1I_1}{E_uI_u}\right) \end{bmatrix}\begin{pmatrix} C_{u,1}\sin\Omega_{v,u}\xi_{ul} + C_{u,2}\cos\Omega_{v,u}\xi_{ul} \\ +C_{u,3}\sinh\Omega_{v,u}\xi_{ul} + C_{u,4}\cosh\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&+ \begin{bmatrix} M^*_u\left(\dfrac{\bar{m}_1}{\bar{m}_u}\right)\ell^*_{mu}\Omega^5_{v,u} \\ -k^*_{Tu}\left(\dfrac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)\Omega_{v,u} - \Omega^3_{v,u} \end{bmatrix}\begin{pmatrix} C_{u,1}\cos\Omega_{v,u}\xi_{ul} \\ -C_{u,2}\sin\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&+ \begin{bmatrix} M^*_u\left(\dfrac{\bar{m}_1}{\bar{m}_u}\right)\ell^*_{mu}\Omega^5_{v,u} \\ -k^*_{Tu}\left(\dfrac{E_1I_1}{E_uI_u}\right)\left(\ell^*_{tu} - \ell^*_{mu}\right)\Omega_{v,u} + \Omega^3_{v,u} \end{bmatrix}\begin{pmatrix} C_{u,3}\cosh\Omega_{v,u}\xi_{ul} \\ +C_{u,4}\sinh\Omega_{v,u}\xi_{ul} \end{pmatrix} \\
&-\sigma_u\varepsilon_u\Omega^3_{v,u+1}\begin{pmatrix} -C_{u+1,1}\cos\Omega_{v,u+1}\xi_{ur} + C_{u+1,2}\sin\Omega_{v,u+1}\xi_{ur} \\ +C_{u+1,3}\cosh\Omega_{v,u+1}\xi_{ur} + C_{u+1,4}\sinh\Omega_{v,u+1}\xi_{ur} \end{pmatrix} = 0
\end{aligned} \tag{7d}$$

where

$$M^*_u = \frac{M_u}{\bar{m}_1 L},\ J^*_u = \frac{J_u}{\bar{m}_1 L^3},\ k^*_{Ru} = \frac{k_{Ru}L}{E_1I_1},\ k^*_{Tu} = \frac{k_{Tu}L^3}{E_1I_1},$$
$$\ell^*_u = \frac{\ell_u}{L},\ \ell^*_{mu} = \frac{\ell_{mu}}{L},\ \ell^*_{tu} = \frac{\ell_{tu}}{L},\ \sigma_u = \frac{E_{u+1}}{E_u},\ \varepsilon_u = \frac{I_{u+1}}{I_u} \tag{8a,b,c,d,e,f,g,h,i}$$

Writing Eqs. (7a)-(7d) in matrix form, one has

$$\left[B_u\right]\left\{C_u\right\} = 0 \tag{9}$$

where

$$\left\{C_u\right\} = \left\{C_{u,1}\quad C_{u,2}\quad C_{u,3}\quad C_{u,4}\quad C_{u+1,1}\quad C_{u+1,2}\quad C_{u+1,3}\quad C_{u+1,4}\right\} \tag{10}$$

In the above Eqs. (9) and (10), the symbols, [ ] and { }, denote the rectangular matrix and column vector, respectively. The coefficient matrix $[B_u]$ is given by Eq. (A1) as one may see from Appendix A at the end of this paper.

2. Coefficient Matrix $[B_0]$ for the Left End of the Entire Hybrid Beam

If the left-end support of the beam is "pinned" as shown in Fig. 1, then the boundary conditions are

$$Y_0(0) = Y''_0(0) = 0 \tag{11a,b}$$

From Eqs. (5), (11a) and (11b), one obtains

$$C_{0,2} + C_{0,4} = 0 \tag{12a}$$

$$-C_{0,2} + C_{0,4} = 0 \tag{12b}$$

or in matrix form

$$[B_0]\{C_0\} = 0 \tag{13}$$

where

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \tag{14}$$

$$\{C_0\} = \{C_{0,1} \quad C_{0,2} \quad C_{0,3} \quad C_{0,4}\} \tag{15}$$

Similarly, if the left-end support of the beam is "clamped", one obtains the following boundary coefficient matrix

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \tag{16}$$

3. Coefficient Matrix $[B_{n+1}]$ for the Right End of the Entire Hybrid Beam

If the right-end support of the beam is "pinned" as shown in Fig. 1, then the boundary conditions are

$$Y_{n+1}(L) = Y''_{n+1}(L) = 0 \tag{17a,b}$$

where n is the total number of (intermediate) rigid bodies.

From Eqs. (5), (17a) and (17b), one obtains

$$\begin{aligned} &C_{n+1,1}\sin\Omega_{v,n+1} + C_{n+1,2}\cos\Omega_{v,n+1} + C_{n+1,3}\sinh\Omega_{v,n+1} \\ &\quad + C_{n+1,4}\cosh\Omega_{v,n+1} = 0 \end{aligned} \tag{18a}$$

$$\begin{aligned} &-C_{n+1,1}\sin\Omega_{v,n+1} - C_{n+1,2}\cos\Omega_{v,n+1} + C_{n+1,3}\sinh\Omega_{v,n+1} \\ &\quad + C_{n+1,4}\cosh\Omega_{v,n+1} = 0 \end{aligned} \tag{18b}$$

or

$$[B_{n+1}]\{C_{n+1}\} = 0 \tag{19}$$

where

$$[B_{n+1}] = \begin{array}{c} \begin{array}{cccc} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{array} \\ \begin{bmatrix} \sin\Omega_{v,n+1} & \cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \\ -\sin\Omega_{v,n+1} & -\cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array} \tag{20}$$

$$\{C_{n+1}\} = \{C_{n+1,1} \quad C_{n+1,2} \quad C_{n+1,3} \quad C_{n+1,4}\} \tag{21}$$

In Eq. (20), q denotes the total number of equations for the integration constants given by

$$q = 4(n+1) \tag{22}$$

Similarly, if the right-end support of the beam is "free", one obtains the following boundary coefficient matrix

$$[B_{n+1}] = \begin{array}{c} \begin{array}{cccc} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{array} \\ \begin{bmatrix} -\sin\Omega_{v,n+1} & -\cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \\ -\cos\Omega_{v,n+1} & \sin\Omega_{v,n+1} & \cosh\Omega_{v,n+1} & \sinh\Omega_{v,n+1} \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array} \tag{23}$$

The integration constants relating to the left-end and right-end supports of the hybrid beam are defined by Eqs. (15) and (21), respectively, while those relating to the intermediate rigid bodies are defined by Eqs. (10). The associated coefficient matrices are given by $[B_0]$ (cf. Eq. (14) or (16)), $[B_u]$ (cf. Appendix A), and $[B_{n+1}]$ (cf. Eq. (20) or (23)). From the last equations concerned one may see that the identification number for each element of the foregoing coefficient matrices is shown on the top side and right side of each matrix. Therefore using the numerical assembly technique, one may obtain a matrix equation for all the integration constants of the entire hybrid beam

$$[\bar{B}]\{\bar{C}\} = 0 \tag{24}$$

Non-trivial solution of Eq. (24) requires that its coefficient determinant is equal to zero, i.e.,

$$|\bar{B}| = 0 \tag{25}$$

which is the frequency equation for the present problem.

In this paper, the incremental search method is used to find the natural frequencies of the vibrating system, ω_v (v = 1, 2, …). With respect to each natural frequency ω_v, one may obtain the corresponding integration constants from Eq. (24). Substituting the last integration constants into displacement functions of the associated elastic beam segments and replacing the space occupied by each rigid-body by a straight line, one determines the corresponding mode shape of the entire hybrid beam, $Y^{(v)}(\xi)$.

III. NUMERICAL RESULTS AND DISCUSSIONS

Before the free vibration analysis of a hybrid beam composed of multiple elastic beam segments and multiple elastic-supported rigid bodies is performed, the reliability of the theory and the computer program developed for this paper are

Table 1. The lowest three natural frequency parameters of a hybrid beam composed of two elastic beam segments and one rigid body as shown in Fig. 2.

Boundary conditions	Methods	$\omega_v\sqrt{\rho_1 A_1 L^4/(E_1 I_1)}$		
		$v = 1$	$v = 2$	$v = 3$
P-P	Present	8.1278	35.0234	88.9239
	Ref. [2]	8.1278	35.023	88.924

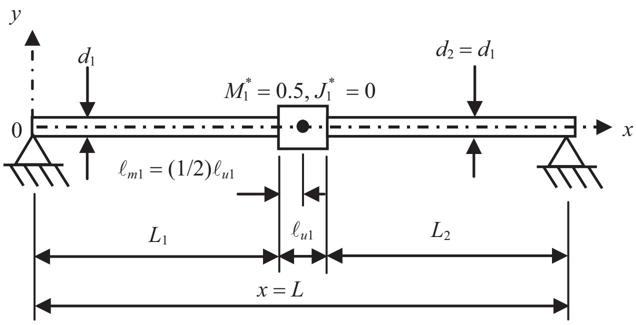

Fig. 2. A hybrid beam composed of two "elastic beam segments" and one "rigid body" in the pinned-pinned (support) conditions.

confirmed by comparing the present results with those obtained from the existing literature.

1. Reliability of Presented Theory and Developed Computer Program

The first example studied is a hybrid beam composed of two elastic beam segments and one rigid body in the pinned-pinned (support) condition as shown in Fig. 2. The non-dimensional lengths of the two elastic beam segments with the same material and cross section are $L_1^* = L_1/L = 0.3$ and $L_2^* = L_2/L = 0.65$, respectively. The non-dimensional length, mass and rotary inertia of the rigid body are $\ell_{u1}^* = \ell_{u1}/L = 0.05$, $J_1^* = 0$ and $\bar{M}_1^* = M_1/(\bar{m}_1 \times L_1 + \bar{m}_2 \times L_2) = 0.5$, respectively. The non-dimensional distance between center of gravity (c.g.) of the rigid body and its left joint is $\ell_{mu}^* = (1/2)\ell_{u1}^*$. The lowest three natural frequency parameters for the hybrid beam are shown in Table 1. From Table 1 one sees that the results of the present paper are in excellent agreement with those of Ref. [2].

2. Free Vibration Analysis of a Hybrid Beam Composed of Four "Elastic Beam Segments" and Three "Elastic-Supported Rigid Bodies" in the Pinned-Pinned (Support) Conditions

Fig. 3 shows the example studied in this paper, it is a hybrid beam composed of four "elastic beam segments" and three "elastic-supported rigid bodies" in the pinned-pinned (support) conditions. The total length of entire hybrid beam is $L = 2\,\text{m}$, the cross-sections of all elastic beam segments are circular, but

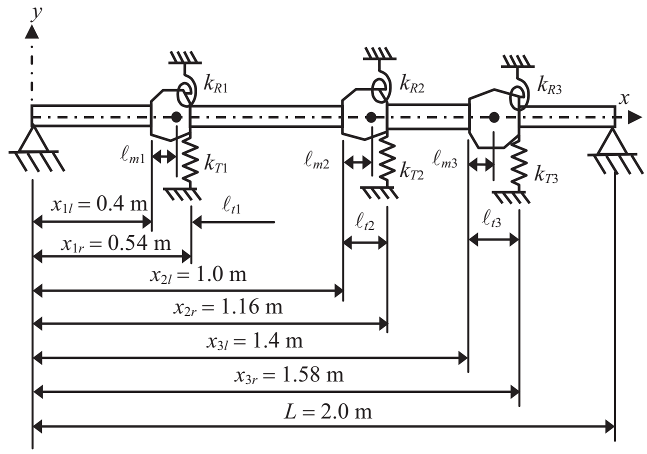

Fig. 3. A hybrid beam composed of four "elastic beam segments" and three "elastic-supported rigid bodies" in the pinned-pinned conditions.

the diameters of the 1st, 2nd and 4th elastic beam segments are equal to 0.05 m, i.e., $d_i = 0.05$ m ($i = 1, 2, 4$), and that of the 3rd elastic beam segment is 0.06 m, i.e., $d_3 = 0.06$ m. The material for 1st, 2nd and 4th elastic beam segments is steel with Young's modulus $E_i = 2.068 \times 10^{11}$ N/m^2, mass density $\rho_i = 7850$ kg/m^3 ($i = 1, 2, 4$). Three cases with different kinds of material for the 3rd elastic beam segment are studied: For the first case, the material of the 3rd elastic beam segment is *steel* with $E_3 = 2.068 \times 10^{11}$ N/m^2 and $\rho_3 = 7850$ kg/m^3. For the second case, the material of the 3rd elastic beam segment is *copper* with $E_3 = 1.05 \times 10^{11}$ N/m^2 and $\rho_3 = 8970$ kg/m^3. For the third case, the material of the 3rd beam segment is *aluminum* with $E_3 = 0.72 \times 10^{11}$ N/m^2 and $\rho_3 = 2790$ kg/m^3.

For convenience, based on the first elastic beam segment, four *reference* parameters are introduced: *reference* mass $\hat{M} = \bar{m}_1 L = \rho_1(\pi/4)d_1^2 L$ kg, *reference* rotary inertia $\hat{J} = \bar{m}_1 L^3$ kg · m^2, *reference* stiffness for translational spring $\hat{k}_T = E_1 I_1/L^3 = E_1(\pi/64)d_1^4/L^3$ N/m and *reference* stiffness for rotational spring $\hat{k}_R = E_1 I_1/L$ N · m/rad. With respect to the last four *reference* parameters, four non-dimensional parameters are also are introduced: $M^* = M/\hat{M}$, $J^* = J/\hat{J}$, $k_R^* = k_R/\hat{k}_R$ and $k_T^* = k_T/\hat{k}_T$.

Furthermore, the non-dimensional parameters for the three "rigid bodies" are as follows: positions $\xi_{1l} = x_{1l}/L = 0.2$, $\xi_{2l} = 0.5$ and $\xi_{3l} = 0.7$; lengths $\ell_1^* = \ell_1/L = 0.07$, $\ell_2^* = 0.08$ and $\ell_3^* = 0.09$; masses $M_1^* = 0.2$, $M_2^* = 0.3$ and $M_3^* = 0.4$; rotary inertias $J_1^* = 0.02$, $J_2^* = 0.03$ and $J_3^* = 0.04$; translational springs $k_{T1}^* = 20$, $k_{T2}^* = 30$, $k_{T3}^* = 40$; rotational springs $k_{R1}^* = 10$, $k_{R2}^* = 8$, $k_{R3}^* = 8$; The distances between left joint and center of gravity for each of the rigid bodies are $\ell_{m1}^* = (2/3)\ell_1^*$, $\ell_{m2}^* =$

Table 2. The lowest four natural frequencies of a hybrid beam composed of four "elastic beam segments" and three "elastic-supported rigid bodies" in the pinned-pinned (support) conditions as shown in Fig. 3.

Cases	Materials for 3rd beam segment	Natural frequencies, ω_v (rad/sec)			
		ω_1	ω_2	ω_3	ω_4
1	Steel	238.3129	660.15542	1282.34191	1804.72916
2	Copper	228.3144	624.60176	1234.40876	1635.87796
3	Aluminum	237.2785	620.59035	1223.22105	1597.74993

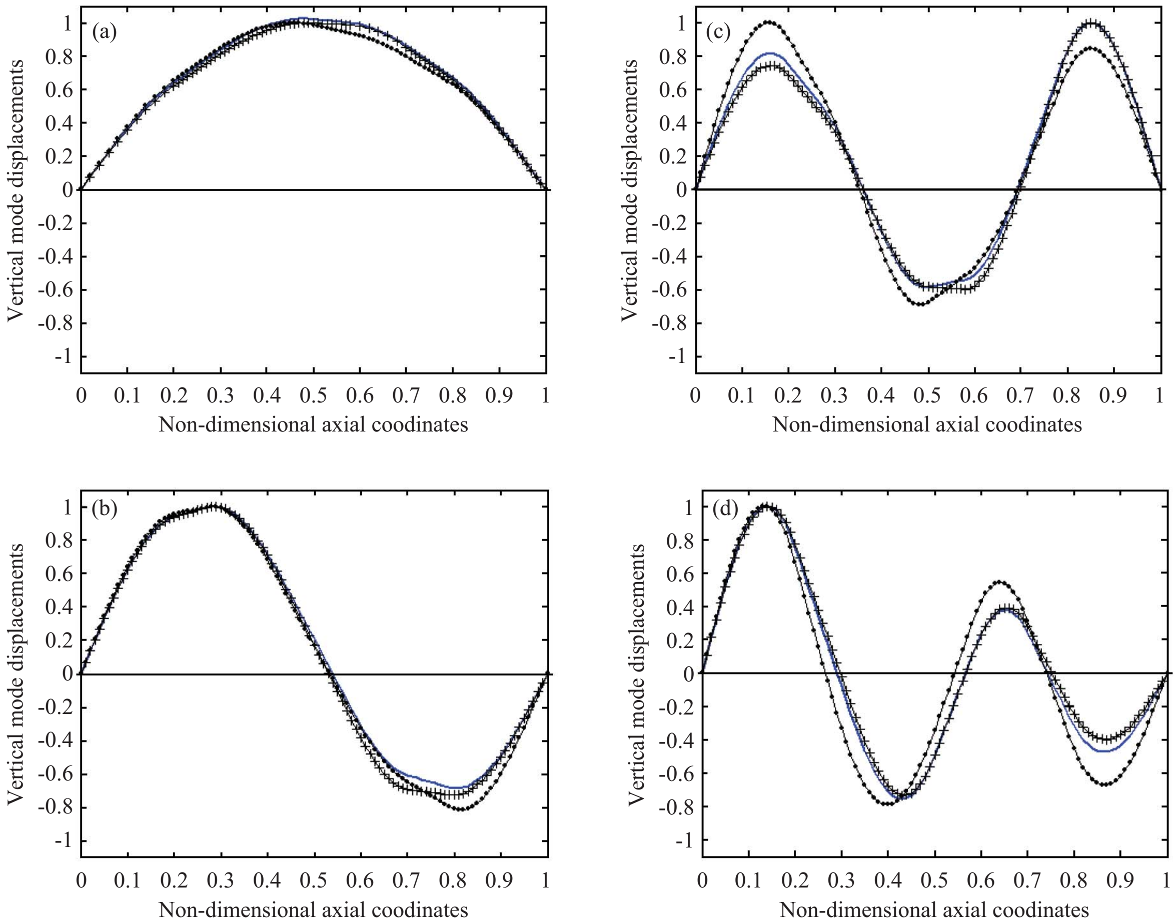

Fig. 4. The lowest four mode shapes for the hybrid beam composed of four elastic beam segments and three elastic-supported rigid bodies in the pinned-pinned (support) conditions as shown in Fig. 3: (a) 1st, (b) 2nd, (c) 3rd and (d) 4th mode shapes. –•–•–•–•–•–•– case 1 (material of the 3rd elastic beam segment being *steel*), ——— case 2 (material being *copper*) and +++++++ case 3 (material being *aluminum*).

$(2/3)\ell_2^*$ and $\ell_{m3}^* = (1/2)\ell_3^*$; the distances between left joint for each of the rigid bodies and attaching point for each of the supporting translational springs are $\ell_{t1}^* = \ell_1^* = 0.07$, $\ell_{t2}^* = \ell_2^* = 0.08$ and $\ell_{t3}^* = \ell_3^* = 0.09$.

The lowest four natural frequencies of the hybrid beam with three kinds of material for the 3rd elastic beam segment are shown in Table 2. From Table 2 one sees that the values of ω_v (v = 1 to 4) obtained from case 1 shown in 1st row (with material of the 3rd elastic beam segment to be *steel*) are higher than the corresponding ones obtained from case 2 shown in 2nd row (with material of the 3rd elastic beam segment to be *copper*), this is reasonable because the *copper* beam segment has lower stiffness (Young's modulus) and higher mass density. Furthermore, from Table 2 one also sees that the values of ω_v (v = 1 to 4) obtained from case 1 shown in 1st row (with material of the 3rd elastic beam segment to be *steel*) are also higher than the corresponding ones obtained from case 3 shown in 3rd row (with material of the 3rd elastic beam segment to be *aluminum*), this is due to the stiffness effect of the *aluminum* beam segment to be greater than its inertia effect for the current hybrid beam, although the mass density ρ and Young's modulus E of *aluminum* are approximately equal to one third (1/3) of the corresponding ones of *steel*, respectively.

Corresponding to the four natural frequencies listed in Table 2, the lowest four mode shapes of the hybrid beam are shown in Figs. 4(a)-(d), respectively, where (a), (b), (c) and (d) refer to the 1st, 2nd, 3rd and 4th mode shapes of the hybrid beam, respectively. Besides, the curves –•–•–•–•–•–•–, ——— and +++++++ denote the mode shapes of the hybrid beam with materials of its 3rd elastic beam segment to be *steel*, *copper* and *aluminum*, respectively. It is noted that the space occupied by each rigid body is replaced by a straight line in each mode shape because the deformation of each rigid body is nil during vibrations.

IV. CONCLUSIONS

Based on the foregoing investigations, one obtains the following conclusions:

1. Since the literature regarding the "exact" natural frequencies and the associated mode shapes for a hybrid beam composed of more than two elastic beam segments and two rigid bodies is rare, the exact-solution method presented in this paper will be significant in this aspect.
2. The numerical results of this paper reveal that the natural frequencies and associated mode shapes of a hybrid beam are significantly dependent on the materials of its elastic beam segments. Therefore, compared with the conventional beam with single material and without composing of rigid bodies, a hybrid beam such as that studied in this paper can provide a lager range of variations of natural frequencies and mode shapes. This should be useful for the practical applications.

NOMENCLATURE

d_i	diameter of the *i*th beam segment
E_i	Young's modulus of the *i*th beam segment
i	numbering for the *i*th beam segment
I_i	moment of inertia of cross-sectional area of the *i*th beam segment
j	$\sqrt{-1}$
J_u	rotary inertia of the *u*th rigid body
J_u^*	non-dimensional rotary inertia of the *u*th rigid body $J_u^* = \dfrac{J_u}{\bar{m}_1 L^3}$
k_{Ru}	stiffness of rotational spring supporting the *u*th rigid body
k_{Ru}^*	non-dimensional stiffness of rotational spring supporting the *u*th rigid body $k_{Ru}^* = \dfrac{k_{Ru} L}{E_1 I_1}$
k_{Tu}	stiffness of translational spring supporting the *u*th rigid body
k_{Tu}^*	non-dimensional stiffness of translational spring supporting the *u*th rigid body $k_{Tu}^* = \dfrac{k_{Tu} L^3}{E_1 I_1}$
L	total length of the entire hybrid beam
ℓ_u	length of the *u*th rigid body
ℓ_u^*	non-dimensional length of the *u*th rigid body $\ell_u^* = \dfrac{\ell_u}{L}$
ℓ_{mu}	distance between center of gravity (c.g.) of the *u*th rigid body and its left joint
ℓ_{mu}^*	non-dimensional distance between center of gravity (c.g.) of the *u*th rigid body and its left joint $\ell_{mu}^* = \dfrac{\ell_{mu}}{L}$
ℓ_{tu}	distance between attaching point of the translational springs and the left joint of the supported *u*th rigid body
ℓ_{tu}^*	non-dimensional distance between attaching point of the translational springs and the left joint of the supported *u*th rigid body $\ell_{tu}^* = \dfrac{\ell_{tu}}{L}$
$\bar{m}_i$	mass per unit length of the *i*th beam segment
M_u	mass of the *u*th rigid body
M_u^*	non-dimensional mass of the *u*th rigid body $M_u^* = \dfrac{M_u}{\bar{m}_1 L}$
n	total number of (intermediate) rigid bodies
q	total number of equations for the integration constants
v	the *v*th vibration mode
x_{ul}	coordinate for left joint of the *u*th rigid body
$y_i(x, t)$	transverse displacement at position *x* and time *t* for the *i*th beam segment
$Y_i(x)$	amplitude function of $y_i(x, t)$
$\beta_{v,i}$	dimensional frequency parameter for the *i*th beam segment corresponding to the *v*th vibration mode $\beta_{v,i}^4 = \dfrac{\omega_v^2 \bar{m}_i}{E_i I_i}$
ε_u	ratio of moment of inertia of cross-sectional area of the right adjacent beam segment of the *u*th rigid body, I_{u+1}, to that of the left one, I_u, i.e., $\varepsilon_u = I_{u+1}/I_u$
ξ_{ul}	non-dimensional coordinate for left joint of the *u*th rigid body ($= x_{ul}/L$)
ξ_{ur}	non-dimensional coordinate for right joint of the *u*th rigid body ($= x_{ur}/L$)
ρ_i	mass density of the *i*th beam segment
σ_u	ratio of Young's modulus of the right adjacent beam segment of the *u*th rigid body, E_{u+1}, to that of the left one, E_u, i.e., $\sigma_u = E_{u+1}/E_u$
ω_v	the *v*th natural frequency
$\Omega_{v,i}$	non-dimensional frequency parameter for the *i*th beam segment corresponding to the *v*th vibration mode

APPENDIX A

The coefficient matrix $[B_u]$ for Eq. (9) is given by

$$\left[B_u\right]= \begin{array}{c} \begin{matrix} 4u\text{-}3 & 4u\text{-}2 & 4u\text{-}1 & 4u & 4u+1 & 4u+2 & 4u+3 & 4u+4 \end{matrix} \\ \left[\begin{matrix} \mathrm{s}\theta_{ul}+\Omega_{v,u}\ell_u^*\mathrm{c}\theta_{ul} & \mathrm{c}\theta_{ul}-\Omega_{v,u}\ell_u^*\mathrm{s}\theta_{ul} & \mathrm{sh}\theta_{ul}+\Omega_{v,u}\ell_u^*\mathrm{ch}\theta_{ul} & \mathrm{ch}\theta_{ul}+\Omega_{v,u}\ell_u^*\mathrm{sh}\theta_{ul} & -\mathrm{s}\theta_{ur} & -\mathrm{c}\theta_{ur} & -\mathrm{sh}\theta_{ur} & -\mathrm{ch}\theta_{ur} \\ \Omega_{v,u}\mathrm{c}\theta_{ul} & -\Omega_{v,u}\mathrm{s}\theta_{ul} & \Omega_{v,u}\mathrm{ch}\theta_{ul} & \Omega_{v,u}\mathrm{sh}\theta_{ul} & -\Omega_{v,u+1}\mathrm{c}\theta_{ur} & \Omega_{v,u+1}\mathrm{s}\theta_{ur} & -\Omega_{v,u+1}\mathrm{ch}\theta_{ur} & -\Omega_{v,u+1}\mathrm{sh}\theta_{ur} \\ \alpha_{ua}\mathrm{s}\theta_{ul}-\eta_{ua}\mathrm{c}\theta_{ul} & \alpha_{ua}\mathrm{c}\theta_{ul}+\eta_{ua}\mathrm{s}\theta_{ul} & \alpha_{ub}\mathrm{sh}\theta_{ul}-\eta_{ub}\mathrm{ch}\theta_{ul} & \alpha_{ub}\mathrm{ch}\theta_{ul}-\eta_{ub}\mathrm{sh}\theta_{ul} & -\lambda_u\mathrm{c}\theta_{ur}+\tau_u\mathrm{s}\theta_{ur} & \lambda_u\mathrm{s}\theta_{ur}+\tau_u\mathrm{c}\theta_{ur} & \lambda_u\mathrm{ch}\theta_{ur}-\tau_u\mathrm{sh}\theta_{ur} & \lambda_u\mathrm{sh}\theta_{ur}-\tau_u\mathrm{ch}\theta_{ur} \\ \delta_u\mathrm{s}\theta_{ul}+\kappa_{ua}\mathrm{c}\theta_{ul} & \delta_u\mathrm{c}\theta_{ul}-\kappa_{ua}\mathrm{s}\theta_{ul} & \delta_u\mathrm{sh}\theta_{ul}+\kappa_{ub}\mathrm{ch}\theta_{ul} & \delta_u\mathrm{ch}\theta_{ul}+\kappa_{ub}\mathrm{sh}\theta_{ul} & \phi_u\mathrm{c}\theta_{ur} & -\phi_u\mathrm{s}\theta_{ur} & -\phi_u\mathrm{ch}\theta_{ur} & -\phi_u\mathrm{sh}\theta_{ur} \end{matrix}\right] \end{array} \begin{matrix} \\ 4u-1 \\ 4u \\ 4u+1 \\ 4u+2 \end{matrix} \tag{A1}$$

where

$$m_u^*=\frac{M_u}{\bar{m}_1 L},\ J_u^*=\frac{J_u}{\bar{m}_1 L^3},\ k_{Ru}^*=\frac{k_{Ru}L}{E_1 I_1},\ k_{Tu}^*=\frac{k_{Tu}L^3}{E_1 I_1},\ \ell_u^*=\frac{\ell_u}{L},\ \ell_{mu}^*=\frac{\ell_{mu}}{L},\ \ell_{tu}^*=\frac{\ell_{tu}}{L},\ \sigma_u=\frac{E_{u+1}}{E_u},\ \varepsilon_u=\frac{I_{u+1}}{I_u} \tag{A2a-i}$$

$$\alpha_{ua}=-\Omega_{v,u}^2+k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right),\ \alpha_{ub}=\Omega_{v,u}^2+k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right) \tag{A3a,b}$$

$$\eta_{ua}=J_u^*\left(\frac{\bar{m}_1}{\bar{m}_u}\right)\Omega_{v,u}^5-k_{Ru}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\Omega_{v,u}-k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right)^2\Omega_{v,u}+\ell_{mu}^*\Omega_{v,u}^3 \tag{A4a}$$

$$\eta_{ub}=J_u^*\left(\frac{\bar{m}_1}{\bar{m}_u}\right)\Omega_{v,u}^5-k_{Ru}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\Omega_{v,u}-k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right)^2\Omega_{v,u}-\ell_{mu}^*\Omega_{v,u}^3 \tag{A4b}$$

$$\kappa_{ua}=M_u^*\left(\frac{\bar{m}_1}{\bar{m}_u}\right)\ell_{mu}^*\Omega_{v,u}^5-k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right)\Omega_{v,u}-\Omega_{v,u}^3,\ \kappa_{ub}=M_u^*\left(\frac{\bar{m}_1}{\bar{m}_u}\right)\ell_{mu}^*\Omega_{v,u}^5-k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right)\left(\ell_{tu}^*-\ell_{mu}^*\right)\Omega_{v,u}+\Omega_{v,u}^3 \tag{A5a,b}$$

$$\delta_u=M_u^*\left(\frac{\bar{m}_1}{\bar{m}_u}\right)\Omega_{v,u}^4-k_{Tu}^*\left(\frac{E_1 I_1}{E_u I_u}\right),\ \lambda_u=\sigma_u\varepsilon_u\left(\ell_u^*-\ell_{mu}^*\right)\Omega_{v,u+1}^3,\ \tau_u=\sigma_u\varepsilon_u\Omega_{v,u+1}^2,\ \phi_u=\sigma_u\varepsilon_u\Omega_{v,u+1}^3 \tag{A6a-d}$$

$$\mathrm{s}\theta_{ul}=\sin\Omega_{v,u}\xi_{ul},\ \mathrm{c}\theta_{ul}=\cos\Omega_{v,u}\xi_{ul},\ \mathrm{sh}\theta_{ul}=\sinh\Omega_{v,u}\xi_{ul},\ \mathrm{ch}\theta_{ul}=\cosh\Omega_{v,u}\xi_{ul} \tag{A7a-d}$$

$$\mathrm{s}\theta_{ur}=\sin\Omega_{v,u+1}\xi_{ur},\ \mathrm{c}\theta_{ur}=\cos\Omega_{v,u+1}\xi_{ur},\ \mathrm{sh}\theta_{ur}=\sinh\Omega_{v,u+1}\xi_{ur},\ \mathrm{ch}\theta_{ur}=\cosh\Omega_{v,u+1}\xi_{ur} \tag{A8a-d}$$

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