A NUMERICAL STUDY OF CUBIC PARABOLAS ON RAILWAY TRANSITION CURVES

Tsung-I Shen
Department of Civil Engineering, National Taipei University of Technology, Taipei, Taiwan, R.O.C

Che-Hao Chang
Department of Civil Engineering, National Taipei University of Technology, Taipei, Taiwan, R.O.C

Kuan-Yung Chang
Department of Civil Engineering, National Taipei University of Technology, Taipei, Taiwan, R.O.C

Cho-Chien Lu
Department of Civil Engineering, National Kaohsiung University of Applied Sciences, Kaohsiung, Taiwan, R.O.C.,
cclu@kuas.edu.tw

Recommended Citation

A NUMERICAL STUDY OF CUBIC PARABOLAS ON RAILWAY TRANSITION CURVES

Tsung-I Shen[1], Che-Hao Chang[1], Kuan-Yung Chang[1], and Cho-Chien Lu[2]

Key words: railway, transition curve, route alignment, cubic parabola.

ABSTRACT

This paper mainly focuses on the design of transition curves of the cubic parabola type in track alignment design. Equations for the transition curves are first derived from the theory of cubic parabolas using calculus techniques. They are then analyzed using numerical analysis methods. The proposed formulation is evaluated by comparison of its calculated results with data in 684 actual cases of transition curves. The accuracy of the proposed method is verified in this study by comparing the estimated results with collected data from actual engineering projects, and the applicability of the new method to the engineering practice of track alignment design is justified. The chainage and coordinates of the control points as well as other curve points and the tangential angle can be easily calculated using the transition curve equations.

I. INTRODUCTION

Upon satisfying the basic requirements of safety, stability, and comfort, rail systems seek to optimize speed and efficiency [16, 19]. Railway alignment design is important because it dictates the spatial arrangement of the railways and thus the locations of civil infrastructure and mechanical and electrical facilities. Alignment design code [11, 12, 15] is the basic standard for railway alignment design in Taiwan.

Trains in operation are exposed to the risk of derailment resulting from severe vibration induced by centrifugal forces if the following three situations occur: (1) the train enters a curve from a straight line; (2) the train travels from a curve to another curve of different radius; (3) the train leaves a curve and enters a straight line. These sudden changes can also cause discomfort to passengers.

The consistency and continuity of track alignment design are essential in improving the travel speed and efficiency of transportation. The quality of alignment design affects the arrangement of stations and facilities along the track line. Alignment comprises straight lines, circular arcs, and transition curves [6, 7, 18]. A good alignment design utilizes smooth shapes. Prior to the use of computers in engineering (before 1975), alignment design mainly involved adopting circular arcs for ease of construction, and due to the limited development of information technology and computers. However, circular curves can no longer meet the requirement of passenger comfort in modern rail systems. Therefore, transition curves must be placed between two circular curves and where straight lines meet circular curves. According to relevant railway alignment regulations and research [2, 5, 11, 13], the majority of transition curves used in rail engineering are designed based on the cubic parabola. A cubic parabola is a nonlinear function curve, and its calculation is difficult and cumbersome for coordinates of any given point on the curve using a geometric mathematical method. The coordinates that include the N coordinate and the E coordinate are a datum coordinate system. Track alignment design serves as an important reference for the rest of the construction; therefore, engineering practice demands the ability to calculate coordinates of any point on the alignment. The present study attempts to use numerical methods to acquire numerical solutions of cubic parabola type transition curves.

Numerical analysis is a method of obtaining approximate solutions for a particular problem using mathematical algorithms [3, 4, 14]. Due to the complexity of most engineering problems, analytical solutions often cannot be obtained. Numerical analysis is therefore necessary to reach an approximation. Newton's method is a typical numerical algorithm used to find the approximate roots of nonlinear functions. Newton's method involves finding the roots of the Taylor series expansion of a function using an iterative process, in order to estimate its actual roots. The main concept is to start with an initial guess for a root of the function, then substitute this root into the function. The resulting value is again used as a root and substituted into the function. Eventually, the actual root is reached after many iterations. The current study utilizes Newton's numerical method to analyze the cubic parabola type of transition curves, and proposes a numerical
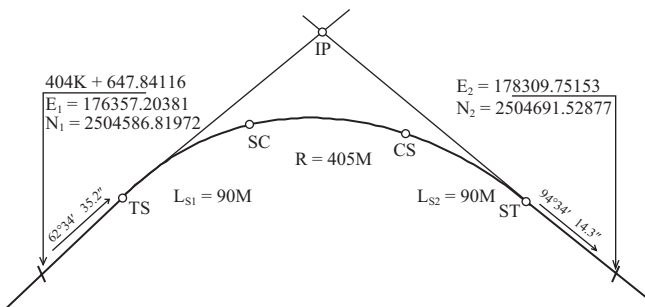
Fig. 1. A typical curve of route alignment.

solution strategy for nonlinear equations describing such transition curves [8, 13]. Following the discussion of basic theories of transition curves, curve alignment formulas are derived. The algorithm and the procedure of the derivation are summarized. Actual alignment designs of some major domestic transport engineering projects are collected. These cases are analyzed using the proposed formulations, and the results are compared with the actual designs to evaluate the correctness and applicability of the proposed equations.

II. THE ROLE OF RAILWAY TRANSITION CURVES

A transition curve is a segment with a gradually changing radius connecting a straight rail section with a curved section. It has three functions: (1) ease the transition and prevent accidents caused by sharp changes in direction; (2) reduce shaking of vehicles in operation; (3) make the transition smoother, more stable, and safer.

The radius of a transition curve changes with its length [1]. A typical route alignment is shown in Fig. 1. There are four control points on this alignment: (1) Tangent to Spiral, TS; (2) Spiral to Curve, SC; (3) Curve to Spiral, CS; and (4) Spiral to Tangent, ST [6, 7, 18]. The radius of curvature of the straight section can be considered infinite, gradually changing to the radius of the circular curve through the transition curve.

In addition to transition curves, to ensure traffic safety and provide comfort to passengers, superelevation will also be introduced into the alignment design to help offset centrifugal forces developed from the tilting plane. It is well known nowadays that simultaneous introduction of the transition curve and superelevation ramp is the only effective solution for straight track–circular curve railway connections; this has been applied in practice for years. Furthermore, it is also known that both should be defined with the same function and possess the same length [9].

Route alignment can be divided into horizontal alignment and vertical alignment. In track engineering, straight lines are favored in horizontal alignment. However, when constrained by topography and the surrounding environment, plane curves are required to change the direction of the route. There are three types of transition curves that are commonly used in

railway horizontal alignment: (1) Clothoid; (2) Half-Sine; and (3) Cubic Parabola. Railway regulations and alignment standards in Taiwan state that transition curves must be designed as cubic parabolas. The current study presents basic theories of the cubic parabola and goes on to discuss its application in engineering practice.

III. NUMERICAL MODELS OF TRANSITION CURVES

The proposed numerical model of transition curves involves computing engineering quantities at any point on the curve based on a geometric mathematical model. These quantities include chainage, the N-coordinate, the E-coordinate, and the tangential angle. The N-E coordinate system, introduced previously, is the datum coordinate system in engineering. The tangential angle is defined as the angle between the tangent line at a given point and the tangent line of the route (the line connecting the TS control point and IP control point). A geometric mathematical model can distinctly express the geometric relationship between the radius of curvature R and the arc length in the track alignment. However, it is not easy to directly calculate the coordinates of a random point on the curve. To use the design outcome from the geometric mathematical model, engineers must set up measuring equipment at the starting point of the curve, refer to the IP point as the backsight point, then set out positions of other points on the curve according to geometric relationships involving deflection angle, chord length, etc. It is difficult to verify results obtained using this approach, and systematic errors often accumulate. A total station is a measuring instrument widely used in modern route surveying. Modern total stations contain programs for coordinate-based numerical methods to aid in setting-out and surveying operations. In engineering practice, chainage, N- and E-coordinates, and tangential angle at every chainage point must be prepared prior to surveying. In light of this, geometric numerical models can no longer meet the design requirements of curved routes. It is therefore important to develop new mathematical models that can describe geometric relationships and process engineering information.

The purpose of this study is to develop a numerical model for transition curves, which allows direct calculation of coordinates and other engineering quantities at any given point on the curve. We start from the curvature of transition curves, and then cover (in order) cubic parabolic equations, length of transition curves, horizontal and vertical distances of transition curves, and the use of Newton's method to solve for length of the curve.

1. Curvature of Transition Curves

Assume that the purpose of a transition curve in between a straight section and a circular arc section is to provide a linear change of curvature. A typical transition curve is shown in Fig. 2. In this study, "*r*" denotes the radius of curvature of
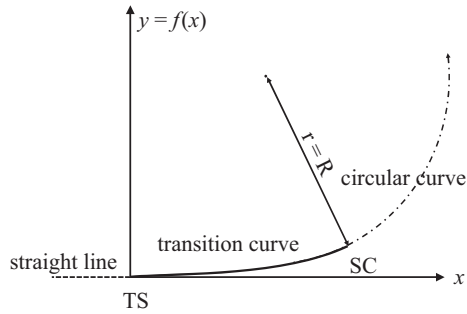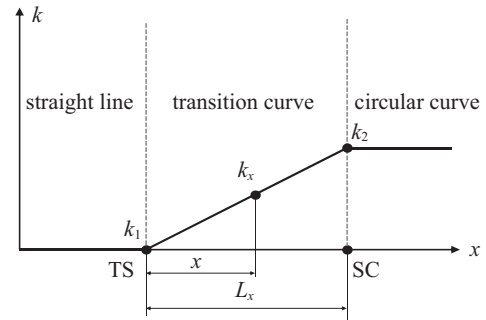
Fig. 2. A typical introduction curve.

Fig. 3. Curvature change of the transition curve.

the transition curve. The radius of curvature at the intersection of the transition curve and the tangent is infinitely large ($R = \infty$), and the radius of curvature at the intersection of the transition curve and the circular curve is the same as that of the circular curve itself ($r = R$). Fig. 3 shows the curvature change of the transition curve. In this figure, K_1 represents the curvature of the tangent, which is expressed mathematically as (1a); K_2 represents the curvature of the circular curve, and the mathematical expression is given as (1c); and K_x is the curvature of the transition curve. The curvature at a point that is of distance x away from the start point (TS) can be calculated from (1b).

$$k_1 = (1/\infty) = 0 \tag{1a}$$

$$k_x = \frac{x}{RL_x} \tag{1b}$$

$$k_2 = \frac{1}{R} \tag{1c}$$

2. Equations of Cubic Parabola Transition Curves

The relationship between the curvature and the y coordinate of the curve can be derived by integrating the curvature expression. In general, calculus textbooks, the curvature (k) is given as (2). Assuming the transition curve is gentle enough, $(dy/dx)^2$ is then close to zero, and the curvature becomes Eq. (3).

$$k_x = \frac{d^2 y / dx^2}{\left[1 + \left(dy/dx\right)^2\right]^{\frac{3}{2}}} \tag{2}$$

$$k_x = \frac{d^2 y}{dx^2} = \frac{x}{RL} \tag{3}$$

Twice integrating Eq. (3) gives Eq. (4), which is the equation of the cubic parabola transition curve:

$$y = \frac{x^3}{6RL} \tag{4}$$

3. Equation of the Arc Length of the Transition Curve

In railway engineering design, the radius of curvature of the circular curve as well as the start and end point of the tangent can be treated as constants. The curvature of the transition curve, which connects the tangent and the circular curve, increases linearly. Therefore, the arc length of the transition curve must be determined first before solving for other details of the curve. In general calculus textbooks, the arc length can be calculated as:

$$L_x = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \tag{5}$$

Integrating Eq. (3) gives the expression for dy/dx, and substituting this into Eq. (5) gives Eq. (6), which allows the calculation of arc length of cubic parabola transition curves.

$$L_x = \int \left(1 + \frac{x^4}{4R^2L^2}\right)^{\frac{1}{2}} dx \tag{6}$$

An analytical equation cannot be easily obtained by integrating Eq. (6). This study utilizes McLaughlin's binomial theorem to find the approximate solution of Eq. (6).

In order to expand Eq. (6) to a McLaughlin binomial series, we take "a" as that given in Eq. (7) and substitute it into (6) to arrive at Eq. (8).

$$a = \frac{x^4}{4R^2L^2} \tag{7}$$

$$f(a) = (1 + a)^{\frac{1}{2}} \tag{8}$$

McLaughlin's binomial series can be tabulated as shown in Table 1.

Finally, is expanded to Eq. (9):

$$f(a) = f(0) + \sum \frac{f^{(n)}(0)}{n!} a^n = 1 + \frac{1}{2}a - \frac{1}{8}a^2 + \frac{1}{16}a^3 - \frac{5}{128}a^4 \tag{9}$$

Table 1. McLaughlin's Binomial Series.

n order	$f^{(n)}(a)$	$f^{(n)}(0)$	$\dfrac{f^{(n)}(0)}{n!}$
original	$f(a)=(1+a)^{\frac{1}{2}}$	1	-
1st. order	$\dfrac{1}{2}(1+a)^{-\frac{1}{2}}$	$\dfrac{1}{2}$	$\dfrac{1}{2}a$
2nd. order	$\left(-\dfrac{1}{2}\right)\dfrac{1}{2}(1+a)^{-\frac{3}{2}}$	$-\dfrac{1}{4}$	$-\dfrac{1}{8}a^2$
3rd. order	$\left(-\dfrac{3}{2}\right)\left(-\dfrac{1}{2}\right)\dfrac{1}{2}(1+a)^{-\frac{5}{2}}$	$\dfrac{3}{8}$	$+\dfrac{1}{16}a^3$
4th. order	$\left(-\dfrac{5}{2}\right)\left(-\dfrac{3}{2}\right)\left(-\dfrac{1}{2}\right)\dfrac{1}{2}(1+a)^{-\frac{7}{2}}$	$-\dfrac{15}{16}$	$-\dfrac{5}{128}a^4$

Eq. (10) can be obtained by substituting "*a*" in Eq. (7) into Eq. (9):

$$f(x)=1+\frac{1}{2}\left(\frac{x^4}{4R^2L^2}\right)-\frac{1}{8}\left(\frac{x^4}{4R^2L^2}\right)^2$$
$$+\frac{1}{16}\left(\frac{x^4}{4R^2L^2}\right)^3-\frac{5}{128}\left(\frac{x^4}{4R^2L^2}\right)^4 \qquad (10)$$

Integrating Eq. (10) with respect to *x* gives the equation for the arc length of the transition curves:

$$L_x = \int f(x)dx$$
$$= \int\left[1+\frac{1}{2}\frac{x^4}{(4R^2L^2)}-\frac{1}{8}\frac{x^8}{(4R^2L^2)^2}\right.$$
$$\left.+\frac{1}{16}\frac{x^{12}}{(4R^2L^2)^3}-\frac{5}{128}\frac{x^{16}}{(4R^2L^2)^4}\right]dx$$
$$= x+\frac{1}{10}\frac{x^5}{(4R^2L^2)}-\frac{1}{72}\frac{x^9}{(4R^2L^2)^2}$$
$$+\frac{1}{208}\frac{x^{13}}{(4R^2L^2)^3}-\frac{5}{2176}\frac{x^{17}}{(4R^2L^2)^4}$$
$$= x\left[1+\frac{1}{10}\left(\frac{x^4}{4R^2L^2}\right)-\frac{1}{72}\left(\frac{x^4}{4R^2L^2}\right)^2\right.$$
$$\left.+\frac{1}{208}\left(\frac{x^4}{4R^2L^2}\right)^3-\frac{5}{2176}\left(\frac{x^4}{4R^2L^2}\right)^4\right] \qquad (11)$$

4. Equations for *x* Coordinate and *y* Coordinate of the Transition Curve

The equation for the *x* coordinate (*x*) of the transition curve

(12a) can be obtained by re-arranging Eq. (11). Eq. (4) for the *y* coordinate (*y*) has been derived previously. The equations for the *x* coordinate and *y* coordinate are summarized in Eqs. (12a) and (12b).

$$x = L_x /\left[1+\frac{1}{10}\left(\frac{x^4}{4R^2L^2}\right)-\frac{1}{72}\left(\frac{x^4}{4R^2L^2}\right)^2\right.$$
$$\left.+\frac{1}{208}\left(\frac{x^4}{4R^2L^2}\right)^3-\frac{5}{2176}\left(\frac{x^4}{4R^2L^2}\right)^4\right] \qquad (12a)$$

$$y = \frac{x^3}{6RL} \qquad (12b)$$

5. Solving the Arc Length Equation Using Newtown's Method

The equation for the *x* coordinate (12a) contains 4 variables, which makes it difficult to solve. As a result, a numerical method is used in this study. First, we express *dy/dx* as *K* (13a). The square of *K* can then be expressed as (13b). Differentiating *dy/dx* gives us Eq. (13c). Substituting (13) into Eq. (11), we obtain Eq. (14), and substituting (13) into Eq. (2) gives Eq. (15).

$$K = \frac{dy}{dx} = \frac{x^2}{2RL} \qquad (13a)$$

$$K^2 = \frac{x^4}{4R^2L^2} \qquad (13b)$$

$$\frac{d^2y}{dx^2} = \frac{x}{RL} \qquad (13c)$$

$$L_x = x\left(1+\frac{1}{10}K^2-\frac{1}{72}K^4+\frac{1}{208}K^6-\frac{5}{2176}K^8\right) \qquad (14)$$

$$\frac{1}{R} = \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{\frac{3}{2}}} = \frac{x/RL}{\left[1+(x^2/2RL)^2\right]^{\frac{3}{2}}} = \frac{x/RL}{\left(1+K^2\right)^{\frac{3}{2}}} \qquad (15)$$

Multiplying (14) by (15) gives:

$$\frac{L_x}{R} = \frac{x^2/RL}{(1+K^2)^{\frac{3}{2}}}\left(1+\frac{1}{10}K^2-\frac{1}{72}K^4+\frac{1}{208}K^6-\frac{5}{2176}K^8\right) \qquad (16)$$

Multiplying both sides by 1/2:

$$\frac{L_x}{2R} = \frac{x^2/2RL}{\left(1+K^2\right)^{\frac{3}{2}}}\left(1+\frac{1}{10}K^2-\frac{1}{72}K^4+\frac{1}{208}K^6-\frac{5}{2176}K^8\right) \qquad (17)$$

Substituting K into the equation above:

$$\frac{L_x}{2R} = \frac{K}{\left(1+K^2\right)^{\frac{3}{2}}}\left(1+\frac{1}{10}K^2-\frac{1}{72}K^4+\frac{1}{208}K^6-\frac{5}{2176}K^8\right) \tag{18}$$

After expanding Eq. (18) and rearranging, we obtain the nonlinear Eq. (19)

$$2925RK^9 - 6120RK^7 + 17680RK^5 - 127296RK^3$$

$$- 12729460RK + 636480L_x(1+K^2)^{3/2} = 0 \tag{19}$$

The R in Eq. (19) is the radius of curvature of the circular curve. L_x is calculated according to the standard from three factors: superelevation, cant deficiency, and speed. R and L_x can both be treated as known variables, leaving K as the unknown for which the equation needs to be solved. This study uses Newtown's method to solve for K in the nonlinear equation. Eq. (20) is set as the objective function. The K value at the iteration step of K_n, K_{n-1}, K is the solution. The values K_1, K_2, and K_3 are calculated according to Eqs. (21a), (21b), and (21c), respectively.

$$f(K) = 2925RK^9 - 6120RK^7 + 17680RK^5 - 127296RK^3$$

$$- 12729460RK + 636480L_x(1+K^2)^{3/2} \tag{20}$$

$$K_1 = K - \frac{f(K)}{f'(K)} \tag{21}$$

$$K_2 = K_1 - \frac{f(K_1)}{f'(K_1)} \tag{21b}$$

$$K_3 = K_2 - \frac{f(K_2)}{f'(K_2)} \tag{21c}$$

Rearranging Eq. (14) leads to Eq. (22a). K in (22a) is computed using Newtown's method, while L_x is calculated according to the standard. The x coordinate and y coordinate, (x) and (y), of the transition curve can be solved by substituting L_x into (22a) and (22b) [3, 4, 8, 14].

$$x = \frac{L_x}{\left(1+\dfrac{K^2}{10}-\dfrac{K^4}{72}+\dfrac{K^6}{208}-\dfrac{5K^8}{2176}\right)} \tag{22a}$$

$$y = \frac{x^3}{6RL_x} \tag{22b}$$

IV. CASE STUDY: THE DESIGN OF RAILWAY TRANSITION CURVES IN THE TAIWAN RAILWAY RECONSTRUCTION BUREAU

The Railway Reconstruction Bureau (R.R.B.) is the main executive organization for major railway construction in Taiwan. This study gathered actual cases of basic railway route alignment from the R.R.B. as the basis for analysis. Transition curves in these alignments are all formulated as cubic parabolas [11, 12, 15].

In Taiwan, in order to ensure traffic safety, the R.R.B. reviews design drawings of track route alignment. Alignment designs by consulting firms may differ due to the usage of different application software. This review mechanism can ensure the quality of design of the curve parameters.

This study collected data on transition curves from 684 actual R.R.B. engineering projects. Each transition curve can be described by four control points, including (1) Tangent to Spiral, TS; (2) Spiral to Curve, SC; (3) Curve to Spiral, CS; and (4) Spiral to Tangent, ST [7, 18]. Each point on the curve has its chainage, coordinates (including the N coordinate and the E coordinate), and tangential angle. We compare data collected on transition curves with those calculated by the method proposed herein.

1. Application of the Numerical Models

The reasons for developing equations for the cubic parabola transition curves are (1) to provide the client with a tool to verify a consulting firm's design product, and (2) to provide site engineers with a convenient way of obtaining coordinates of any given point on the curve. The client examines two aspects of the design product: (1) values of various key parameters on the curve drawings and (2) overall route alignment and coordinate details.

The outcome of this research leads to rapid and accurate calculation of coordinates for any given point on the cubic parabola transition curve. The calculation procedure is summarized in the following five steps:

Step 1: Obtain the tangential angle (Δ) of the curve. This is the angle between the two tangent lines at the start and end of the alignment. The radius (R) of the circular curve is then decided based on the topography and surrounding environment.

Step 2: The total length of the transition curve, L_s, is evaluated. The length is taken as the maximum of three criteria specified in rail system alignment codes or standards. The three factors involved in the determination of the curve length are design speed (V), superelevation (C), and cant deficiency (C_d).

Step 3: K is computed using a numerical method, following which x of this point can be calculated. In order to use the transition curve equation proposed in this paper, the radius of the circular curve (R) and total length of the transition curve (L_s) must be determined first.

Both parameters are treated as given in the Eq. (20). The x coordinate and the y coordinate, (x) and (y), are the unknowns that must be solved.

Step 4: Calculate the x coordinate and y coordinate, (x) and (y), from the equations of the transition curve. The x value is determined first, then y is calculated [10].

Step 5: This final step is to transform the coordinate (x, y) from the x–y coordinate system to the N–E coordinate system. This relates two 2–D Cartesian coordinate systems through a rotation followed by a translation. The rotation is defined by one rotation angle (α) and the translation is defined by two origin shift parameters (x_o, y_o). The transform equation is shown as Eqs. (23a) and (23b).

$$E = x\cos(\alpha) - y\sin(\alpha) + x_o \qquad (23a)$$

$$N = x\sin(\alpha) + y\cos(\alpha) + y_o \qquad (23b)$$

2. Verification of the Numerical Models

In our research, equations for cubic parabola transition curves are written into a computer program. Error analysis is performed on all 684 transition curves collected from actual cases. For each transition curve, four control points are taken as the baseline for comparison. They are (1) Tangent to Spiral, TS; (2) Spiral to Curve, SC; (3) Curve to Spiral, CS; (4) Spiral to Tangent, ST. Each of the four points contains (1) chainage (S), (2) N coordinate (N), (3) E coordinate (E), and (4) tangential angle (θ). There are 10944 comparisons analyzed in total.

Each comparison includes the following: (1) difference in the chainage (ΔS); (2) difference in the N coordinate (ΔN); (3) difference in the E coordinate (ΔE); and (4) difference in the deflection angle ($\Delta\theta$). The difference in chainage is in units of cm, the differences in the coordinates are in mm, and the difference in the deflection angle is in seconds.

Table 2 presents the results of the comparison, from which it can be deduced that the proposed equations of cubic parabola transition curves and the developed computer program can be applied to route alignment design in rail engineering. The mean difference of the tangential angle ($\Delta\theta$) is 0.00", and the standard deviation is 1.08". This demonstrates that the curve design generated by the proposed equations in this study is in agreement with the traditional alignment design using a non-coordinate method. The mean difference in the chainage is 0.01 cm, and the standard deviation is 0.50 cm. Results show consistency with the traditional method of alignment design by setting out pegs on the curve. This means the total route length calculated from the new method does not differ from the traditional method. Furthermore, the proposed coordinate method can provide better and more accurate control over costs. Differences in the vertical coordinates between the proposed method and actual data have a mean difference of 0.01 mm and a standard deviation of 2.18 mm, while the

Table 2. Differences in curve details between calculated results and actual data.

Differences in curve details				
	Chainage difference (cm)	ΔE (mm)	ΔN (mm)	$\Delta\theta$ (")
Mean (μ)	0.01	0.10	-0.10	0.00
Standard deviation (σ)	0.50	2.18	4.51	1.08

mean difference and standard deviation in the horizontal coordinates are -0.01 mm and 4.51 mm, respectively. This shows the high precision of the proposed method and verifies its application in the engineering practice of alignment design [17].

V. CONCLUSION

The theory behind the cubic parabola-based transition curves discussed in this paper is not new; however, clear explanations of related concepts are indeed rare in textbooks and published literature. The main design concept of the curved route proposed here is based on coordinates, and has two advantages: (1) it is convenient to survey using modern instruments; and (2) measured results are more easily verified. McLaughlin's Binomial Series was first used to construct the equation of the transition curve's arc length. Newton's method was subsequently applied to solve for K in the equations containing radius of curvature R, arc length of transition curve L_x, and K. The arc length of the transition curve is set equal to the total length of the curve ($L_x = L_s$), and L_x and R are constants.

In engineering practice, the radius of circular arcs (R) and total length of the curve (L_s) are regulated in the route alignment design code. From these two quantities, the value of K can be calculated. As a result, K and R can be seen as constants when solving for the x and y coordinates. The x coordinate changes with the arc length, L_x, while the y coordinate varies with both x and L_x.

K represents the gradient (dy/dx) at a given point on the transition curve. The tangential angle can be obtained by taking the arc tangent in accordance with the K value at the point of interest. The tangential angle can be used in surveying at the TS control point.

The proposed transition curve equations can completely describe the features of the curve and calculate coordinates of any point on the curve, including pegs. Railway alignment personnel can conveniently and rapidly review and verify design drawings and details provided by consulting firms.

Although the research outcomes can be directly applied to railway alignment practice, the following recommendations are offered to facilitate computerization in this area:

1. A standard protocol of operation should be in place to regulate procedures of route alignment design. This can prevent

civil construction from being carried out too hastily, such as before the route alignment drawings and relevant information are confirmed. Otherwise, clearance problems may arise, in which case infrastructure in the clearance region must be removed. This results in a waste of time and resources, or modification of the alignment design to accommodate the structures already built, meaning the optimized design cannot be achieved.

2. In addition to horizontal alignment, the complete track alignment design also includes vertical alignment and earthwork estimation. This research focuses on horizontal alignment only. Vertical alignment and earthwork calculations will be investigated in future work.

3. Further work will be carried out on the visualization of simulation results from numerical analysis in the hopes of speeding up the decision making process in rail engineering route alignment design.

REFERENCES

1. Australian Rail Track Corporation LTD., "Survey definition of alignment and kilometrage," *Track of Engineering Standard-NSW*, TEP 25, RIC Standard: C 4601 (2005).

2. Bonnett, C. F., *Practical Railway Engineering*, Imperial College Press (2005).

3. Burden, R. L. and Faires, J. D., *Numerical Analysis*, Thomson Brooks/Cole (2011).

4. Cheney, W. and Kincaid, D., *Numerical Mathematics and Computing*, Thomson Brooks/Cole (2008).

5. Esveld, C., *Modern Railway Track*, Delft University of Technology (2001).

6. Kavanagh, B. F., *Surveying with Construction Applications*, Prentice Hall (2009).

7. Kavanagh, B. F., *Surveying: Principles and Applications*, Prentice Hall (2008).

8. Kincaid, D. and Cheney, W., *Numerical Analysis: Mathematics of Scientific Computing*, Thomson Brooks/Cole (2002).

9. Levien, R., *The Euler Spiral: A Mathematical History*, Technical Report No. UCB/EECS-2008-111, Electrical Engineering and Computer Sciences, University of California at Berkeley, CA, U.S.A. (2008).

10. Long, X.-Y., Wei1, Q.-C., and Zheng, F.-Y., "Dynamic analysis of railway transition curves," *Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid Transit*, Vol. 224, No. 1, pp. 1-14 (2010).

11. Railway Reconstruction Bureau, *Railway Operations Planning Guide*, Railway Reconstruction Bureau, Ministry of Transportation and Communications (2010). (in Chinese)

12. Railway Reconstruction Bureau, *The Planning Report of Zuoying Railway Underground Project*, Railway Reconstruction Bureau, Ministry of Transportation and Communications (2009). (in Chinese)

13. Shen, T.-I., *The Application Study of Transitional Curve in Railway Engineering,* Master Thesis, Department of Civil Engineering, National Taipei University of Technology, Taipei, Taiwan (2001). (in Chinese)

14. Süli, E. and Mayers, D. F., *An Introduction to Numerical Analysis*, Cambridge University Press (2003).

15. Taipei City Railway Reconstruction Bureau, *The Planning Report of Kaohsiung Railway Underground Project*, Taipei City Railway Reconstruction Bureau (2000). (in Chinese)

16. Vermeij, I. D. J., "Design of a high speed track," *HERON*, Vol. 45, No.1, pp. 9-23 (2000).

17. Wolf, P. R. and Ghilani, C. D., *Adjustment Computations: Statistics and Least Squares in Surveying and GIS*, John Wiley and Sons Inc. (1997).

18. Wolf, P. R. and Ghilani, C. D., *Elementary Surveying: An Introduction to Geomatics,* Prentice Hall (2006).

19. Zboiński, K. and Woźnica, P., "Optimization of the railway transition curves' shape with use of vehicle-track dynamical model," *Archives of Transport*, Vol. 22, No. 3, pp. 387-407 (2010).