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SOLVING INHOMOGENEOUS PROBLEMS BY SINGULAR BOUNDARY METHOD

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Key words: singular boundary method, inhomogeneous equation, noisy boundary, meshless method.

ABSTRACT

This study makes the first attempt to extend the singular boundary method (SBM) to inhomogeneous problems in conjunction with the dual reciprocity method (DRM). The SBM is a new boundary-type meshless method and utilizes the fundamental solution to calculate the homogeneous solution of the governing equation of interest, where the inverse interpolation technique is designed to evaluate the origin intensity factor while overcoming the singularity of the fundamental solution at the origin. In this study, the DRM is employed to evaluate the particular solution of Poisson equation with multiquadratic functions. The efficiency and accuracy of the proposed SBM-DRM scheme are tested to the three benchmark inhomogeneous Poisson problems. We also demonstrate the stability of the SBM-DRM scheme in dealing with noisy boundary data.

I. INTRODUCTION

Compared with the finite element method and the finite difference method, the boundary element method (BEM) [1, 17, 20] only requires the boundary discretization in the solution of homogeneous problems. However, the BEM encounters two troublesome problems: 1) boundary-only discretization of inhomogeneous problems without inner nodes, 2) mathematically complex and computationally expensive evaluation of singular or hyper-singular integrals. To overcome the second issue, a variety of novel boundary-type methods have been proposed in recent decades; for instance, the method of fundamental solutions (MFS) [11, 12], the boundary knot method (BKM) [6, 10], the regularized meshless method (RMM) [5, 26], the modified method of fundamental solution (MMFS) [22, 25], the boundary collocation method (BCM) [2, 3], and the singular boundary method (SBM) [7, 9].

The MFS, first introduced by Kupradze and Aleksidze [18], has successfully been applied to a large number of engineering problems [11, 13]. One necessary task when using the MFS is to approximate a solution by using a linear combination of fundamental solutions of the given differential operator. However, due to the singularity of the fundamental solution, the MFS requires a controversial fictitious boundary outside the physical domain, which limits its practical application to complex-shaped boundary or multiply connected domain problems. To avoid this drawback, Chen and Tanaka [10] presents an alternative method, boundary knot method, which replaces the singular fundamental solutions with nonsingular general solutions. However, an ill-conditioning matrix would arise as severely as the MFS while the number of the boundary knots increasing. Recently, Chen *et al*. [5] and Young *et al*. [26] propose a novel meshless method, called the regularized meshless method (RMM), to remedy the singularities of the fundamental solution by employing the desingularization of subtracting and adding-back technique. In addition, the condition number of the RMM interpolation matrix does not increase as rapidly as those of the MFS and the BKM. On the other hand, the original RMM requires a uniform distribution of nodes which severely reduces its applicability. Although Song and Chen [23] bring a weighted method to calculate the diagonal elements of interpolation matrix, its stability has yet to be proved. Following the RMM, Sarler [22] developed the modified method of fundamental solution to solve potential flow problems, which involves a complex integral in the calculation of the diagonal elements.

Inspired by the innovative RMM by Chen *et al*. [5] and Young *et al*. [26], Chen [7] proposes a novel singular boundary method, which uses the fundamental solution of the governing equation of interest as the basis function but collocates source knots in coincidence with response knots on the physical boundary. The singularity of fundamental solution is eliminated by a simple novel numerical desingularization technique called inverse interpolation technique (IIT). Later, Chen *et al*. [9] further improve the SBM by adding a constant term in the approximate representation to guarantee its uniqueness and

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stability. Meanwhile, Chen also makes some further investigations on the role of the constant for BCM in Chen *et al*. [4].

To the best of our knowledge, the SBM has not yet been extended to solving inhomogeneous problems. This paper makes a first attempt to investigate the efficiency and stability of the improved SBM for solving Poisson problems.

The structure of this paper is as follows. In Section 2 we introduce the SBM for solving homogeneous problems. In Section 3, we present the evaluation of the particular solution of inhomogeneous equation through the use of the DRM. In Section 4, we numerically examine the accuracy and efficiency of the proposed approach for a variety of Poisson problems. To show the stability of our approach, we artificially add noisy data on the boundary. In Section 5, we conclude our study with some remarks.

II. THE SBM FORMULATION OF HOMOGENEOUS LAPLACE EQUATION

Without loss of generality, we consider the Laplace equation on a two-dimensional domain

$$\nabla^2 u = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} u(x) &= g(x), \quad x \subset \Gamma_D, \\ \frac{\partial u(x)}{\partial \boldsymbol{n}} &= h(x), \quad x \subset \Gamma_N, \end{aligned} \tag{2}$$

where ∇^2 denotes Laplace operator, and $\boldsymbol{n}$ is the unit outward normal vector, g and h are given functions, Ω denotes the computational domain and $\partial\Omega = \Gamma_D \cup \Gamma_N$ represents the whole physical boundary.

The SBM approximates the solution $u(x)$ by a linear combination of basis functions

$$u(\boldsymbol{x_i}) = \sum_{j=1,\, i\neq j}^{N} \alpha_j u^*(\boldsymbol{x_i}, \boldsymbol{x_j}) + \alpha_i u_{ii}, \ \ \boldsymbol{x} \in \Gamma_D, \tag{3}$$

$$q(\boldsymbol{x_i}) = \frac{\partial u(\boldsymbol{x_i})}{\partial n_{x_i}} = \sum_{j=1,\, i\neq j}^{N} \alpha_j \frac{\partial u^*(\boldsymbol{x_i}, \boldsymbol{x_j})}{\partial n_{x_i}} + \alpha_i q_{ii}, \ \ \boldsymbol{x} \in \Gamma_N, \tag{4}$$

where N is the number of the source points, $\{x_j\}_{j=1}^N$ are the source points on the boundary, and $\{\alpha_j\}_{j=1}^N$ are the unknown coefficients to be determined. u^* is the fundamental solution of the Laplace operator

$$u^*(x, s_j) = -\frac{1}{2\pi}\ln\left(\left\|x - s_j\right\|_2\right) \qquad x \in R^2 \tag{5}$$

In the SBM, we assume that there exists an origin intensity factor, u_{ii} and q_{ii} in Eqs. (3) and (4), when the collocation point coincides with source points. The origin intensity factor is numerically determined by the so-called inverse interpolation technique (IIT), where a sample solution u_t satisfying the governing equation are imperative, and some sample points x_k^t are located inside the physical domain. It follows that the u_{ii} and q_{ii} can be respectively calculated by

$$u_{ii} = \frac{u_t(x_i) - \sum_{j=1,\, j\neq i}^{N} \beta_j u^*(x_i, x_j)}{\beta_i}, \qquad x_i \in \Gamma_D \tag{6}$$

$$q_{ii} = \frac{\dfrac{\partial u_t(x_i)}{\partial \boldsymbol{n}} - \sum_{j=1,\, j\neq i}^{N} \beta_j \dfrac{\partial u^*(x_i, s_j)}{\partial \boldsymbol{n}}}{\beta_i}, \qquad x_i \in \Gamma_N \tag{7}$$

in which $\{\beta_j\}_{j=1}^N$ can be obtained by the following system of linear equations

$$\left\{G(x_k^t, s_j)\right\}\left\{\beta_j\right\} = \left\{u_t(x_k^t)\right\}, \tag{8}$$

where x_k^t is the sample points inside. And it should be noted that the number of the sample points should be larger than N.

Based on our extensive numerical experiments, we observe that when the physical domain is circular centered at the origin with the source points uniformly distributed, the origin intensity factor in the Eqs. (3) and (4) evaluated by the IIT is similar to the one attained from the following expression

$$u_{ii} = -\sum_{j=1, i\neq j}^{N} u^*(x_i, x_j),\ x_i \in \Gamma_D \tag{9}$$

$$q_{ii} = \sum_{j=1, i\neq j_i}^{N} \frac{\partial u^*(x_i, x_j)}{\partial \boldsymbol{n}},\ x_i \in \Gamma_N \tag{10}$$

However, the SBM may obtain incorrect solutions in some potential problems, especially for those with a constant potential [9]. Hence, a constant term is added into the solution expression to warrant the uniqueness of the approximate solution. As a result, the expression of the SBM with an augmented constant term can be written as

$$u(\boldsymbol{x_i}) = \sum_{j=1,\, i\neq j}^{N} \alpha_j u^*(\boldsymbol{x_i}, \boldsymbol{x_j}) + \alpha_i u_{ii} + \alpha_{N+1} \tag{11}$$

with the constraint condition

$$\sum_{j=1}^{N} \alpha_j = 0\,.$$

III. THE SBM-DRM FOR INHOMOGENEOUS EQUATIONS

For inhomogeneous problems, such as Poisson problem, the solution is generally divided into two parts, namely, the homogeneous solution and the particular solution. The homogeneous solution can be approximated by the SBM, while the approximate particular solution can be evaluated by the DRM, which is introduced by Nardini and Brebbia [19]. Golberg [14], Golberg and Chen [15], and Chen and Tanaka [10], respectively couple the DRM with the MFS and the BKM to solve inhomogeneous problems. On the other hand, Wen and Chen [24] proposes the method of particular solution to eliminate the superposition by assembling the homogeneous and inhomogeneous interpolation matrices.

In general, we consider the following Poisson equation

$$\nabla^2 u = f(x, y), \tag{12}$$

subjected to the following boundary conditions

$$u(x) = g(x), \quad x \subset \Gamma_D,$$
$$\frac{\partial u(x)}{\partial \boldsymbol{n}} = h(x), \quad x \subset \Gamma_N. \tag{13}$$

The solution of the problem can be split into homogeneous solution u_h and particular solutions u_p

$$u = u_h + u_p\,. \tag{14}$$

The particular solution u_p is acquired from the governing equation only

$$\nabla^2 u_p = f(x, y) \tag{15}$$

without satisfying the boundary conditions. Approximate particular solution $\hat{u}_p$ in (15) can be obtained by a series of radial basis function φ

$$u_p \approx \hat{u}_p = \sum_{j=1}^{N+L} \beta_j \varphi(r_j), \tag{16}$$

where β_j are unknown coefficients to be determined, L denotes the number of the interior nodes, and $r_j = \|x - x_j\|$ represents the Euclidean distance. Then the Eq. (15) can be recast as

$$\sum_{j=1}^{N+L} \beta_j \nabla^2 \varphi(r_j) = f(x, y)\,. \tag{17}$$

In this study, we select the multiquadric (MQ) as radial basis function φ

$$\varphi = (r^2 + c)^{\frac{1}{2}}, \quad c > 0, \tag{18}$$

where c is the shape parameter in the MQ function.

After the approximate particular solution is obtained, the approximate homogeneous solution $\hat{u}_h$ has to satisfy the following governing and boundary condition equations

$$\nabla^2 \hat{u}_h = 0$$
$$\hat{u}_h(x) = g(x) - \hat{u}_p(x), \quad x \subset \Gamma_D,$$
$$\frac{\partial \hat{u}_h(x)}{\partial \boldsymbol{n}} = h(x) - \frac{\partial \hat{u}_p(x)}{\partial \boldsymbol{n}}, \quad x \subset \Gamma_N. \tag{19}$$

The solution of Eq. (19) can be obtained by the SBM detailed in Section 2. The efficiency and accuracy of the proposed SBM-DRM technique for solving Poisson problems will be examined in the following section.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the efficiency and accuracy of the SBM-DRM are demonstrated by the three benchmark Poisson problems with various inhomogeneous terms and boundary conditions.

The average relative error, *Rerr*, and the maximum relative error, *Mrerr*, are defined by

$$Rerr(u) = \sqrt{\frac{1}{NT}\sum_{i=1}^{NT}\left|\frac{u(i) - \bar{u}(i)}{\bar{u}(i)}\right|^2}, \tag{20}$$

$$Mrerr(u) = \max_{1 \le i \le NT}\left|\frac{u(i) - \bar{u}(i)}{\bar{u}(i)}\right|, \tag{21}$$

where $\bar{u}(i)$ and $u(i)$ are the analytical and numerical solutions at $\boldsymbol{x_i}$, respectively, and NT is the total number of test points in the domain. When $\bar{u}(i)$ is smaller than 1e-6, we take the value of $|u(i) - \bar{u}(i)|$ instead of $\left|\frac{u(i) - \bar{u}(i)}{\bar{u}(i)}\right|$ to avoid the divergence induced by the small value of $\bar{u}(i)$. Unless otherwise specified, we choose shape parameter $c = 1$ in MQ radial basis function.

Example 1: We consider the Poisson problem in an irregular subjected to the Dirichlet boundary conditions

$$\nabla^2 u(x, y) = 1, \quad (x, y) \in \Omega,$$
$$u(x, y) = \frac{x^2 + y^2}{4}, \quad (x, y) \in \partial\Omega, \tag{22}$$

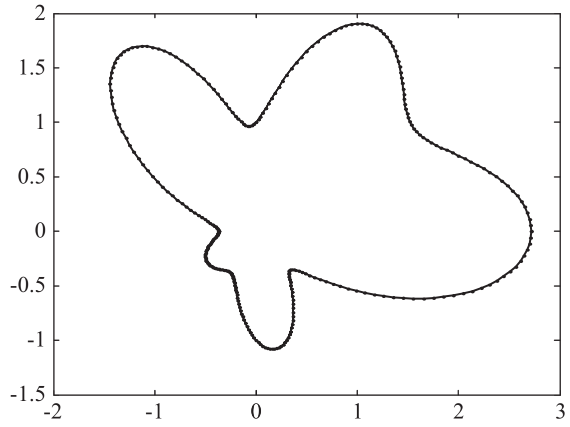

Fig. 1. The shape of irregular domain for Example 1.

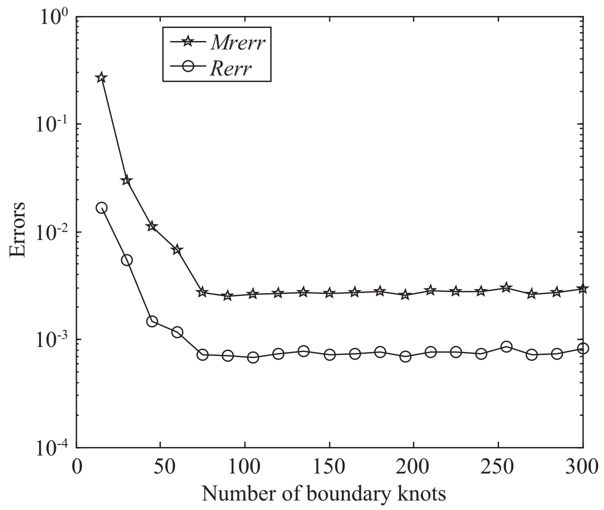

Fig. 2. The *Rerr* and *Mrerr* versus the numbers of the boundary nodes in Example 1.

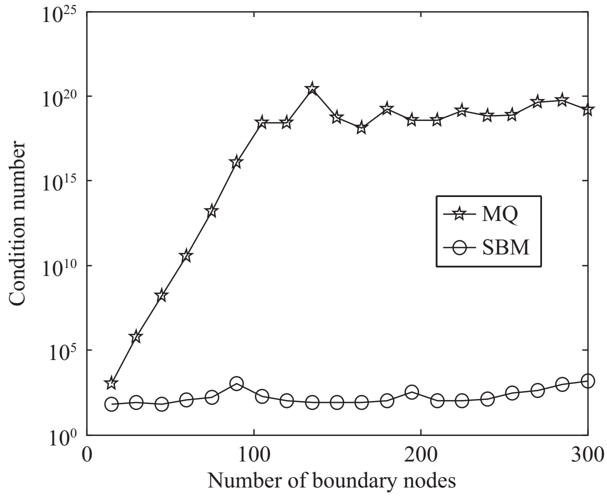

Fig. 3. Condition numbers of the interpolation matrices in the SBM and the DRM with MQ radial basis function versus the number of the boundary nodes in Example 1.

and the exact solution is

$$u(x, y) = \frac{x^2 + y^2}{4}, \quad (x, y) \in \Omega \cup \partial\Omega. \tag{23}$$

In this example we choose the uniform testing nodes (NT = 1322) in the computational domain which is shown in Fig. 1.

Fig. 2 presents *Rerr* and *Mrerr* in terms of the number of boundary nodes for solving Poisson problem in an irregular domain. In Fig. 2, we can see that the numerical solution becomes more accurate as the number of boundary nodes increases and the curves only oscillate slightly which indicates the stability of the solution is quite good. We also observe that the accuracy is improved rapidly for $N < 80$ and improved very little for $N > 80$. This may be largely due to the severely ill-conditioned interpolation matrix of the DRM using MQ radial basis function when a large number of boundary nodes are employed as shown in Fig. 3. The interpolation matrix of the SBM for the homogeneous solution has far smaller condition number than that of the DRM.

We also note that the average and maximum relative errors have very similar varying trend regarding the number of boundary nodes. It is found that the SBM-DRM can successfully solve this Poisson problem with the irregular domain.

Example 2: Consider the following Poisson equation

$$\nabla^2 u(x, y) = -2 \sin x \sin y \tag{24}$$

in a unit square domain $[0,1] \times [0,1]$ with the mixed boundary conditions

$$u_y(x, 0) = \sin x, \quad u_y(x, 1) = \sin x \cos 1,$$
$$u(0, y) = 1, \quad u(1, y) = \sin 1 \sin y + 1. \tag{25}$$

The exact solution is given by

$$u(x, y) = \sin x \sin y + 1. \tag{26}$$

The number of test points is evenly distributed as 50*50 in the domain of interest.

Fig. 4 illustrates the average relative error and the maximum relative error versus the numbers of boundary knots. The approximate result of the SBM-DRM is remarkable in this problem. It is seen from Fig. 4 that the accuracy of the method converges very fast and has little oscillation due to its small condition number. Similar to the example 1, the accuracy also has some enhancement after a certain number of boundary nodes are used.

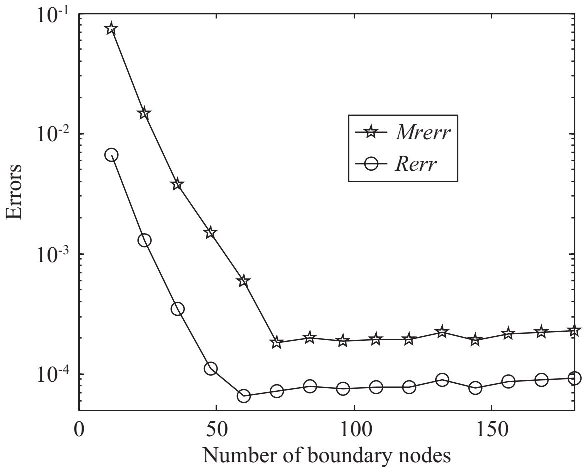

Fig. 4. The errors *Rerr* and *Mrerr* versus the numbers of the boundary nodes in Example 2.

In the finial example, the superior stability of the SBM will be exhibited in comparison with the MFS with different fictitious boundaries and the BKM by adding the noise into the boundary conditions, at the same time, the influence from parameter *c* and the number of the boundary nodes will be largely reduced when approximated by the SBM.

Example 3: Consider a Dirichlet problem with noisy boundary condition in a unit square domain [-0.5,0.5] × [-0.5,0.5], whose governing equation is given by

$$\nabla^2 u = -(\sin x + \sin y)\,, \tag{27}$$

and the exact solution and the boundary condition are given by

$$u = \sin x + \sin y\,. \tag{28}$$

In order to compare the stability of the SBM, the MFS with different fictitious boundaries and the BKM, the boundary data of this case have ±1%, ±2% noise, respectively. The noisy data is added to boundary conditions in the following way.

$$\tilde{u} = ((rand(1,1)\text{-}rand(1,1))*p + 1)*u \tag{29}$$

where *rand*(*M*, *N*) returns an *M*-by-*N* matrix containing pseudo-random values drawn from a uniform distribution on the unit interval in the MATLAB programming, and *p* is the noise level of the boundary data, namely, 1%, and 2%.

In comparison, the BKM employs the nonsingular harmonic solution as the basis function, introduced by Hon and Wu [16] and further improved by Chen *et al*. [8], which outperforms the traditional Bessel function.

$$H(\boldsymbol{x}, \boldsymbol{y}_j) = e^{-\gamma(x^2-y^2)}\cos(2\gamma xy) \tag{30}$$

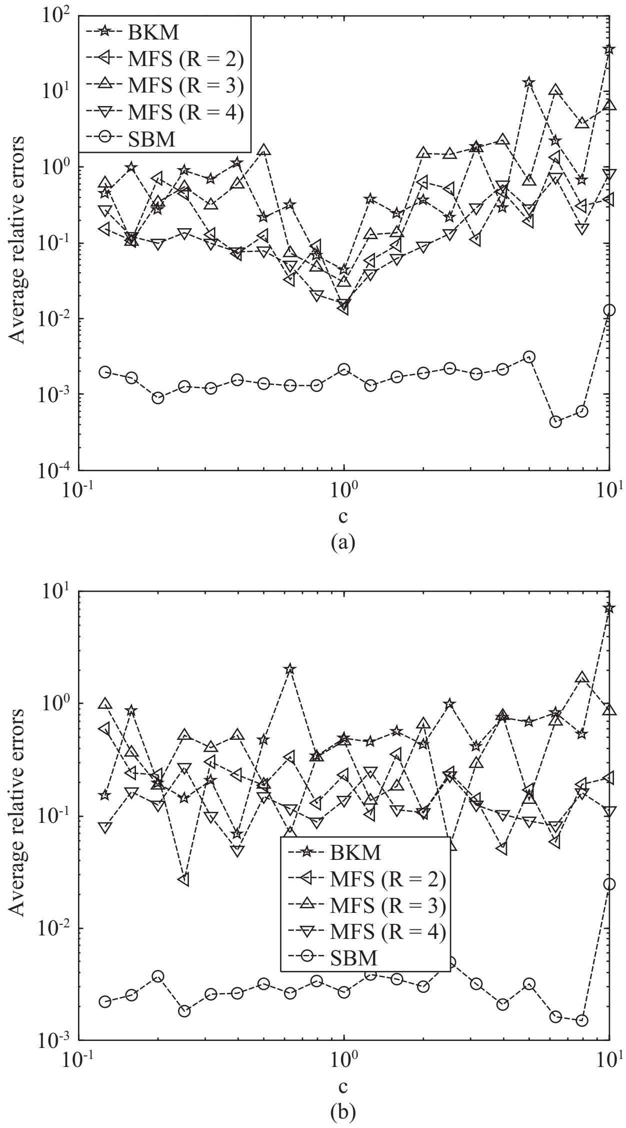

Fig. 5. The average relative error, *Rerr*, versus the parameter *c* in MQ function with ±1% (a) and ±2% (b) noisy data by using 60 boundary nodes in Example 3.

where γ is a parameter chosen as 0.2 in this example, and $x = x_i - x_j$, $y = y_i - y_j$, $\boldsymbol{x}_i = (x_i, y_i)$ denotes the collocation point, $\boldsymbol{y}_j = (x_j, y_j)$ the source point.

On the other hand, in order to test the stability of the MFS, we take different fictitious boundaries into account. We collocate the fictitious boundary on a circle with radius R which is variable, that is, 2, 3 and 4 in this example.

This example examines the accuracy through 2500 uniformly distributed testing nodes in computational domain.

Figs. 5 and 6 present respectively the average relative errors (*Rerr*) versus MQ function's parameter *c* and the number of boundary nodes with the same noise level.

Fig. 5 shows that both the BKM-DRM and the MFS-DRM with different fictitious boundaries are very sensitive to the parameter *c* in dealing with Poisson problems with noisy data.

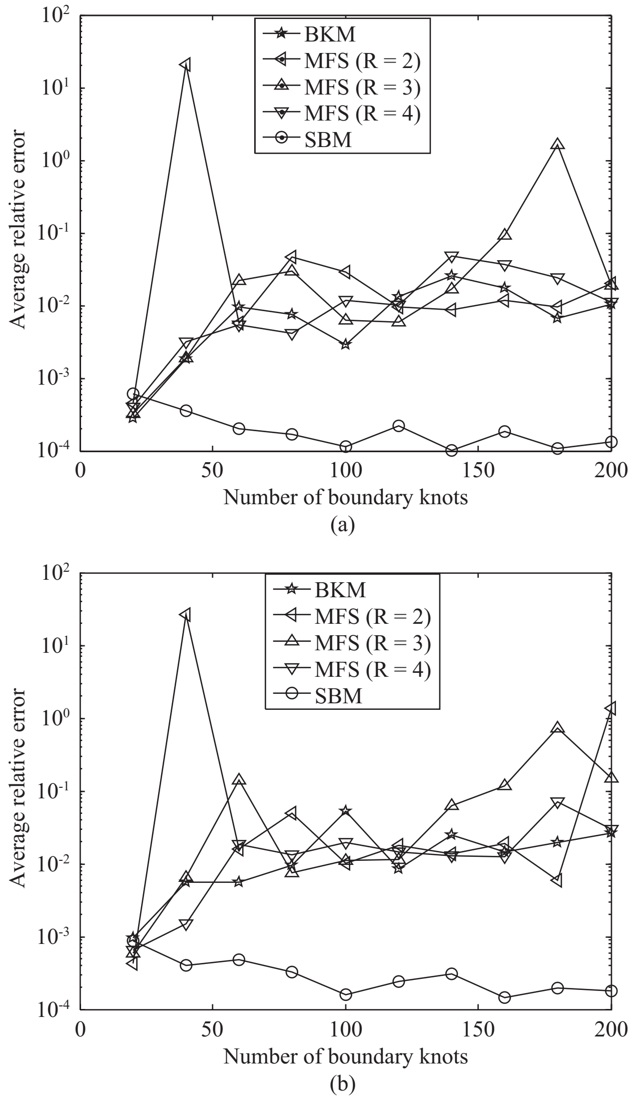

Fig. 6. The average relative error, *Rerr*, against the numbers of the boundary nodes with ±1% (a) and ±2% (b) noisy data in Example 3.

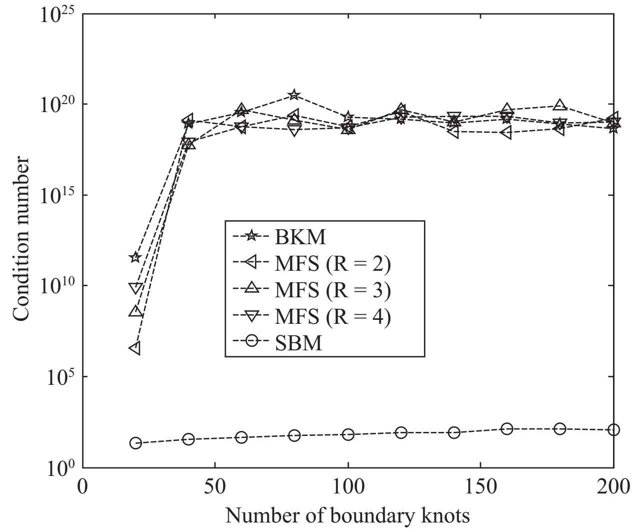

Fig. 7. The condition number of the interpolation matrices in different methods versus the number of the boundary nodes in Example 3.

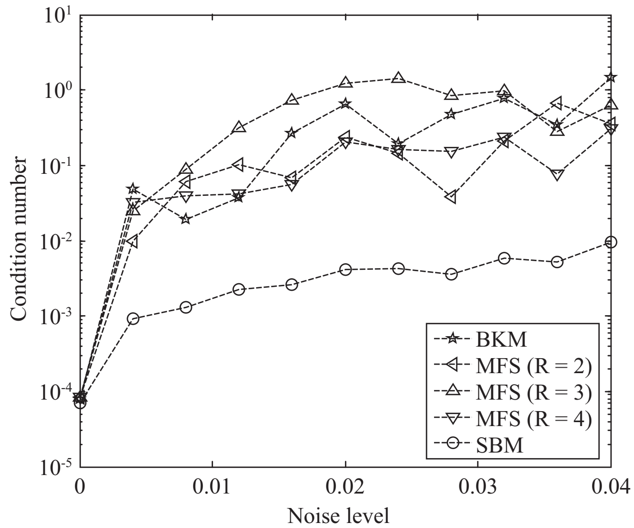

Fig. 8. The average relative error, *Rerr*, with respect to the different noise level by using 60 boundary nodes in Example 3.

These curves oscillate so dramatically that it is very difficult to find the appropriate parameter c to get accurate results, while the SBM-DRM performs much better than the other two methods in term of stability. Furthermore, the accuracy almost remains in the same level with varying parameter c.

It is obvious from Fig. 6 that the results of the BKM and the MFS are rapidly deteriorated with the increasing boundary nodes. In contrast, the SBM performs far more stable than the other two approaches.

Fig. 7 displays condition number curves of the SBM, the BKM and the MFS. The SBM has much smaller condition number than the BKM and the MFS. Thus, the SBM has the best computational stability.

Fig. 8 shows that the accuracy of the SBM, the MFS and the BKM are in the same level when there is no noisy data in the boundary. However, as the noisy level increases, all the results of the BKM, the MFS and the SBM are deteriorating in some extent, but the SBM is much less sensitive to the noisy boundary data than the other two methods due to relatively much smaller condition number of its interpolation matrix as shown in Fig. 7. It is worthy of noting that the condition number of the interpolation matrix of the DRM has little influence on the sensitivity of the resulting solution regarding the noisy boundary data, since the evaluation of the particular solution of the inhomogeneous problem does not involve boundary conditions at all.

V. CONCLUSION

This study extends the SBM in conjunction with the DRM to solve inhomogeneous problems. Though the first two examples, the feasibility of the method has been demonstrated in problems with various irregular domains and different boundary conditions. In the third example we focus on the stability which would be affected by the shape parameter of the MQ, the number of boundary knots and the boundary data from measurement. The SBM notably performs much better and stable than the MFS and the BKM, largely due to the relatively much smaller condition number of its interpolation matrix.

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