Journal of Marine Science and Technology

Volume 21 | Issue 3 Article 7

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Recommended Citation
Tseng, Wei-Kuo; Guo, Jiunn-Liang; and Liu, Chung-Ping (2013) "A COMPARISON OF GREAT CIRCLE, GREAT ELLIPSE, AND GEODESIC SAILING," *Journal of Marine Science and Technology*: Vol. 21: Iss. 3, Article 7.
DOI: 10.6119/JMST-012-0430-5
Available at: https://jmstt.ntou.edu.tw/journal/vol21/iss3/7

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Acknowledgements

This work was supported in part by the National Science Council of Taiwan, Republic of China, under grant NSC 98-2410-H-019-019- and NSC 99-2410-H-019-023-.

Journal of Marine Science and Technology, Vol. 21, No. 3, pp. 287-299 (2013)
DOI: 10.6119/JMST-012-0430-5

A COMPARISON OF GREAT CIRCLE, GREAT ELLIPSE, AND GEODESIC SAILING

Wei-Kuo Tseng, Jiunn-Liang Guo, and Chung-Ping Liu

Key words: great circle, great ellipse, geodesic, sailing.

ABSTRACT

An analytical and numerical comparison of great circle (GC) sailing, great elliptic (GE) sailing, and geodesic (Geod) sailing is presented. The comparison between GC and GE sailing addresses some problems whether the navigator and navigational software developers promptly have to use GE sailing or use hybrid sailing mixed with features of the GC sailing and GE sailing. This fact found here presents that the formulae tackling relationship of latitude and longitude of GC sailing also can be suited to the GE sailing except some calculation of GE sailing involving distance and course. The validity of effectiveness of proposed GE sailing has been verified with numerical tests and compared with extremely accurate geodetic methods (Vincenty's method). The numerical tests calculate the standard deviation of large sample of distance differences comparing GE sailing and Andoyer-Lambert method to Geod sailing. The result reveals that the mean and the standard deviation of distance differences of GE is one half and one sixth of Andoyer-Lambert method. The significance gives the assertion that the accuracy of GE sailing is better than Andoyer-Lambert method which (UK) Royal Navy and (US) Naval Oceanographic Office preferred spheroidal mathematical solution. We also give a dynamic programming recursive algorithm attaining any requirement of accuracy for distance calculation of GE sailing and more compact computational procedure of intermediate points along the GE. The course of GE sailing can be obtained from the proposed course reduction of GC sailing.

I. INTRODUCTION

In traditional navigation, the computations are simplified by the use of a spherical Earth model. It is well know that more accurate results can be obtained by the adoption of a spheroidal Earth and the calculation of geodesics distance and course. Vincenty's formulae [14] are two related iterative methods of nested equations used in geodesy to calculate the distance between two points on the surface of a spheroid, developed by Thaddeus Vincenty in 1975. They are based on the assumption that the figure of the Earth is an oblate spheroid, and hence are more accurate than methods such as great circle. The direct method computes the location of a point which is a given distance and course from another point. The inverse method computes the distance and course between two given points. They have been widely used in geodesy because they are very accurate to within 0.5 mm on the spheroidal Earth.

The discrepancies between the results on the GC sailing and the Geod sailing are in order of 0.27% according to Tobler [10], and in the order of 0.5% according to Earle [4]. Despite these discrepancies the use of the spherical model in traditional navigation for most practical purpose is considered satisfactory. Nevertheless for the case of sailing computations in GIS navigational systems such as ECDIS the computation has to be conducted on the spheroid in order to eliminate these significant errors but without seeking the sub-meter accuracies pursued in the other geodetic application. Seeking extremely high accuracy for marine navigation purpose does not offer any real benefit and require more computing power and processing time. For these reasons and before proceeding with the adoption of any geodetic computational method on the spheroid for sailing calculation it is required to adopt a realistic accuracy standard in order not only to eliminate the significant errors of the spherical model but also to avoid the exaggerate and unrealistic requirement of sub-meter accuracy.

In reality these discrepancies of distances calculated on the WGS-84 ellipsoid by the Vicenty's method [14] and GC sailing can reach maximum value 38.777908 nautical miles (71.81669 km) along the Equator around the Earth and minimum value 2.517774 nautical miles (4.662917 km) passing two Poles along one meridian around the Earth (made of comparison by this paper). In practice very accurate results can be obtained by calculation of the GE sailing or Andoyer-Lambert method [6] rather than the geodesic. In reality this discrepancies of distance around one quarter of the Earth between GE sailing (5405.18004 nm) and Geodesic (5405.17622 nm) computed by Vicenty's algorithm can only

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reach maximum value 7.0643 meters from one point at the Equator to another points at latitude 45 degrees and longitude 90 degrees away. The difference will vanish when both departure and destination are at the Equator or the same meridian. Around the Earth, the maximum different value is 4 times 7.0643 meters about 28.25732 meters. This discrepancy is still acceptable for the practical purposes of marine navigation.

The Andoyer-Lambert method [6] also provides very accurate solutions. In this method distance and bearing are pre-computed on an auxiliary sphere of radius equal to the semi-major axis of the spheroid on which the positions are located. Corrections are then made to obtain the corresponding spheroidal values. In fact, the Andoyer-Lambert method is just another type of the GE sailing. There are some drawbacks existed in this method that the bearing is the approximately value, the arc of auxiliary between two points equal to 0 or 180 degrees on the auxiliary sphere will give some problems of calculation divided by zero, and the calculated distances are not enough accurate. The GE sailing overcomes those drawbacks and gives the waypoints directly along the GE.

Comparatively, the discrepancies of distances between the great elliptic (GE) sailing and Geod sailing should be able to fulfill the requirement of meters accuracy. The numerical algorithm of the GE sailing is also more computer-efficient than the Geod sailing, and therefore is nice alternative instead the Geodesic. Even though the GE sailing is nice alternative, nevertheless the application of the GE sailing has to be considered the similarities and differences between the GE and GC sailing. Are the implements of the GE sailing holistic better than the implements of the GC sailing on navigation and (GIS)? There are some misconceptions about the navigational solutions of GE sailing and GC sailing. In this paper, these myths are discussed and demystified so that navigators and software developers in navigational industry can better understand what the real implements are. An analytical and numerical comparison among GC sailing, GE sailing, and Geod sailing is presented. The comparison between GC sailings and GE sailing addresses the confusing problems whether the navigator and navigational software developers promptly have to use GE sailing or use hybrid sailing mixed with features of the GC sailing and GE sailing. This paper discovers that the functional relationship between latitude and longitude of the GC sailing is the same as the GE sailing. This fact found here presents that the formulae tackling relationship of latitude and longitude of the GC sailing also can be suited to the GE sailing except some calculation of GE sailing involving distance and course. The answer suggested by this paper is that the hybrid sailing can be applied.

In dealing with GE sailing or GC sailing, many papers [2, 3, 5, 11-13, 15] demonstrate the finding of solutions for positions on a GE sailing or GC sailing. The mathematical derivations of those articles are a little bit tedious and abstruse hardly suited to coding, Pallikaris (2009) [8]. Our previously work [11] gave a mathematic unclosed form which lacks a convenient anti-derivative; the computation of the integral has to be carried out by numerical integration. There are no insightful comparisons of the GC equation, GE equation, and geodesic.. For those reasons, we revisit the topic of GE sailing.

The paper provides a more straightforward and compact mathematical derivation of the vector solution for GE sailing. The functional relationships among those parameters involving GE sailing and GC sailing are rearranged and more discussed comparing our previously work [11]. We also give a dynamic programming recursive numerical algorithm attaining discretionary accuracy for calculation of arc length and develop the more compact calculation of the geodetic coordinates of intermediate points along GE arc. Additionally, we give the reduction of spherical course computing the course of GE sailing.

Compare GE sailing with GC sailing, this paper discovers that the functional relationship between latitude and longitude of the GC sailing is the same as the GE sailing. This fact found here presents that the formulae tackling relationship of latitude and longitude of the GC also can be suited to the GE such as waypoints, vertices, and node of passing Equator except some problems of GE sailing involving distance and course. Applying vector methods to navigation problems gives some advantage for GE sailing to both syntax of programming algorithms and commercial mathematics software.

In the mathematical derivation, we take a direct scenario to produce the GE equation determining a great ellipse by a point and its course. We also provide different mathematical derivation for vertices and nodes along a GE arc or great circle. Finally, we give dynamic programming recursive algorithms that satisfy any requirement of accuracy for distance calculation of GE sailing and the complete set of the proposed algorithm for the great elliptic sailing, and then the readers should comprehensively grasp the meaning of geometry.

II. VECTORS INVOLVED IN DERIVATION

Using geodetic latitude, a point P on the surface of the Earth can be represented as a vector function of longitude λ and geodetic latitude φ.

$$\vec{P}(\varphi,\lambda) = (x \quad y \quad z)$$

$$= (N\cos\varphi\cos\lambda, \quad N\cos\varphi\sin\lambda, \quad N(1-e^2)\sin\varphi) \quad (1)$$

where e is eccentricity, $N = a/(1-e^2\sin^2\varphi)^{1/2}$ is the radius of curvature of the prime vertical, and a is the semi-major axis.

A moving point P on the surface of a spheroid along a path is associated with some vectors as the following. The unit velocity vector $\vec{T}_V$ tangent to the path which characterizes its moving direction, the north vector $\vec{T}_N$ tangent to the meridian

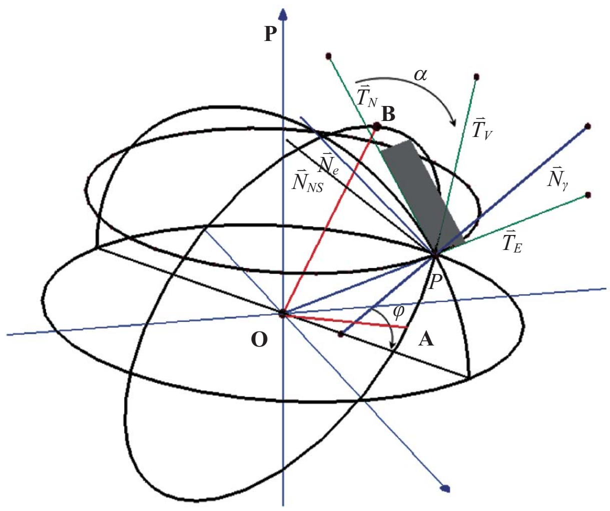

Fig. 1. The geometric relationship among tangent plane and related vectors.

which points the north pole and the east vector $\vec{T}_E$ tangent to parallel that points to the east, and the course α which is the angle between the meridian plane and the normal plane containing $\vec{T}_V$ at point *P*.

The above three vectors all lie in the tangent plane at the tangency *P* to spheroid. The tangent plane at the surface is perpendicular to the normal of *P*. The normal vector, often simply called the "normal," to a surface is a vector perpendicular to it. The normal plane is the plane determined by a unit tangent vector to point *P* and the normal to the point *P* on the spheroid. Then the normal section is defined as the intersection of normal plane and spheroid. The normal $\vec{N}_P$ is a vector perpendicular to tangent plane at point *P* on the spheroid. We show all those aforementioned important vectors as the following. Fig. 1 describes the geometric relationship among those important vectors and tangent plane at point *P*. In Fig. 1, we have the unit normal to the meridian at point *P*.

$$\vec{N}_p = (\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi) . \tag{2}$$

Partial differentiate vector function (2) with respect to latitude to obtain the north tangent unit vector (3). The same operation obtains the east tangent vector, and then normalizes it to give the east tangent unit vector (4).

$$\vec{T}_N = (-\sin\varphi\cos\lambda, -\sin\varphi\sin\lambda, \cos\varphi) . \tag{3}$$

$$\vec{T}_E = (-\sin\lambda, \cos\lambda, 0) . \tag{4}$$

Since the two vectors $\vec{T}_E$ and $\vec{T}_N$ form an orthogonal basis for the set of all vectors in the tangent plane at point *P* on a spheroid, the velocity vector $\vec{T}_V$ is a linear combination of $\vec{T}_E$ and $\vec{T}_N$ [12], which is shown in Eq. (5).

$$\vec{T}_V = \sin\alpha \cdot \vec{T}_E + \cos\alpha \cdot \vec{T}_N . \tag{5}$$

III. THE GREAT ELLIPTIC EQUATION

Geodesic is defined to be the shortest path between two points on the Earth's surface [14]. On the sphere, the geodesics are great circles. A great circle is the intersection of a sphere and a plane passing its origin. On the spheroid, the great ellipse is defined to be the intersection of a spheroid and a plane passing its origin. The great ellipse is not the shortest path between two points on the Earth's surface. The flattening of the Earth is very small, and therefore the great ellipses are very similar to the Geodesics. The discrepancies in the computed distances, courses, and waypoints betweens geodesic and great elliptic sailing are practically negligible for navigation [8]. For computational convenience, we develop two scenarios for determining a GE equation as the follow:

(1) Determine a GE Equation by a point on a spheroid and its course angle.
(2) Determine a GE Equation by specific two points on a spheroid.

Direct Scenario: Determine a great ellipse by a point and its course angle on a spheroid.

Let the vector of a given point be $\vec{A}$ and its course angle be α_a.

$$\begin{aligned}\vec{A} &= N(\cos\varphi_a\cos\lambda_a,\quad \cos\varphi_a\sin\lambda_a,\quad (1-e^2)\sin\varphi_a) \\ &= N(x_a,\quad y_a,\quad (1-e^2)z_a) .\end{aligned} \tag{6}$$

The velocity vector can be obtained by Eq. (7):

$$\vec{T}_{Va} = \sin\alpha_a \cdot \vec{T}_{Ea} + \cos\alpha_a \cdot \vec{T}_{Na} . \tag{7}$$

The velocity vector $\vec{T}_{Va}$ is linear combination of the northern tangent vector $\vec{T}_{Na}$ and the eastern tangent vector $\vec{T}_{Ea}$ at departure *A*. The last two vectors are the following:

$$\vec{T}_{Na} = (-\sin\varphi_a\cos\lambda_a, -\sin\varphi_a\sin\lambda, \cos\varphi_a) . \tag{8}$$

$$\vec{T}_{Ea} = (-\sin\lambda_a, \cos\lambda_a, 0) \tag{9}$$

Expanding Eq. (9) yields:

$$\vec{T}_{Va} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}^T = \begin{bmatrix} -\sin\varphi_a\cos\lambda_a\sin\alpha_a - \sin\lambda_a\cos\alpha_a \\ -\sin\varphi_a\sin\lambda\sin\alpha_a + \cos\lambda_a\cos\alpha_a \\ \cos\varphi_a\sin\alpha_a \end{bmatrix}^T . \tag{10}$$

Since the three vectors $\vec{A}$, $\vec{T}_{Va}$, and $\vec{P}$ are coplanar, let a plane equation containing the three vectors be

$$\vec{N}_e \cdot \vec{P} = l \cdot x + m \cdot y + z = 0 \tag{11}$$

Note that $\vec{N}_e = (l, m, 1)$ is the normal to the plane of a great ellipse (see Fig. 1):

$$(l, m, 1) = (q/s, r/s, 1) \tag{12}$$

where $(q, r, s) = \vec{A} \times \vec{T}_{Va}$.

This function of vector cross product exists or can be written as user-defined function in some programming languages. Further expansion of Eq. (12) into trigonometric terms is unnecessary for computer evaluation. Expanding Eq. (11) gives the following.

$$l \cdot \cos\varphi \cos\lambda + m \cdot \cos\varphi \sin\lambda + (1-e^2)\sin\varphi = 0\,. \tag{13}$$

Rearranging and rewriting Eq. (13) as a tangent function, we arrive at

$$\tan\varphi = -(l \cdot \cos\lambda + m \cdot \sin\lambda)/(1-e^2)\,. \tag{14}$$

From Eq. (14), the geographic latitude of any point along the great ellipse can be identified once the longitude is specified. The longitude can also be expressed in terms of latitude as Eq. (15).

$$\sin(\lambda - \Delta) = \frac{(1-e^2)}{\sqrt{l^2 + m^2}} \tan\varphi = \frac{\tan\varphi}{\sqrt{l'^2 + m'^2}} \tag{15}$$

where $\Delta = \tan^{-1}(\frac{l}{m})$ and $(l', m') = (l, m)/(1-e^2)$.

Inverse Scenario: Determining a great ellipse by two points on a spheroid.

Let $\vec{A}$ and $\vec{B}$ be the vectors of the departure and the destination.

$$\begin{aligned}\vec{B} &= N(\cos\varphi_b \cos\lambda_b,\quad \cos\varphi_b \sin\lambda_b,\quad (1-e^2)\sin\varphi_b) \\ &= N(x_b,\quad y_b,\quad (1-e^2)z_b)\,,\end{aligned} \tag{16}$$

Since the three vectors $\vec{A}$, $\vec{B}$, and $\vec{P}$ are coplanar, let a plane equation containing the three vectors be

$$\vec{N}_e \cdot \vec{P} = l \cdot x + m \cdot y + z = 0 \tag{17}$$

Note that $\vec{N}_e = (l, m, 1)$ is the normal to the plane of a great ellipse.

$$(l, m, 1) = (q/s, r/s, 1) \tag{18}$$

where $(q, r, s) = \vec{A} \times \vec{B}$.

The normal vector $\vec{N}_e = (l, m, 1)$ to the plane parallels the cross product $\vec{A} \times \vec{B}$. Solving for (l, m), we find the following.

$$l = -(1-e^2)(\frac{y_b z_a - y_a z_b}{x_a y_b - x_b y_a}), m = -(1-e^2)(\frac{x_a z_b - x_b z_a}{x_a y_b - x_b y_a}) \tag{19}$$

Expanding Eq. (11) also gives the following.

$$l \cdot \cos\varphi \cos\lambda + m \cdot \cos\varphi \sin\lambda + (1-e^2)\sin\varphi = 0\,. \tag{20}$$

Rearranging and rewriting Eq. (20) as a tangent function, we arrive at

$$\tan\varphi = -(l \cdot \cos\lambda + m \cdot \sin\lambda)/(1-e^2)\,. \tag{21}$$

Letting eccentricity=0 obtains Eq. (22) describing a great circle (GC) on a sphere.

$$\tan\varphi = -(l' \cdot \cos\lambda + m' \cdot \sin\lambda)\,. \tag{22}$$

where $(l', m') = (l, m)/(1-e^2)$.

Comparing the Eq. (22) with Eq. (21) gives the fact that the GC equation is the same as the GE equation. Therefore, using the implements of GC sailing can give the relationship between latitude and longitude of the waypoints along a great ellipse instead of GE sailing. Substituting Eq. (19) into Eq. (21) and expanding into trigonometric terms gives the concisely formula appropriate for both the GE sailing and the GC sailing.

$$\tan\varphi = \tan(\varphi_a)\frac{\sin(\lambda_b - \lambda)}{\sin(\lambda_b - \lambda_a)} + \tan(\varphi_b)\frac{\sin(\lambda - \lambda_a)}{\sin(\lambda_b - \lambda_a)} \tag{23}$$

What will be your latitude on passing mid-longitude? Apply double angle formulae of trigonometric identities into Eq. (23) and some manipulations to give Mid-longitude Equation which was discussed in the Journal of Navigation by our previous work [13].

$$\tan\varphi = \frac{\tan(\varphi_a) + \tan(\varphi_b)}{2\cos(\lambda_m)} \tag{24}$$

where $\lambda_m = (\lambda_b - \lambda_a)/2$ is the mid-longitude of the departure and destination.

IV. THE VERTEX AND NODE OF A GREAT ELLIPSE

The semi minor axis of a great ellipse equals the distance between the vertex of the great ellipse and the center of the spheroid. A vertex (N or S vertex), whose latitude and

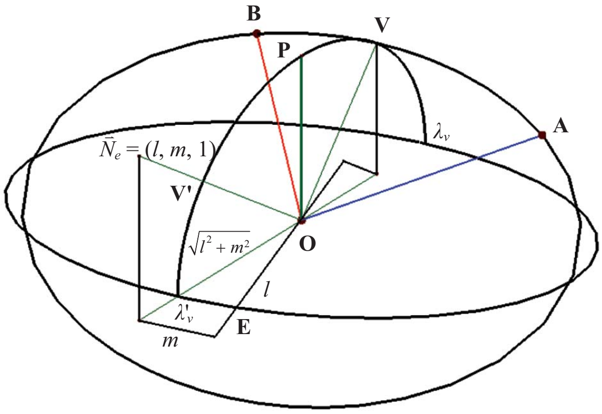

Fig. 2. The vertex latitude presented by the normal vector of a great ellipse.

longitude are denoted as φ_v and λ_v, is shown in Fig. 2. The geocentric latitude of vertex equals to co-latitude of the normal vector to the plane of a great ellipse; the meaning is shown in Fig. 2. The longitude of vertex can be obtained by the Y-axis component of normal vector to GE dividing the X-axis component of it in Eq. (27).

The tangent of the geocentric latitude angle $\mathbf{VO\lambda_V}$ is equal to the cotangent of an angle $\mathbf{V'O\lambda'_V}$ between normal vector to GE and equatorial plane. Substituting Eq. (1) into tangent and cotangent trigonometric functions obtains the follow.

$$\frac{z}{\sqrt{x^2+y^2}} = \sqrt{\frac{N^2(1-e^2)^2\sin^2\varphi_v}{N^2\cos^2\varphi_v\cos^2\lambda_v + N^2\cos^2\varphi_v\sin^2\lambda_v}}$$

$$= \sqrt{l^2+m^2}\ . \tag{25}$$

Rearrange Eq. (25) and apply trigonometric identity to yield the following.

$$\tan\varphi_v = \pm\frac{\sqrt{l^2+m^2}}{1-e^2} = \sqrt{l'^2+m'^2} \tag{26}$$

In trigonometry, the two-argument function atan2 is a variation of the arctangent function. The atan2 function is useful in many applications involving vectors in Euclidean space, such as finding the direction from one point to another. For any real arguments x and y not both equal to zero, atan2(y, x) is the angle in radians between the positive x-axis of a plane and the point given by the coordinates (x, y) on it. The angle is positive for counter-clockwise angles (upper half-plane, $y > 0$), and negative for clockwise angles (lower half-plane, $y < 0$).

The longitude of vertex is opposite direction of image vector of the normal to a great elliptic plane in the equatorial plane (X-Y plane), therefore the longitude of vertex is opposite of longitude of the normal.

$$\lambda_v = \text{atan2}(-m, -l)\ . \tag{27}$$

Note: atan2 has the conventional ordering of arguments, namely atan2(y, x). This is not universal, Excel for instance uses atan2(x, y). Be warned. It returns a value in the range -PI < atan2 <= PI.
Further note: If your calculator/programming language is so impoverished that only atan is available then use:

atan2(y, x) = atan(y/x),	$x > 0$.
atan2(y, x) = atan(y/x)+PI,	$x < 0$, $y >= 0$.
atan2(y, x) = PI/2,	$X = 0$, $y > 0$.
atan2(y, x) = atan(y/x)-PI,	$X < 0$, $y < 0$.
atan2(y, x) = -PI/2,	$x = 0$, $y < 0$.
atan2(0, 0),	The value is undefined.

The unit vector of the vertex is denoted as the vector $\vec{V}_{vertex}$. Setting the geodetic latitude $\varphi = 0$ in Eq. (14) gives the ascending and descending nodes where the great ellipse intersects with the equator at longitude λ_e:

$$\lambda_e = \text{atan2}(-l, m) \tag{28}$$

which is equivalent to $\lambda_e = \lambda_v \pm \pi/2$.

The unit vector of the node is given by:

$$\vec{V}_{node} = (-m, l, 0)/\sqrt{l^2+m^2}\ . \tag{29}$$

In a recent paper [5] dealing with GE sailing, substituting two identities of its Eq. (7) back into its Eqs. (3) and (4) can not lead to alternatives for GE equation as the following Eq. (31). In [5], its Eq. (11) is also too tedious; we give more compact expression as Eq. (31).

Substituting Eq. (26) and Eq. (27) into Eq. (15) yields a different presentation for the GE equation and GC equation as the following:

$$\sin(\lambda - \lambda_e) = \frac{\tan\varphi}{\tan\varphi_v} \tag{30}$$

Since the longitude difference between the longitude of the nodes and the longitude of the vertices is 90 degrees, then using this relation obtains

$$\cos(\lambda - \lambda_v) = \frac{\tan\varphi}{\tan\varphi_v} \tag{31}$$

which is the same as the trigonometric identities of Napier's mnemonic Rule for Right-Angle Triangle [6] for conventional technique of navigation. The above both formulae can not be only applied to GE sailing but also applied to GC sailing.

V. THE COURSE FUNCTION OF GREAT ELLIPIC SAILING

In navigation, a course is the intended path of a vehicle over the surface of the Earth. For sea travel, it is the intended sailing path of a vessel or the direction of a line drawn on a chart representing the intended sailing path, expressed as the angle measured from a specific reference datum clockwise from 0° through 360° to the line.

The course is the angle between the meridian plane and normal plane containing velocity vector at point *P*. The normal plane usually is slightly different from the GE plane at point *P*.

By above definition, the course can be obtained by the inner dot of the unit velocity vector $\vec{T}_V$ and the unit parallel tangent vector $\vec{T}_E$ dividing the inner dot of the vector $\vec{T}_V$ and the unit meridian tangent vector $\vec{T}_N$ at point *P* as the follow.

$$\alpha = \operatorname{atan2}(\vec{T}_E \cdot \vec{T}_V, \vec{T}_N \cdot \vec{T}_V) \tag{32}$$

where $\vec{T}_V = \dfrac{\vec{N}_e \times \vec{N}_P}{\left|\vec{N}_e \times \vec{N}_P\right|}$.

In the sphere Earth model, the course can be obtained as

$$\alpha_C = \operatorname{atan2}(\vec{T}_E \cdot \vec{T}_V, \vec{T}_N \cdot \vec{T}_V) \tag{33}$$

where $\vec{T}_V = \dfrac{\vec{N}_C \times \vec{N}_P}{\left|\vec{N}_C \times \vec{N}_P\right|}$.

Note that $\vec{N}_C = (l', m', 1)$ is the normal to the plane of a great circle (see Fig. 1).

Expanding the Eq. (32) and Eq. (33) gives reduction of the course of GC sailing instead of the course of GE sailing.

$$\tan\alpha = \tan\alpha_C \frac{(1-e^2\sin^2\varphi)}{(1-e^2)} \tag{34}$$

The unit normal vector (see Fig. 1) to the normal section containing the velocity vector at point *P* on the GE is the cross product of the unit normal and velocity vector at point *P*, that is

$$\vec{N}_{NS} = \vec{N}_p \times \vec{T}_V \tag{35}$$

The velocity vector is orthogonal combination of the north tangent vector and the east tangent vector forming orthogonal basis, and then we derive another approach for the solution of course. Any waypoint on the path satisfies Eq. (36).

$$\vec{T}_V = \sin\alpha \cdot \vec{T}_E + \cos\alpha \cdot \vec{T}_N = \begin{bmatrix} -\sin\lambda\sin\alpha - \cos\lambda\sin\varphi\cos\alpha \\ \cos\lambda\sin\alpha - \sin\lambda\sin\varphi\cos\alpha \\ \cos\varphi\cos\alpha \end{bmatrix}^T . \tag{36}$$

Since velocity vector is orthogonal to the normal vector of the great ellipse, the inner product of the two vectors equals to zero.

$$\vec{N}_e \cdot \vec{T}_V = 0 \tag{37}$$

where $\vec{N}_e$ is normal to a great ellipse.

Expanding Eq. (37) gives

$$(-l\sin\lambda + m\cos\lambda)\sin\alpha - (l\cos\lambda\sin\varphi + m\sin\lambda\sin\varphi + \cos\varphi)\cos\alpha = 0. \tag{38}$$

Rearranging Eq. (38) and incorporating Eq. (14), we have

$$\tan\alpha = \cos\varphi \frac{1+(1-e^2)\tan^2\varphi}{(l\sin\lambda - m\cos\lambda)} . \tag{39}$$

When eccentricity e = 0, Eq. (39) can be reduced into Eq. (40):

$$\tan\alpha_C = \frac{\sec\varphi}{l'\sin\lambda - m'\cos\lambda} \tag{40}$$

Expanding Eq. (40) and transforming into Eq. (41) by trigonometric identity.

$$\tan\alpha_C = \frac{\sec\varphi}{\tan\varphi_v \sin(\lambda_V - \lambda)} \tag{41}$$

Substituting Eq. (40) and into Eq. (39) also obtains the important reduction of spherical course to the course of GE sailing.

$$\tan\alpha = \tan\alpha_C \frac{(1-e^2\sin^2\varphi)}{(1-e^2)} \tag{42}$$

Some relationships can be found aforementioned between GC sailing and GE sailing as Table 1.

VI. THE CALCULATION OF THE DISTANCE OF THE GREAT ELLIPTIC SAILING

The distance calculation of the GE sailing can be conducted by the use of standard geodetic Formula (46) for the length of the meridian arc after proper replacing the eccentricity e of the meridian ellipse with the eccentricity ε of the great ellipse. This is better understood if we consider a great ellipse as an inclined version of meridian ellipse. The semi-minor axis of a great ellipse is measured from the vertex to the origin of the spheroid. The eccentricity $\varepsilon\,(0 \le \varepsilon \le e)$ of a great ellipse is:

$$\varepsilon = \frac{a^2 - \left|P(\varphi_V, \lambda_V)\right|^2}{a^2} = \frac{\sqrt{1-e^2}}{\sqrt{1-e^2\sin^2\varphi_v}} e\sin\varphi_v \tag{43}$$

Table 1. Relationships of waypoints and courses between the Great Circle Sailing and the Great Elliptic Sailing.

Lon-Lat	$\tan\varphi = \tan(\varphi_1)\dfrac{\sin(\lambda_2-\lambda)}{\sin(\lambda_2-\lambda_1)} + \tan(\varphi_2)\dfrac{\sin(\lambda-\lambda_1)}{\sin(\lambda_2-\lambda_1)}$	O
Vertex	$\tan\varphi_v = \pm\dfrac{\sqrt{l^2+m^2}}{1-e^2} = \sqrt{l'^2+m'^2}$ $\lambda_v = \text{atan2}(-m,-l) = \text{atan2}(-m',-l')$	O
Node	$\lambda_e = \text{atan2}(-l,m) = \text{atan2}(-l',m')$	O
Course	$\tan\alpha_C = \dfrac{\sec\varphi}{\tan\varphi_v \sin(\lambda_V-\lambda)}$ $\tan\alpha = \tan\alpha_C \dfrac{(1-e^2\sin^2\varphi)}{(1-e^2)}$	X

O: no difference, X: need correction. The vector (l, m, 1) is the normal to the plane of a great ellipse and the other vector $(l', m', 1)$ is the normal to the plane of a great circle.

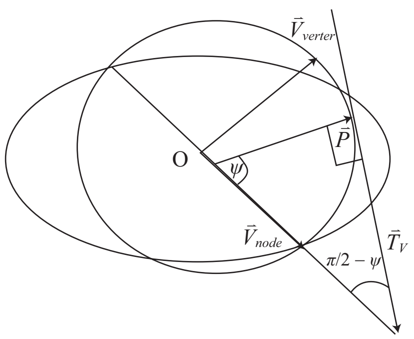

Fig. 3. The angle from the node to point *P* on a great ellipse.

Eqs. (22), (23), (30) or (31) can build the track of a GE sailing. Once the latitude and longitude are given, the velocity vector $\vec{T}_V$ can be derived. The sine of geodetic angle ψ is the inner product of the normal vector to the equator at a node (major axis of a great ellipse) and the normal vector to a great ellipse at a point P can be represented as the following (Fig. 3). The geodetic angle ψ is equivalent to the geodetic latitude used in the calculation of the length of the meridian arc.

$$\sin(\psi) = \vec{T}_V \cdot \vec{V}_{node} \tag{44}$$

The angle also equals to the angle between the velocity vector and the vector of vertex as

$$\cos(\psi) = \vec{T}_V \cdot \vec{V}_{vertex} \tag{45}$$

where $\vec{T}_V = \dfrac{\vec{N}_e \times \vec{N}_P}{\left|\vec{N}_e \times \vec{N}_P\right|}$ is the velocity vector along a great ellipse.

Distance from node of equator to one point $\vec{P}(\varphi,\lambda)$ on the GE sailing is given by:

$$L(\psi) = a(1-\varepsilon^2)\int_0^{\psi}\frac{d\theta}{(1-\varepsilon^2\sin^2\theta)^{3/2}} \tag{46}$$

The above equation can be transformed to an elliptic integral of the second type, which can not be evaluated in closed form. The integral of distance lacks convenient anti-derivative. The binomial expansion series of integrant can discover the analytic solution term by term. The closed form of general differential equation is usually unavailable. But the power series representation is always a welcome solution. Expanding the RHS of the Eq. (46) by binomial theorem as rapidly convergent series yields Eq. (47).

$$L(\psi) = a(1-\varepsilon^2)\int_0^{\psi}\left[\sum_{i=0}^{n}(-1)^i\begin{pmatrix}-\frac{3}{2}\\ i\end{pmatrix}(\varepsilon^2\sin^2\theta)^i\right]d\theta + R \tag{47}$$

By the Mean Value Theorem for definite integral applied to (47), we obtain the error bound.

$$R < R_n = (1-\varepsilon^2)(-1)^{n+1}\begin{pmatrix}-\frac{3}{2}\\ n+1\end{pmatrix}(\varepsilon^2\sin^2\psi)^{n+1}\left|\psi\right|$$

Integrating Eq. (47) termwisely by parts gives the relevant reduction formula which can enables us to handle positive integral powers of sine in Eq. (47).

$$\int\sin^n\theta d\theta = -\frac{\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\int\sin^{n-2}\theta d\theta \tag{48}$$

Apply dynamic programming algorithm to integrate Eq. (47) as Table 2. When applicable, the method takes much less time than naive methods and attains discretionary accuracy.

An approximation to Eq. (46) is provided in [9], the first two terms of the expansion is:

$$L(\psi) = a\left[(1-\frac{\varepsilon^2}{4})\psi - \frac{3\varepsilon^2}{8}\sin(2\psi)\right] \tag{49}$$

This approximation causes the error at most up to 836.0592 meters when φ equals to 90 degrees. Since the values of powers of ε are very small, we can attain any requirement of accuracy by retaining some terms of powers of eccentricity ε. For example, since the fifth term in Eq. (47) is less than 0.049186cmeter, holding the first four terms arrives at the sub meter accuracy. If the point P is the same semi-sphere of the departure, then the distance can be computed as the following.

$$\text{Dist}(\psi) = \left|L(\psi) - L(\psi_a)\right| \tag{50}$$

Table 2. Dynamic programming of calculation of the ellipse arc.

Function Ellipse_Arc(θ, n, a, ε)
Input: positive even integer n such that $n \geqq 0$,
geodetic angle θ.
$\varpi = 1$, $Ellipse\_Arc = \theta$, $A = 1$, $B = \theta$, $C = \sin\theta\cos\theta$,
eps=small number
    Do while ($A \cdot B > eps$ and $2\varpi \leq n$)
    $A = A \cdot (2\varpi + 1)/(2\varpi) \cdot \varepsilon^2, B = -\dfrac{C}{2\varpi} + \dfrac{2\varpi - 1}{2\varpi} B$
    $Ellipse\_Arc = Ellipse\_Arc + A \cdot B$
    $C = C \cdot \sin^2\theta$
    $\varpi = \varpi + 1$
End Do
$Ellipse\_Arc = a \cdot (1 - \varepsilon^2) \cdot Ellipse\_Arc$
End Ellipse_Arc

If the point P is the opposite semi-sphere of the departure, then the distance can be computed as the following.

$$\text{Dist}(\psi) = |L(\psi_a)| + |L(\psi)| \tag{51}$$

VII. THE PROPOSED SIMPLE ALGORITHM FOR THE GREAT ELLIPTIC SAILING, NUMERICAL TESTS AND COMPARISON

The track of a GE sailing can be plotted by connecting successive intermediate points along a great ellipse. In the following algorithm we selected integer longitude between successive intermediate points along a great ellipse. The complete set of the proposed algorithm for the great elliptic sailing is listed in Table 3. From Comparing the proposed algorithm with another algorithm (from Part I to Part IV) provided by Pallikaris [6], we found that our proposed algorithm is more easily, simpler, shorter, more logical and more intuitive than another algorithm.

For comparison, the results of the computation for the great ellipse, the great circle and the geodesics (using Andoyer-Lambert method [6] and the method of Vicenty [14]) departed from A point (Lat 0 N, long 120 E) to successive latitudes in 1 degree increments up to 90 degrees (GE and GC) passing one quarter of the Earth (see Fig. 4) (longitude difference is 90 degrees E) are shown in Table 4.

The differences of distances between the GE and Geodesics are far less than the differences of distances between the GC sailing and Geodesics. The maximum value of GC-Geod is about -17.95 km occurred at the Equator where the difference of GE-Geod is 0. The maximum value of GE-Geod is about 7.06 meters occurred at about latitude 45 degrees and the difference of GC-Geod is about 9.5864 km (Fig. 6).

The differences of distances between the Andoyer-Lambert parametric method and Geodesics are decreasing from latitude 0 degree shown in Table 5. The value attains to maximum about -7.051 meters along the meridian.

Table 3. The complete set of the proposed algorithm for the great elliptic sailing.

Input A,B $A = (\varphi_a, \lambda_a)$, $B = (\varphi_b, \lambda_b)$
Applying the great circle sailing
Step 1: Transform to Cartesian Coordinates
$\vec{A}' = (\cos\varphi_a \cos\lambda_a, \cos\varphi_a \sin\lambda_a, \sin\varphi_a)$
$\vec{B}' = (\cos\varphi_b \cos\lambda_b, \cos\varphi_b \sin\lambda_b, \sin\varphi_b)$
Step 2: Calculate the normal to great ellipse
$(q, r, s) = \vec{A}' \times \vec{B}'$, $\vec{N}_C = (l'.m'.1) = (q/s, q/s, 1)$,
$\vec{N}_e = (l'(1-e^2), m'(1-e^2), 1)$
Step 3: Calculate the latitude of vertex , vector of node passing the Equator, and eccentricity of the great ellipse.
$\varphi_v = \pm\text{atan}(\sqrt{l'^2 + m'^2}), \vec{V}_{node} = (-m', l', 0)/\sqrt{l'^2 + m'^2}$,
$\varepsilon = \dfrac{\sqrt{1-e^2}}{\sqrt{1 - e^2\sin^2\varphi_v}} e \sin\varphi_v, \lambda_J = \lambda_a, J = 0,$
Loop Steps:
Do while($\lambda_J < \lambda_b$)
Calculate the latitude given longitude and the course of the GE sailing
Method 1: $\varphi_J = \text{atan}(-l' \cdot \cos\lambda_J - m' \cdot \sin\lambda_J)$
Method 2: $\varphi_J = \text{atan}\left[\tan(\varphi_a)\dfrac{\sin(\lambda_b - \lambda_J)}{\sin(\lambda_b - \lambda_a)} + \tan(\varphi_b)\dfrac{\sin(\lambda_J - \lambda_a)}{\sin(\lambda_b - \lambda_a)}\right]$
Calculate the normal vector and velocity vector of moving point P
$\vec{N}_p = (\cos\varphi_J \cos\lambda_J, \cos\varphi_J \sin\lambda_J, \sin\varphi_J)$
$\vec{T}_V = \dfrac{\vec{N}_e \times \vec{N}_P}{|\vec{N}_e \times \vec{N}_P|}$
Calculate the northern tangent vector and eastern tangent vector
$\vec{T}_N = (-\sin\varphi_J \cos\lambda_J, -\sin\varphi_J \sin\lambda_J, \cos\varphi_J)$
$\vec{T}_E = (-\sin\lambda_J, \cos\lambda_J, 0)$
Calculate the course at point P and distance from A to point P
$\alpha = \text{atan2}(\vec{T}_N \cdot \vec{T}_V, \vec{T}_E \cdot \vec{T}_V), \psi_J = \pi/2 - \cos^{-1}(\vec{T}_V \cdot \vec{V}_{node}),$
$L_J = L(\psi_J)$, distance $= \begin{cases} |L(\psi_J) - L(\psi_a)|, \text{sgn}(\varphi_a) = \text{sgn}(\varphi_J) \\ |L(\psi_J)| + |L(\psi_a)|,, \text{sgn}(\varphi_a) \neq \text{sgn}(\varphi_J) \end{cases}$
$\lambda_{J+1} = \lambda_J + 1, J = J + 1$
End do

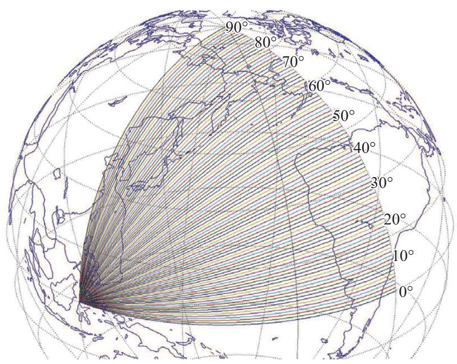

Fig. 4. The 91 passages of GE sailing passing one quarter.

Table 4. Comparison between great circle, great ellipse, and geodesics.

Lat.	Distance (nautical mile)			Distance Differences			
	Geod	GE	GC	GE-Geod (Meter)		GC-Geod (KM)	
0	5409.6945	5409.6945	5400	0.0000	0.000000%	17.9542	0.1792%
10	5409.4228	5409.4232	5400	0.8227	0.000008%	17.4510	0.1742%
20	5408.6399	5408.6415	5400	2.9082	0.000029%	16.0012	0.1597%
30	5407.4389	5407.4417	5400	5.2856	0.000053%	13.7768	0.1376%
40	5405.9628	5405.9665	5400	6.8456	0.000068%	11.0432	0.1103%
44	5405.3344	5405.3382	5400	7.0545	0.000070%	9.8793	0.0987%
45	5405.1762	5405.1800	5400	7.0643	0.000071%	9.5864	0.0958%
46	5405.0180	5405.0218	5400	7.0569	0.000070%	9.2934	0.0928%
50	5404.3887	5404.3924	5400	6.8570	0.000069%	8.1279	0.0812%
60	5402.9064	5402.9093	5400	5.3110	0.000053%	5.3827	0.0538%
70	5401.6958	5401.6973	5400	2.9296	0.000029%	3.1405	0.0314%
80	5400.9045	5400.9049	5400	0.8301	0.000008%	1.6751	0.0167%
90	5400.6294	5400.6294	5400	0.0000	0.000000%	1.1657	0.0117%

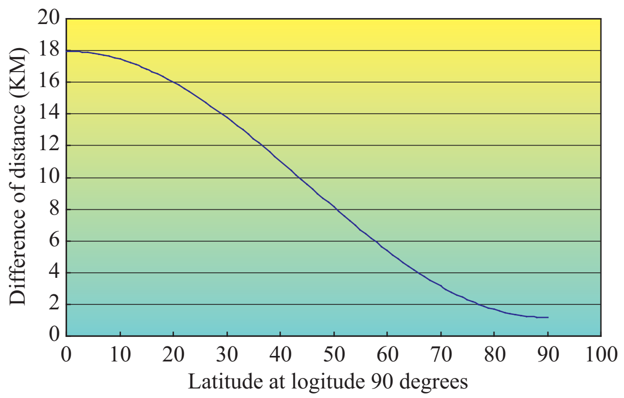

Fig. 5. Distance differences between Great Circle and Geodesics.

Fig. 5 depicts the differences of distances between the GC sailing and Geod sailing from the same departure (Latitude 0, longitude 0) to the distinct destinations (between equator and successive latitudes in 1 degree increments up to 90 degrees and longitude 90 degrees away). The difference reaches to maximum about 17.95 KM at the Equator. The discrepancy is diminishing toward Poles. The minimum value is 1.165729 KM at Poles. This is not acceptable for practical purposes of navigation and ECDIS.

Calculation of shortest sailing paths on the ellipsoid by a geodetic inverse and direct method involves formulae that are too much complex. By above analysis, the GE sailing is a nice, simpler, and straightforward alternative. The method can satisfy the requirement of meter accuracy.

Fig. 6 depicts the discrepancies of those distances between the GE sailing and the Geod sailing from the same departure (Latitude 0, longitude 0) to the distinct destinations (from equator to successive latitudes in 1 degree increments up to 90° and longitude 90°). There are no differences occurring at Lat 0° and Lat 90° where the two great ellipses coincide with

Table 5. Comparison between Andoyer-Lambert method and geodesics.

Latitude	Geod	Andoyer_Lambert	Geod-Lambert
0	5409.6945	5409.6945	0.0000
10	5409.4228	5409.4228	-0.0019
20	5408.6399	5408.6399	-0.0813
30	5407.4389	5407.4387	-0.4149
40	5405.9628	5405.9622	-1.1732
50	5404.3887	5404.3874	-2.4006
60	5402.9064	5402.9043	-3.9471
70	5401.6958	5401.6928	-5.4883
80	5400.9045	5400.9009	-6.6295
90	5400.6294	5400.6256	-7.0508

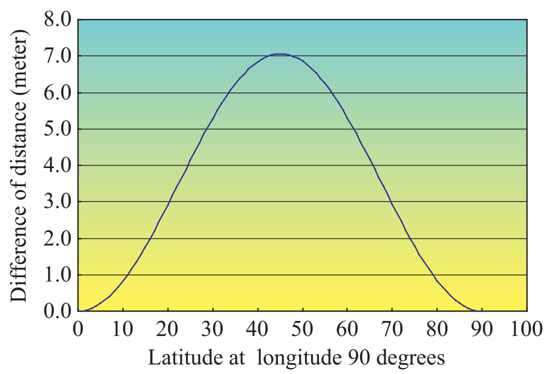

Fig. 6. Distance differences between Great Ellipse and Geodesics.

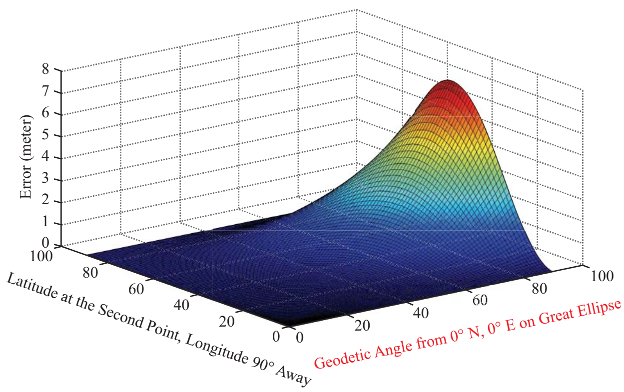

Fig. 7. Distance differences between great ellipse and Geodesics for different geodetic angles on great ellipses.

the geodesics. The difference reaches to maximum value approximately 7.06 meters at about Latitude 45°.

Fig. 7 depicts the differences of distances between the GE and Geod sailing from the same departure (Latitude 0°, longitude 0°) to the distinct destinations along a great ellipse (departing from the same departure to successive latitudes in 1° increments up to 90 degrees and the same difference of longitude equal to 90°) at successive geodetic angles of the great ellipse in 1 degree increment up to 90°, i.e. The

Table 6. The means and standard deviation of distance differences comparing Andoyer-Lambert method$_1$ and GE sailing$_2$ to Geodesics. (Unit: m, Sample: 8281)

Lat	10	20	30	40	45	50	60	70	80	90	Total
STD_1	0.21	0.76	1.42	1.96	2.13	2.25	2.37	2.48	2.64	2.73	2.51
STD_2	0.21	0.76	1.38	1.78	1.84	1.78	1.38	0.76	0.22	0	1.19
$Mean_1$	0.434	1.573	2.993	4.195	4.590	4.815	4.772	4.286	3.748	3.520	3.159
$Mean_2$	0.142	0.502	0.912	1.181	1.219	1.183	0.916	0.505	0.143	0.000	0.603

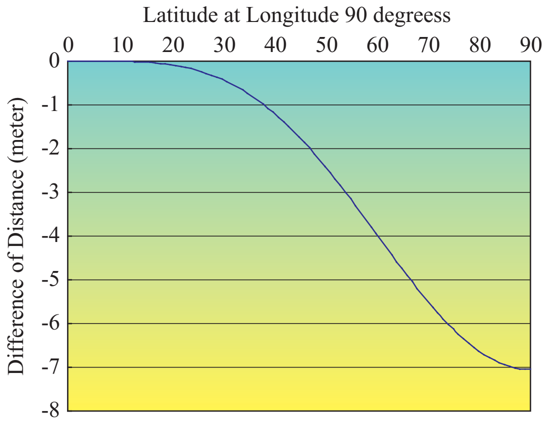

Fig. 8. Distance differences between Andoyer-Lambert Method and Geodesics.

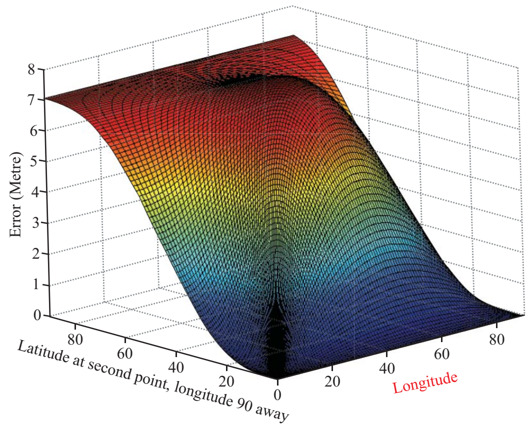

Fig. 9. Distance differences between Andoyer-Lambert method and Geodesics.

coordinates of destinations range between integer latitude 0°-90° N and integer longitude 0°-90° E.

Fig. 8 depicts the discrepancies of distances between the values computed by the Andoyer-Lambert method [6] and the true geodesic distances from the same departure (Latitude 0, longitude 0) to the distinct destinations (from equator to successive latitudes in 1 degree increments up to 90° and longitude 90° away). There are no differences occurring at Lat 0° where the correction of Andoyer-Lambert method is equal to 0. The difference reaches to maximum value approximately -7.05 meters at about Latitude 90°.

Fig. 9 depicts the differences of distances between the values computed by Andoyer-Lambert Method and geodesic distance from the same departure (Latitude 0°, longitude 0°) to the distinct destinations at successive latitudes in 1 degree increment up to 90 degree on the meridians starting from longitude 0 degree at successive longitude in 1 degree increment up to 90 degree. The coordinates of destinations range between integers of latitude 0°-90° N and integers of longitude 0°-90° E. From comparison between Fig. 7 and Fig. 9, the curve surface of Fig. 7 is smoother than the Fig. 9. The fact means that the numerical fitting of more accurate computation applied to GE sailing is more appropriate than Andoyer-Lambert Method.

The accuracies of the GE sailing in terms of variance achieved are assessed and compared to Andoyer-Lambert method in the Table 6. The mean differences and standard deviations are computed for the 8281 lines (91X91) by data extracted from the dataset plotting Fig. 7 and Fig. 9. The difference standard deviations for the 8281 lines between true geodesic distances and computed values by Andoyer-Lambert method are increasing when destination (vertex) is toward North Pole.

The mean difference between true geodesic distance and computed value of GE sailing is one sixth of the difference mean of Andoyer-Lambert method with one half standard deviation of Andoyer-Lambert method. We can assert that the accuracy of GE is better than Andoyer-Lambert method.

The statistics hypothesis testing can test whether the GE sailing is better than Andoyer-Lambert method. Null hypothesis: The two methods have the same accuracy. Alternative hypothesis: The accuracy of Andoyer-Lambert method is worse than GE sailing. The observed level of significance is very small calculated by statistics method, therefore we reject null hypothesis. We accept the alternative hypothesis that the accuracy of Andoyer-Lambert method is worse than GE sailing.

Since the coefficients of Eq. (21) and Eq. (22) are the same, then the latitudes and longitudes of GC and GE have same value. Instead of the GC sailing, plotting the positions of the GE to the chart does not give different positions. Only for calculating distance and course, the GE equation generates significant effect.

The numerical examples we used as based older numerical tests conducted by Pallikaris (2009). The first numerical example very long distance with difference of longitude about

Table 7. Determining route from Sydney to Valparaiso.

G.C. Distance (N.M.)	6124.02416097770
G.E. Distance (N.M.)	6129.12072590703
Geodesic Distance (N.M.)	6129.11244819428
Diff. of GC-Geodesic (m)	-9423.50792510500
Diff. of GE-Geodesic (m)	15.330324013
Intermediate Waypoints	At a given longitude of leg
Leg interval	Each integer longitude
Vertex (longitude)	140.37062 W
Vertex (latitude)	60.68006 S

WP	Latitude	Longitude	Total distance	Leg distance	Course
0	-33.77017	151.53273	0.00000	0.00000	143.99462
1	-34.30294	152.00000	39.51026	39.51026	143.73428
2	-35.41490	153.00000	122.42280	82.91253	143.16528
3	-36.48898	154.00000	203.13155	80.70875	142.58072
4	-37.52597	155.00000	281.68506	78.55351	141.98135
5	-38.52670	156.00000	358.13576	76.45070	141.36787
6	-39.49208	157.00000	432.53913	74.40337	140.74094
…	…	…	…	…	…
133	-37.78836	-76.00000	5770.56699	75.91550	38.17607
134	-36.76084	-75.00000	5848.57120	78.00421	37.57298
135	-35.69644	-74.00000	5928.71743	80.14622	36.98452
136	-34.59435	-73.00000	6011.05529	82.33787	36.41142
137	-33.45385	-72.00000	6095.62988	84.57458	35.85443
138	-32.99997	-71.61125	6129.12073	33.49085	35.64241

Table 8. Determining route from Yokohama to Valparaiso.

G.C. Distance (N.M.)	9250.05315077852
G.E. Distance (N.M.)	9242.56158336103
Geodesic Distance (N.M.)	9242.55803581660
Diff. of GC-Geodesic (m)	13880.95290947430
Diff. of GE-Geodesic (m)	6.570052283
Intermediate Waypoints	At a given longitude of leg
Leg interval	Integers of longitude
Vertex (longitude)	129.6260160 E
Vertex (latitude)	34.86578657 N

WP	Latitude	Longitude	Total distance	Leg distance	Course
0	34.43630	139.85650	0.00000	0.00000	95.80092
1	34.42414	140.00000	7.15921	7.15921	95.88169
2	34.33461	141.00000	57.10861	49.94940	96.44379
3	34.23674	142.00000	107.16895	50.06034	97.00454
4	34.13051	143.00000	157.34994	50.18099	97.56381
5	34.01589	144.00000	207.66129	50.31136	98.12150
6	33.89284	145.00000	258.11272	50.45142	98.67746
…	…	…	…	…	…
145	-32.13663	-76.00000	9014.12408	52.68847	104.24813
146	-32.34826	-75.00000	9066.56531	52.44123	103.71718
147	-32.55105	-74.00000	9118.76859	52.20328	103.18317
148	-32.74505	-73.00000	9170.74328	51.97468	102.64625
149	-32.93030	-72.00000	9222.49877	51.75549	102.10655
150	-32.99997	-71.61125	9242.56158	20.06281	101.89601

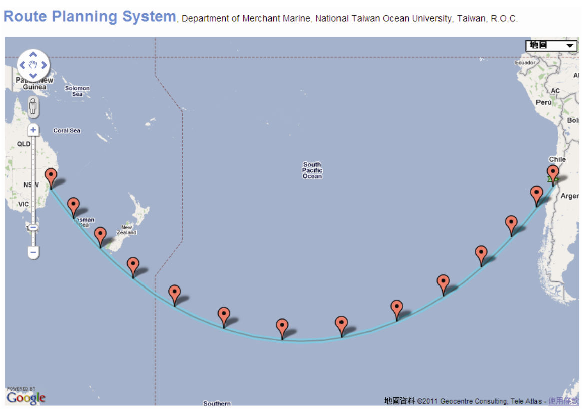

Fig. 10. The track from Sydney Harbor, Australia to Valparaiso, Chile. Note: This chart is plotted by Google Maps API

136° is the sailing from the approaches of Sydney Harbor -Australia (33° 46.21' S, 151° 31.964' E) to the approaches of Valparaiso-Chile (32° 59.998' S, 71° 36.675' W). The results of these calculations are shown in Table 7 and Fig. 10. In this calculation we select successive integers of longitude between two locations along the great ellipse (the difference of longitude between intermediate points can be selected by the user and can be as short as desired).

Comparing the distance of GE sailing calculated by our algorithm mentioned above between Sydney and Valparaiso (6129.120726 nautical miles) with the corresponding true geodesic distance (6129.112448 nautical miles) calculated by built-in geodesicfwd.m function of Matlab (Vicenty's algorithm, Adapted from U.S. National Geodetic Survey (NGS) Fortran program INVERSE.FOR, Version 200208.19 by Stephen J. Frakes, including subroutines GPNHRI and GPNLOA by Robert (Sid) Safford.) gives the fact that even for this extremely long distance with difference of longitude about 137°, the little discrepancy (15.33 m) is still negligible for the practical purposes of navigation. The older numerical test calculated by Pallikaris (2009) [8] gave the bigger discrepancy (0.71 nautical miles, about 1315 meters). The value is 86 times computed value here. This result is too exaggerative to make us doubt. We don't know how the author got it.

The second numerical example of very long navigational route with difference of longitude greater than 145° is the sailing from Valparaiso-Chile (32° 59.998' S, 71° 36.675' W) to Yokohama-Japan (34° 26.178' N, 139° 51.39 E).

The results of these calculations are shown in Table 8. The geodesic is slightly curved than the great ellipse (see Fig. 11). The eccentricity of the Earth exaggeratedly is set to 0.5 for showing the difference between the great ellipse and geodesic (see Fig. 11). In this example the discrepancy between the GE distance computed by our algorithm (9242.561583 nautical miles) and the true geodesic distance computed by Vicenty's algorithm (9242.558036 nautical miles) is a little smaller

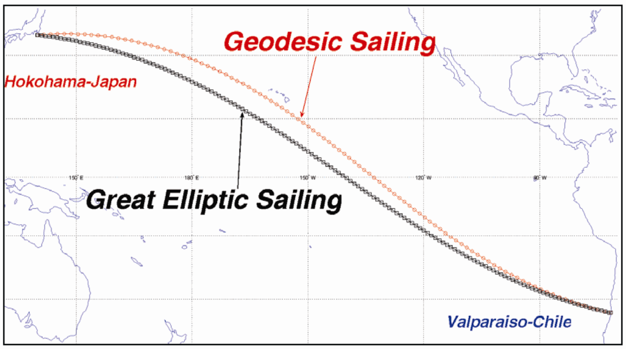

Fig. 11. The track from Yokohama, Japan to Valparaiso, Chile.
Note: To show that the difference between the two, the eccentricity exaggeratedly is set to 0.5.

(6.57 meters) acceptable for the practical purposes of navigation. Because the track passes through the Equator, and then the discrepancy becomes smaller. The older numerical test calculated by Pallikaris (2009) [6] also gave the bigger discrepancy (0.88 nautical miles, about 1356.76 meters). The value is 208 times the value computed here. This result is too exaggerative to make us doubt again.

It is noted that even for these two extreme cases where the differences of longitude between departure and destination points are about 136° and 145°. The resulting discrepancies that are still less than 17 meters are practically diminished in the process of the computation of the intermediate points. Our algorithm computes these coordinates for as many intermediate points as desired that is easier, more accurate and efficient than other methods provided by previous researcher (see Table 7).

VIII. CONCLUSION

In this paper, we have presented a method for computing the position, the distance, and the course of intermediate points along a great ellipse. With basic vector analysis, the mathematical derivations presented here are more straightforward. A variety of expressions are suited to both the syntax of computer algorithms and commercial mathematics software. We have developed a course reduction function instead of the solution of GE. Since the GE equation and GC equation are the same, many formulae tackling the problems of GC sailing also can be applied or reduced to GE sailing.

The differences of distances between the GC sailing and Geodesics passing one quarter of the Earth can reach to maximum about 17.95KM along the Equator. The minimum value is 1.165729 KM along one meridian arc. This is not acceptable for practical purposes of navigation and ECDIS. The GE sailing can overcome those drawbacks of GC sailing and is more computer-efficient than Geodesic mathematics. The proposed algorithm for GE sailing provides extremely high accuracies comparable to those obtained by the computations of geodesics. Numerical tests show that discrepancies between geodesic and GE sailing are practically negligible for navigation and ECDIS.

The numerical tests calculate the mean difference and the standard deviation of large sample of distance differences comparing the values computed by GE sailing and Andoyer-Lambert method to the true geodesic distances. The result reveals that the difference and the standard deviation of distance differences of GE sailing is one half and one sixth of the values computed by Andoyer-Lambert method. The significance gives the assertion that the accuracy of GE sailing is better than Andoyer-Lambert method.

By above analysis, the hybrid sailing mixed with the features of the GC sailing and GE sailing is a nice, simpler, and straightforward alternative. The GC sailing gives waypoints, courses and some parameters used in the GE sailing, and then the GE sailing proceeds to calculate the distance from waypoints to departure point and the course computed by course reduction function of GC. The proposed algorithm can be easily implemented by software language such as C++, Java, Javascript, Matlab and some mathematical packets. The gist of this paper is to facilitate navigators and designers of GIS or electronic chart to design the navigational software more efficiently, accurately, and easily.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Council of Taiwan, Republic of China, under grant NSC 98-2410-H-019-019- and NSC 99-2410-H-019-023-.

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