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Pin-Lin Liu

Department of Electrical Engineering, Chienkuo Technology University, Taiwan, R.O.C., pl@ctu.edu.tw

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FURTHER RESULTS ON ROBUST EXPONENTIAL STABILIZATION FOR TIME-VARYING DELAY SATURATING ACTUATOR SYSTEMS WITH DELAY-DEPENDENCE

Pin-Lin Liu

Key words: Leibniz-Newton formula, linear matrix inequality (LMI), generalized eigenvalue problem (GEVP), delay-dependence.

ABSTRACT

The problem of delay-dependent exponential robust stabilization for a class of uncertain saturating actuator systems with time-varying delay is investigated. Novel exponential stability and stabilization criteria for the system are derived using the Lyapunov-Krasovskii functional combined with Leibniz-Newton formula. The issue of exponential stabilization for time-varying delay systems with saturating actuator using generalized eigenvalue problem (GEVP) approach remains open, which motivates this paper. The designed controller is dependent on the time-delay and its rate of change. All the conditions are presented in terms of linear matrix inequalities (LMIs), which can be solved efficiently by using the convex optimization algorithms. A state feedback control law is also given such that the resultant closed-loop system is stable for admissible uncertainties. Two numerical examples are given to demonstrate the efficiency of the obtained results.

I. INTRODUCTION

Both time-delay and saturating controls are commonly encountered in various engineering systems and are frequently a source of instability. Time delays are frequently encountered in various areas, including physical and chemical processes, economics, engineering, communication, networks and biological systems, etc. The existence of a time delay is often a source of oscillations, instability and poor performance in a

system. Many methods to check the stability of time delay systems [1-26]. Nearly all physical systems are subject to saturation constraints, such as actuator saturation and/or sensor saturation. It is known that actuator saturation may have adverse effects on the performance and stability of a closed-loop system if the controller is designed without considering this kind of nonlinearity. Consequently, a great deal of attention has been focused on the stability analysis and controller design for systems with a saturating actuator [3-6, 9, 10, 12-15, 17-19, 21, 23-25] and references therein. Furthermore, the problem of the stabilization of uncertain systems with state delay has attracted an important amount of interest in recent years [4, 8, 10, 11, 15, 17, 18, 20, 21, 23-25]. The problem of uncertain systems stabilization with saturating control has recently motivated an important effort of research due to its practical importance [4, 10, 15, 17, 18, 21, 23-25] and references therein. The use of Lyapunov functionals is certainly the main approach for deriving sufficient conditions for asymptotic stability. In fact, some of the results are indeed equivalent to the LMIs formulations in view of the Schur complement. Instead of applying the Lyapunov function, properties of comparison theorem and matrix measure with model transformation technique are employed to investigate the problems [4, 13, 19, 21].

Since delay is usually time-varying in many practical systems, many approaches have been developed to derive the delay-dependent stability criteria for saturating actuator systems with time-varying delays, for example, Razumikhin theorem [9, 16, 17], the improved Riccati equation [10, 22, 25], integral inequality matrices [12], and the properly chosen Lyapunov-Krasovskii functionals [12, 14, 15]. Control saturation constraint comes from the impossibility of actuators to drive signal with unlimited amplitude or energy to the plants. However, only few works have dealt with stability analysis and the stabilization of time-varying systems in the presence of actuator saturation [14]. For linear systems with time-varying delays, the reported results are generally based on the assumption that the derivative of time-varying delays is less than one, which is, $0 \leq \dot{h}_d < 1$ [14]. Such restriction is very

*Paper submitted 10/29/10; revised 12/20/11; accepted 01/09/12. Author for correspondence: Pin-Lin Liu (e-mail: lpl@ctu.edu.tw).
Department of Electrical Engineering, Chienkuo Technology University, Taiwan, R.O.C.*

conservative and of no practical signification. In the present paper we fill the gap between the case of the delay derivative not greater than 1 and the fast-varying delay by deriving a new integral operator bound. This bound is an increasing and continuous function of the delay derivative bound $h_d \geq 1$. In the limit case (where $h_d \rightarrow \infty$) which corresponds to the fast-varying delay, the new bound improves the existing one. As a result, improved frequency domain and time domain stability criteria are derived for systems with the delay derivative bound greater than 1.

On the other hand, the decay rates (i.e. convergent rates or convergence rates) are important indices of practical systems, and the exponential stability analysis of time-delay systems has been a popular topic in the past decades; see for examples [11, 14] and their references. Via strict LMI optimization approaches, Liu provides an easy-to-check condition for a delayed system without uncertainties [11, 14]. By similar methodologies as in [14], the exponential stability of saturating actuator systems containing time-varying state delays is discussed. However, to the best of the authors' knowledge, the issue of robust exponential stabilization for saturating actuator systems with time-varying delays remains open, which motivates this paper.

In this paper, we are interested in designing a state-feedback controller for a class of linear time-varying delay systems with actuator saturation. Firstly, an appropriate Lyapunov-Krasovskii functional is constructed and its positive definiteness is proved, by which the constraints on some functional parameters are relaxed. Then, the Leibniz-Newton formula and the convex combination condition of time-varying delay are used to get the new delay-dependent criteria. Through constructing augmented Lyapunov-Krasovskii functionals and using integral inequality matrix, delay-dependent robust exponential stability and stabilization criteria are achieved in terms of linear matrix inequalities (LMIs), which can be solved by various convex optimization algorithms. The obtained results are presented in terms of linear matrix inequalities and are less conservative than some existing stability conditions. To the best of the authors' knowledge, the issue of robust exponential stabilization for time-varying delay systems with saturating actuator using generalized eigenvalue problem (GEVP) approach is a new and open problem in the literatures. Finally, numerical examples are given to illustrate the effectiveness and the benefits of the proposed method.

II. MAIN RESULT

Consider the following time-varying delay system with saturating actuator described by

$$\begin{aligned} \dot{x}(t) = & [A_0 + \Delta A_0(t)]x(t) + [A_1 + \Delta A_1(t)]x(t-h(t)) \\ & + [B + \Delta B(t)]\text{Sat}(u(t)) \end{aligned} \quad (1a)$$

$$x(t) = \phi(t), \quad \forall t \in [-h, 0] \quad (1b)$$

where $x(t) \in R^n$ is the state vector; $u(t) \in R^m$ is the control input vector; x_t is the state at time t denoted by $x_t(s) := x(t+s)$. A_0, A_1 and B are known constant matrices with appropriate dimensions. $\phi(t)$ is a smooth vector-valued initial function.

The time-varying parameter uncertainties $\Delta A_0(t)$, $\Delta A_1(t)$ and $\Delta B(t)$ are assumed to be in the form of

$$[\Delta A_0(t) \quad \Delta A_1(t) \quad \Delta B(t)] = DF(t)[E_0 \quad E_1 \quad E_b] \quad (2)$$

where D, E_0, E_1 , and E_b are known real constant matrices with appropriate dimensions, and $F(t)$ is an unknown, real, and possibly time-varying matrix with Lebesgue-measurable elements satisfying

$$F^T(t)F(t) \leq I, \quad \forall t. \quad (3)$$

Time delay, $h(t)$, is a time-varying continuous function that satisfies

$$0 \leq h(t) \leq h \quad \text{and} \quad \dot{h}(t) \leq h_d \quad (4)$$

where h and h_d are constants.

The saturating function is defined as follows:

$$\text{Sat}(u(t)) = [\text{Sat}(u_1(t)), \text{Sat}(u_2(t)), \dots, \text{Sat}(u_m(t))]^T \quad (5)$$

The operation of $\text{Sat}(u_i(t))$ is linear for $-U_i \leq u_i \leq U_i$ as

$$\text{Sat}(u_i(t)) = \begin{cases} -U_i & \text{if } u_i < -U_i < 0 \\ u_i & \text{if } -U_i \leq u_i \leq U_i \\ U_i & \text{if } u_i > U_i > 0 \end{cases} \quad (6)$$

Throughout this paper we will use the following concept of stabilization for the time-varying delay system with saturating actuator (1).

Definition 1: The time-varying delay system with saturating actuator (1) is said to stable in closed-loop via memoryless state feedback control law if there exists a control law $u(t) = Kx(t)$, $K \in R^{m \times n}$ such that the trivial solution $x(t) \equiv 0$ of the functional differential equation associated to the closed-loop system is uniformly asymptotically stable.

In order to develop our result, by considering a state feedback controls law $u(t) = Kx(t)$ the saturating term $\text{Sat}(Kx(t))$ can be written in an equivalent form:

$$\text{Sat}(Kx(t)) = G(\beta(x))Kx(t), \quad G(\beta(x)) \in R^{m \times n} \quad (7)$$

where $G(\beta(x))$ is a diagonal matrix for which the diagonal elements $\beta_i(x)$ satisfy for $i = 1, 2, \dots, m$.

$$\beta_i(x) = \begin{cases} -\frac{U_i}{(Kx)_i} & \text{if } (Kx)_i < -U_i < 0 \\ 1 & \text{if } -U_i \leq (Kx)_i \leq U_i \\ \frac{U_i}{(Kx)_i} & \text{if } (Kx)_i > U_i > 0 \end{cases} \quad (8)$$

and therefore

$$0 \leq \beta_i(x) \leq 1 \quad (9)$$

The main objective is to find the range of h and guarantee stabilization for the time-varying delay system with saturating actuator (1). When the time delay is unknown, how long time delay can be tolerated to keep the system stable. To do this, two fundamental lemmas are reviewed.

Lemma 1 [11]: For any positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0 \quad (10)$$

Then, we obtain

$$-\int_{t-h(t)}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds \leq \int_{t-h(t)}^t \begin{bmatrix} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s) \end{bmatrix} \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds \quad (11)$$

Lemma 2 [1]: The following matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \quad (12)$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend on affine on x , is equivalent to

$$R(x) < 0, \quad (13a)$$

$$Q(x) < 0, \quad (13b)$$

and

$$Q(x) - S(x)R^{-1}(x)S^T(x) < 0. \quad (13c)$$

Lemma 3 [1]: Given symmetric matrices Ω and D, E , of appropriate dimensions,

$$\Omega + DF(t)E + E^T F^T(t)D^T < 0 \quad (14a)$$

for all F satisfying $F^T(t)F(t) \leq I$, if and only if there exists some $\varepsilon > 0$ such that

$$\Omega + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0 \quad (14b)$$

The nominal unforced time-varying delay saturating actuator system (1) can be written as

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h(t)) \quad (15)$$

Now, we describe our method for determining the stabilization of time-varying delay system (15) in the following Theorem.

Theorem 1: For given positive scalars h, h_d , and α , the nominal time-varying delay unforced system (15) is exponentially if there exist symmetry positive-definite matrices $P = P^T > 0, Q = Q^T > 0, R = R^T > 0, Z = Z^T > 0$ and positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0,$$

which satisfy the following inequalities:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} \\ \Omega_{14}^T & \Omega_{24}^T & \Omega_{34}^T & \Omega_{44} \end{bmatrix} < 0 \quad (16a)$$

and

$$Z - X_{33} \geq 0 \quad (16b)$$

$$Z - Y_{33} \geq 0 \quad (16c)$$

where

$$\Omega_{11} = (A_0 + 0.5\alpha I)^T P + P(A_0 + 0.5\alpha I) + Q + R + e^{-\alpha h} (hX_{11} + X_{13} + X_{13}^T),$$

$$\Omega_{12} = PA_1 + e^{-\alpha h} (hX_{12} - X_{13} + X_{23}^T), \quad \Omega_{14} = hA_0^T Z,$$

$$\Omega_{22} = e^{-\alpha h} [hX_{22} - X_{23} - X_{23}^T + hY_{11} + Y_{13} + Y_{13}^T - (1-h_d)Q],$$

$$\Omega_{23} = e^{-\alpha h} (hY_{12} - Y_{13} + Y_{23}^T), \quad \Omega_{24} = hA_1^T Z,$$

$$\Omega_{33} = e^{-\alpha h} (hY_{22} - Y_{23} - Y_{23}^T - R), \quad \Omega_{44} = -hZ, \quad \Omega_{13} = \Omega_{34} = 0.$$

Proof: Consider the following Lyapunov-Kravoskii functional

$$V(x_t) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) + V_4(t, x(t)) \quad (17)$$

where

$$\begin{aligned} V_1(t, x(t)) &= e^{\alpha t} x^T(t) P x(t) \\ V_2(t, x(t)) &= \int_{t-h(t)}^t e^{\alpha s} x^T(s) Q x(s) ds \\ V_3(t, x(t)) &= \int_{t-h}^t e^{\alpha s} x^T(s) R x(s) ds \\ V_4(t, x(t)) &= \int_{-h}^0 \int_{t+\theta}^t e^{\alpha s} \dot{x}^T(s) Z \dot{x}(s) ds d\theta \end{aligned}$$

Then, the time derivative of $V(x_t)$ with respect to t along the system (15) is

$$\dot{V}(x_t) = \dot{V}_1(t, x(t)) + \dot{V}_2(t, x(t)) + \dot{V}_3(t, x(t)) + \dot{V}_4(t, x(t)) \quad (18)$$

where

$$\begin{aligned} \dot{V}_1(t, x(t)) &= e^{\alpha t} \{ \alpha x^T(t) P x(t) + \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \} \\ &= e^{\alpha t} \{ \alpha x^T(t) P x(t) + [A_0 x(t) + A_1 x(t-h(t))]^T P x(t) \\ &\quad + x^T(t) [A_0 x(t) + A_1 x(t-h(t))] \} \end{aligned}$$

$$\dot{V}_2(t, x(t)) = e^{\alpha t} [x^T(t) Q x(t) - (1-h_d) e^{-\alpha h} x^T(t-h(t)) Q x(t-h(t))]$$

$$\dot{V}_3(t, x(t)) = e^{\alpha t} [x^T(t) R x(t) - x^T(t-h) e^{-\alpha h} R x(t-h)]$$

and

$$\dot{V}_4(t, x(t)) = e^{\alpha t} \left[\dot{x}^T(t) h Z \dot{x}(t) - \int_{t-h}^t e^{\alpha(s-t)} \dot{x}^T(s) Z \dot{x}(s) ds \right]$$

Obviously, for any a scalar $s \in [t-h, t]$, we have $e^{-\alpha h} \leq e^{\alpha(s-t)} \leq 1$, and

$$- \int_{t-h}^t e^{\alpha(s-t)} \dot{x}^T(s) Z \dot{x}(s) ds \leq -e^{-\alpha h} \int_{t-h}^t \dot{x}^T(s) Z \dot{x}(s) ds \quad (19)$$

Alternatively, the following equations are true:

$$\begin{aligned} - \int_{t-h}^t \dot{x}^T(s) Z \dot{x}(s) ds &= - \int_{t-h(t)}^t \dot{x}^T(s) Z \dot{x}(s) ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s) Z \dot{x}(s) ds \\ &= - \int_{t-h(t)}^t \dot{x}^T(s) (Z - X_{33}) \dot{x}(s) ds - \int_{t-h(t)}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds \\ &\quad - \int_{t-h}^{t-h(t)} \dot{x}^T(s) (Z - Y_{33}) \dot{x}(s) ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s) Y_{33} \dot{x}(s) ds \quad (20) \end{aligned}$$

Applying Lemma 1, it can be written that

$$\begin{aligned} - \int_{t-h(t)}^t \dot{x}^T(s) X_{33} \dot{x}(s) ds &\leq \int_{t-h(t)}^t \begin{bmatrix} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s) \end{bmatrix} \\ &\quad \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds \\ &= x^T(t) [h X_{11} + X_{13}^T + X_{13}] x(t) \\ &\quad + x^T(t) [h X_{12} - X_{13} + X_{23}^T] x(t-h(t)) \\ &\quad + x^T(t-h(t)) [h X_{12}^T - X_{13}^T + X_{23}] x(t) \\ &\quad + x^T(t-h) [h X_{22} - X_{23} - X_{23}^T] x(t-h(t)) \quad (21) \end{aligned}$$

Similarly, we have

$$\begin{aligned} - \int_{t-h}^{t-h(t)} \dot{x}^T(s) Y_{33} \dot{x}(s) ds &\leq x^T(t-h(t)) [h Y_{11} + Y_{13}^T + Y_{13}] x(t-h(t)) \\ &\quad + x^T(t-h(t)) [h Y_{12} - Y_{13} + Y_{23}^T] x(t-h) \\ &\quad + x^T(t-h) [h Y_{12}^T - Y_{13}^T + Y_{23}] x(t-h(t)) \\ &\quad + x^T(t-h) [h Y_{22} - Y_{23} - Y_{23}^T] x(t-h) \quad (22) \end{aligned}$$

with the operator for the term $\dot{x}^T(t) h Z \dot{x}(t)$ as follows:

$$\begin{aligned} \dot{x}^T(t) h Z \dot{x}(t) &= [A_0 x(t) + A_d x(t-h(t))]^T h Z [A_0 x(t) + A_1 x(t-h(t))] \\ &= x^T(t) h A_0^T Z A_0 x(t) + x^T(t) h A_0^T Z A_1 x(t-h(t)) \\ &\quad + x^T(t-h(t)) h A_1^T Z A_0 x(t) + x^T(t-h(t)) h A_1^T Z A_1 x(t-h(t)) \quad (23) \end{aligned}$$

Substituting the above Eqs. (19)-(23) into (18), we obtain

$$\begin{aligned} \dot{V}(x_t) &\leq e^{\alpha t} \{ \xi^T(t) \Xi \xi(t) - \int_{t-h(t)}^t e^{-\alpha h} \dot{x}^T(s) (Z - X_{33}) \dot{x}(s) ds \\ &\quad - \int_{t-h}^{t-h(t)} e^{-\alpha h} \dot{x}^T(s) (Z - Y_{33}) \dot{x}(s) ds \} \quad (24) \end{aligned}$$

where

$$\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-h) \end{bmatrix}$$

and

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{12}^T & \Xi_{22} & \Xi_{23} \\ \Xi_{13}^T & \Xi_{23}^T & \Xi_{33} \end{bmatrix}$$

and

$$\begin{aligned}\Xi_{11} &= (A_0 + 0.5\alpha I)^T P + P(A_0 + 0.5\alpha I) + Q + R \\ &\quad + e^{-\alpha h}(hX_{11} + X_{13} + X_{13}^T) + hA_0^T Z A_0, \\ \Xi_{12} &= P A_1 + e^{-\alpha h}(hX_{12} - X_{13} + X_{23}^T) + hA_0^T Z A_1, \\ \Xi_{22} &= e^{-\alpha h}[hX_{22} - X_{23} - X_{23}^T + hY_{11} + Y_{13} + Y_{13}^T - (1-h_d)Q] \\ &\quad + hA_1^T Z A_1, \\ \Xi_{23} &= e^{-\alpha h}(hY_{12} - Y_{13} + Y_{23}^T), \\ \Xi_{33} &= e^{-\alpha h}(hY_{22} - Y_{23} - Y_{23}^T - R), \\ \Xi_{13} &= 0.\end{aligned}$$

Finally, using the Schur complements, with some effort we can show that (24) guarantees of $\dot{V}(x_t) < 0$. It is clear that if $\Xi < 0$, $Z - X_{33} \geq 0$, and $Z - Y_{33} \geq 0$ then, $\dot{V}(x_t) < 0$ for any $\xi(t) \neq 0$. So the nominal time-varying delay unforced systems (15) is exponential stable with decay rate α if linear matrix inequalities (16) are true. This completes the proof.

III. EXTENSION TO EXPONENTIAL STABILIZATION FOR TIME DELAY SATURATING ACTUATOR SYSTEMS

According to the Theorem 1, we describe our method for determining the stabilization of time-varying delay system with saturating actuator (1). The main aim of this paper is to develop delay-dependent conditions for stabilization of the time-varying delay saturating actuator system (1) under the state feedback control law $u(t) = Kx(t)$. More specifically, our objective is to determine bounds for the delay time by using Lyapunov-Krasovskii functional and LMI methods with Leibniz-Newton formula. The following Theorem gives an LMI-based computational procedure to determine state feedback controller. Then we have the following result.

Theorem 2: For any given positive scalars $h > 0$, $h_d > 0$, $\alpha > 0$ and $0 \leq \beta_i(x) \leq 1$. There exists a state feedback controller of the form $u(t) = Kx(t)$ such that the closed-loop system (1) is exponentially stable with decay rate and different values of saturated range, if there exist symmetry positive-definite matrices $W = W^T > 0$, $U = U^T > 0$, $V = V^T > 0$, $S = S^T > 0$, $\varepsilon > 0$, and positive semi-defined matrices

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12}^T & M_{22} & M_{23} \\ M_{13}^T & M_{23}^T & M_{33} \end{bmatrix} \geq 0, \quad N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23}^T & N_{33} \end{bmatrix} \geq 0$$

and a matrix J with appropriate dimension such that the following set of coupled LMIs holds

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & \Psi_{16} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & \Psi_{24} & \Psi_{25} & \Psi_{26} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & \Psi_{34} & \Psi_{35} & \Psi_{36} \\ \Psi_{14}^T & \Psi_{24}^T & \Psi_{34}^T & \Psi_{44} & \Psi_{45} & \Psi_{46} \\ \Psi_{15}^T & \Psi_{25}^T & \Psi_{35}^T & \Psi_{45}^T & \Psi_{55} & \Psi_{56} \\ \Psi_{16}^T & \Psi_{26}^T & \Psi_{36}^T & \Psi_{46}^T & \Psi_{56}^T & \Psi_{66} \end{bmatrix} < 0 \quad (25a)$$

and

$$W - M_{33} \geq 0 \quad (25b)$$

$$W - N_{33} \geq 0 \quad (25c)$$

where

$$\begin{aligned}\Psi_{11} &= W(A_0 + 0.5\alpha I)^T + (A_0 + 0.5\alpha I)W + BG(\beta(x))J \\ &\quad + J^T G^T(\beta(x))B^T + U + V + e^{-\alpha h}(hM_{11} + M_{13} + M_{13}^T), \\ \Psi_{12} &= A_1 W + e^{-\alpha h}(hM_{12} - M_{13} + M_{23}^T), \\ \Psi_{14} &= hWA_0^T + J^T G^T(\beta(x))B^T, \\ \Psi_{15} &= WE_0^T + J^T G^T(\beta(x))E_b^T, \Psi_{16} = \varepsilon D, \\ \Psi_{22} &= -(1-h_d)e^{-\alpha h}U \\ &\quad + e^{-\alpha h}(hM_{22} - M_{23} - M_{23}^T + hN_{11} + N_{13} + N_{13}^T), \\ \Psi_{23} &= e^{-\alpha h}(hN_{12} - N_{13} + N_{23}^T), \\ \Psi_{24} &= hA_1^T W, \Psi_{25} = WE_1^T, \\ \Psi_{33} &= e^{-\alpha h}(hN_{22} - N_{23} - N_{23}^T - V), \\ \Psi_{44} &= -hS, \Psi_{46} = h\varepsilon D, \Psi_{55} = -\varepsilon I, \Psi_{66} = -\varepsilon I, \\ \Psi_{13} &= \Psi_{26} = \Psi_{34} = \Psi_{35} = \Psi_{36} = \Psi_{45} = \Psi_{56} = 0.\end{aligned}$$

The stabilizing memoryless controller gain is given by $K = JW^{-1}$.

Proof: If A_0 and A_1 in (16) are replaced with $A_0 + BG(\beta(x))K + DF(t)(E_0 + E_b G(\beta(x))K)$, and $A_1 + DF(t)E_1$, then (16) for uncertain system (1) is equivalent to the following condition:

$$\bar{\Omega} + \Gamma_d F(t) \Gamma_e + \Gamma_e^T F(t) \Gamma_d^T < 0 \quad (26)$$

$$\text{where } \bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{11} & \Omega_{12} & \Omega_{13} & \bar{\Omega}_{14} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} \\ \bar{\Omega}_{14}^T & \Omega_{24}^T & \Omega_{34}^T & \Omega_{44} \end{bmatrix}, \quad \Omega_{ij}(i, j = 1, 2, 3, 4, i \leq j)$$

are defined in (16), and

$$\begin{aligned} \bar{\Omega}_{11} &= [A_0 + BG(\beta(x))K + 0.5\alpha I]^T P \\ &\quad + P[A_0 + BG(\beta(x))K + 0.5\alpha I] \\ &\quad + Q + R + e^{-\alpha h}(hX_{11} + X_{13} + X_{13}^T), \\ \bar{\Omega}_{14} &= h[A_0 + BG(\beta(x))K]^T Z, \\ \Gamma_d &= [PD \quad 0 \quad 0 \quad hZD]^T \text{ and} \\ \Gamma_e &= [E_0 + E_b G(\beta(x))K \quad E_1 \quad 0 \quad 0]. \end{aligned}$$

By lemma 3, a necessary and sufficient condition for (26) for system (1) is that there exists a positive number $\varepsilon > 0$ such that

$$\bar{\Omega} + \varepsilon^{-1} \Gamma_d^T \Gamma_d + \varepsilon \Gamma_e^T \Gamma_e < 0 \quad (27)$$

Applying the Schur complements, we find that (27) is equivalent to the following condition:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} \\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} \\ \Phi_{14}^T & \Phi_{24}^T & \Phi_{34}^T & \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{15}^T & \Phi_{25}^T & \Phi_{35}^T & \Phi_{45}^T & \Phi_{55} & \Phi_{56} \\ \Phi_{16}^T & \Phi_{26}^T & \Phi_{36}^T & \Phi_{46}^T & \Phi_{56}^T & \Phi_{66} \end{bmatrix} < 0 \quad (28)$$

where

$$\begin{aligned} \Phi_{11} &= [A_0 + BG(\beta(x))K + 0.5\alpha I]^T P \\ &\quad + P[A_0 + BG(\beta(x))K + 0.5\alpha I] + Q + R \\ &\quad + e^{-\alpha h}(hX_{11} + X_{13} + X_{13}^T), \\ \Phi_{12} &= PA_1 + e^{-\alpha h}(hX_{12} - X_{13} + X_{23}^T), \\ \Phi_{14} &= h[A_0 + BG(\beta(x))K]^T Z, \\ \Phi_{15} &= E_0^T + K^T G^T(\beta(x))E_b^T, \quad \Phi_{16} = PD, \\ \Phi_{22} &= -e^{-\alpha h}(1 - h_d)Q + e^{-\alpha h}(hX_{22} - X_{23} - X_{23}^T) \\ &\quad + e^{-\alpha h}(hY_{11} + Y_{13} + Y_{13}^T), \end{aligned}$$

$$\Phi_{23} = e^{-\alpha h}(hY_{12} - Y_{13} + Y_{23}^T),$$

$$\Phi_{24} = hA_1^T Z, \quad \Phi_{25} = E_1^T,$$

$$\Phi_{33} = e^{-\alpha h}(hY_{22} - Y_{23} - Y_{23}^T - R),$$

$$\Phi_{44} = -hZ, \quad \Phi_{46} = h\varepsilon ZD,$$

$$\Phi_{55} = \Phi_{66} = -\varepsilon I,$$

$$\Phi_{13} = \Phi_{26} = \Phi_{34} = \Phi_{35} = \Phi_{36} = \Phi_{45} = \Phi_{56} = 0.$$

Setting the change of variables such that $W = P^{-1}$, $U = P^{-1}QP^{-1}$, $V = P^{-1}RP^{-1}$, $M_{ij} = P^{-1}X_{ij}P^{-1}$, $N_{ij} = P^{-1}Y_{ij}P^{-1}$, $S = Z^{-1}$, $J = KW$. Then, pre- and post-multiplying both sides of (28) by $\text{diag}\{P^{-1}, P^{-1}, P^{-1}, Z^{-1}, I, I\}$ leads to (25a). Applying

$$\begin{bmatrix} Z^{-1} & P^{-1} \end{bmatrix} \begin{bmatrix} Z \\ -X_{33} \end{bmatrix} P^{-1} = W - M_{33},$$

$$\begin{bmatrix} Z^{-1} & P^{-1} \end{bmatrix} \begin{bmatrix} Z \\ -Y_{33} \end{bmatrix} P^{-1} = W - N_{33}$$

yields (25b) and (25c). This completes the proof.

Remark 1: As in the stabilization problem, the maximum allowable delay bound (MADB) \bar{h} which ensures that time-varying delay system with saturating actuator (1) is stabilizable for \bar{h} , decay rate α and the operation range of saturated range $\beta_i(x)$ can be determined by solving the following quasi-convex optimization problem when the other bound of decay rate α and the operation range of saturated range $\beta_i(x)$ are known.

$$\begin{cases} \text{Maximize } \bar{h} \\ \text{Subject to (25)} \end{cases} \quad (29)$$

Inequality (29) is a quasi-convex optimization problem and can be obtained efficiently using MATLAB LMI Toolbox. Then, the controller $K = JW^{-1}$ stabilizes system (1). The determination of the upper bound of the delay for which time-varying delay system with saturating actuator (1) will remain exponential stable can be cast into a generalized eigenvalue minimization problem (GEVP).

To show usefulness of our result, let us consider the following numerical examples.

IV. EXAMPLES

In this section, two numerical examples are presented to compare with the proposed stabilization method with previous results.

Example1: Consider the time-varying delay system with an actuator saturated at level ± 1 described as the follows

$$\dot{x}(t) = [A_0 + \Delta A_0(t)]x(t) + [A_1 + \Delta A_1(t)]x(t - h(t)) + BSat(u(t)) \quad (30)$$

where $A_0 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\Delta A_0(t)$ and $\Delta A_1(t)$ are of the form of (4) with $D = I$, $E_0 = E_1 = \text{diag}\{0.2, 0.2\}$.

Assume the operation range $\beta_i(x)$ is inside the sector $[0.1, 1]$. The problem is to design a state feedback controller to estimate the delay time h such that the above system to be exponentially stable.

Solution: By taking $\alpha = 0$, $h_d = 0.1$ and $\beta_i = 0.1$, we get the Theorem 2 remains feasible for any delay time $h \leq 5.8995$. In case of $\bar{h} = 5.8995$, solving Theorem 2 yields the following set of feasible solutions:

$$W = \begin{bmatrix} 16.8121 & -6.4328 \\ -6.4328 & 10.3395 \end{bmatrix}, U = \begin{bmatrix} 31.5576 & -1.4853 \\ -1.4853 & 36.5608 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.9906 & 0.4628 \\ 0.4628 & 3.9238 \end{bmatrix}, S = \begin{bmatrix} 283.0926 & -37.9065 \\ -37.9065 & 431.2342 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} 2.1567 & 0.6247 \\ 0.6247 & 2.1465 \end{bmatrix}, M_{12} = \begin{bmatrix} 0.9634 & 1.7847 \\ -0.5345 & 1.0269 \end{bmatrix},$$

$$M_{13} = \begin{bmatrix} 0.3863 & 1.1696 \\ -1.6688 & 1.0070 \end{bmatrix}, M_{22} = \begin{bmatrix} 1.9004 & 0.6820 \\ 0.6820 & 2.4718 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix} 3.9680 & -1.1845 \\ -0.0106 & 1.9439 \end{bmatrix}, M_{33} = \begin{bmatrix} 13.3790 & -5.4092 \\ -5.4092 & 6.9862 \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} 0.8714 & -0.1236 \\ -0.1236 & 0.9793 \end{bmatrix}, N_{12} = \begin{bmatrix} -0.5428 & 0.2092 \\ 0.2005 & -0.3245 \end{bmatrix},$$

$$N_{13} = \begin{bmatrix} -2.1082 & 0.8032 \\ 0.8020 & -1.2962 \end{bmatrix}, N_{22} = \begin{bmatrix} 0.5425 & -0.2067 \\ -0.2067 & 0.3352 \end{bmatrix},$$

$$N_{23} = \begin{bmatrix} 2.1126 & -0.8044 \\ -0.8044 & 1.3015 \end{bmatrix}, N_{33} = \begin{bmatrix} 12.5048 & -4.7650 \\ -4.7650 & 7.6997 \end{bmatrix},$$

$$J = [-394.7479 \quad -401.6903], \varepsilon = 1.7024,$$

the corresponding state feedback

$$K = YW^{-1} = [-50.3256 \quad -70.1607].$$

The result obtained, system (30) would be stable if the delay time h is less than 5.8995. Bound of delay time h for various decay rates α and the change of time varying delay h_d (saturated range $\beta_i(x) = 0.1$) is shown in Table 1. From the

Table 1. Bound of delay time h for various decay rate α and h_d (the operation range of saturated range $\beta_i(x) = 0.1$).

$\alpha \backslash h_d$	0.1	0.3	0.5	0.7	0.9
0.1	5.8995	5.2625	5.0345	4.5299	2.9960
0.2	3.8685	3.7436	3.6506	3.3376	2.0560
0.3	3.4645	3.4155	3.2011	2.6568	1.8482
0.4	2.9565	2.9019	2.6618	2.0751	1.6359
0.5	2.6999	2.5011	2.4018	2.0306	1.4498
0.6	2.1904	2.1190	2.0025	1.7061	1.3228
0.7	1.9841	1.9291	1.8026	1.5099	1.2767
0.8	1.7899	1.7219	1.6108	1.4215	1.1920
0.9	1.7629	1.6560	1.4819	1.2780	1.1001

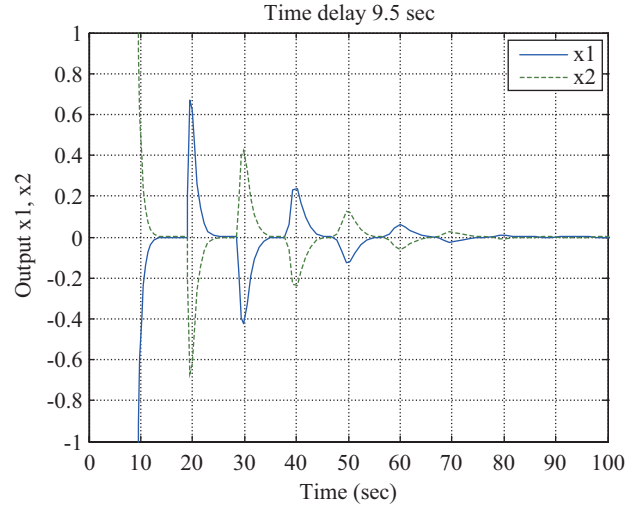


Fig. 1. The simulation of the example 1 for $h = 9.5$ sec.

results of Table 1, if the decay rate α or the change of time varying delay h_d increases the delay time length decreases. We claim that the sharpness of the upper bound of the delay time h varies with the chosen decay α or the change of time varying delay h_d .

Fixing $\alpha = 0$, $h_d = 0$, $\beta_i = 0.1$, Eq. (30) reduces to the system discussed in [10, 17, 21]. Solving the quasi-convex optimization problem (29), according to the Theorem 1, using the software package LMI Toolbox, we obtain the controller $u(t) = [-28.0202 \quad -26.3012]x(t)$ and the corresponding maximum allowed delay $h = 9.5899$. The simulation of the above closed system for $h = 9.5$ is depicted in Fig. 1. An upper bound given by [21] is $h < 0.2841$. On the other hand, the delay bound for guaranteeing asymptotic stability of the system (30) given [10, 17] is $h < 0.3781$ and $h < 0.5522$, respectively. Hence, for this example, the robust stability criterion of this paper is less conservative than the existing results of [10, 17, 21].

Table 2. Maximum allowable delay bounds (MADB) \bar{h} for the operation range of saturated range $\beta_i(x)$ for ($h_d = 0.5, \delta = \sigma = 0.9$).

$\beta_i(x)$	0.1	0.3	0.5	0.7	0.9
[15]	0.5971	0.6941	0.9949	1.4002	2.9698
Theorem 2	1.4871	1.6551	1.7465	1.8027	3.0506

Example 2: This case considers the time-varying delay uncertain system with an actuator saturated at level ± 1 of the form

$$\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t-h(t)) + BSat(u(t)) \quad (31)$$

where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_0 = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}, E_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}.$$

The problem is to design a state feedback controller to estimate the delay time h such that the system (31) to be exponentially stable.

Solution: To begin with, for $h_d = \sigma = \delta = 0$ and $\beta_i(x) = 0.5$, Eq. (31) reduces to the system discussed in [15, 18, 21]. Using Theorem 2, the maximum value of delay time for the nominal system to be asymptotically stable is $h < 14.4088$. By the criterion in [15, 18, 21], the nominal system is asymptotically stable for any h that satisfies $h < 4.3949$, $h < 0.3819$ and $h < 0.6153$, respectively. Hence, for this example, the criteria proposed here significantly improve the estimate of the stability limit compared for the result of [15, 18, 21]. If $h_d = 0$ and $\sigma = \delta = 0.9$ then by solving the quasi-convex optimization problem (31), the maximum upper bound, h , for which the system is $h < 6.2298$. Therefore, we can get the stabilizing state feedback controller for the system (31) is

$$K = \begin{bmatrix} -237.6015 & 116.9965 \\ -62.9404 & -61.5236 \end{bmatrix}. \text{ Finally, the allowable time de-}$$

lay obtained by the operation range of saturated range $\beta_i(x)$ at fixed $h_d = 0.5$ and $\delta = \sigma = 0.9$ is listed in Table 2. Table 2 shows that our results are less conservative than the ones in [15]. It is worth pointing out that our criteria carried out more efficiently for computation. This table also shows that if the $\beta_i(x)$ increases then the delay time length increases.

V. CONCLUSION

In this paper, the problem of robust exponential stability and stabilization criteria for a class of time-varying delay systems with saturating actuator has been considered. A satu-

rating control law is designed and a region is specified in which the stability of the closed-loop system is ensured. A major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. A controller design method to enlarge the estimates is then formulated and solved as an optimization problem with linear matrix inequality (LMI) constraints. The results are obtained based on the Lyapunov-Krasovskii theory in combination with generalized eigenvalue problem (GEVP). Different from the existing ones, our results can overcome the conservatism by choosing suitable scalars for the given exponential decay rate or delays. Numerical examples have also been given to demonstrate the effectiveness of the proposed approach.

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