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Key words: boundary knot method, boundary singularities, singularity subtraction technique, Helmholtz problem.

## ABSTRACT

This paper proposes a boundary knot method (BKM) formulation in conjunction with the singularity subtraction technique to solve Helmholtz problems with boundary singularities. The solution breaks down into the singular solution and the regular solution. The singular solution is derived analytically and satisfies the governing equation and the corresponding boundary conditions containing singularities. Then the regular solution is approximated by the BKM. Numerical results demonstrate the accuracy and efficiency of the proposed technique.

## I. INTRODUCTION

Helmholtz-type equations arise in various scientific and engineering fields involving wave propagation and vibration, for instance, scattering [3], acoustics [7], and radiation [6]. Unlike other problems, there are no dominant numerical techniques to handle these problems. The boundary discretization method is popular in this field since it employs approximate functions which satisfy Helmholtz equation and are in fact a semi-analytical approach, such as the boundary element method (BEM) [1, 36, 38], method of fundamental solutions (MFS) [4, 6, 12, 25, 29], Trefftz method [26-28], boundary knot method (BKM) [10, 13, 40, 41], regularized meshless method (RMM) [8, 11, 43] and boundary node method (BNM) [35, 44, 45]. Among these methods, the BKM is mathematically simple, easy-to-program, integration-free, truly meshless and appears to be very promising in the very accurate and efficient solution of a great variety of Helmholtz equations with smooth boundary [10, 40, 41]. However, Helmholtz problems with boundary singularities also often occur in many engineering applications, such as the shape discontinuity, abrupt change of boundary conditions, and discontinuity in the material properties.

It is well-known that boundary singularities have a severely adverse effect on the accuracy and convergence of standard numerical methods [5, 16, 32, 37]. There are important studies regarding numerical treatment of singularities, such as the finite difference [34], boundary element and finite element methods [17, 18, 42]. For comprehensive studies on the treatment of singularities by meshless methods such as the MFS and Trefftz method, we refer the reader to [2, 14-16, 19-24, 30] and reference therein. Our numerical experiments indicate that the original BKM formulation fails to yield admissible solutions for Helmholtz problems with boundary singularities.

This study is to present a novel BKM strategy for an accurate and efficient solution of problems with boundary singularities. In this paper, we introduce a singularity subtraction technique [32] to remove the ill effect of the boundary singularities, therefore the governing equation and the boundary conditions can simply be approximated by the traditional BKM.

The rest of the paper is organized as follows. In Section 2, Helmholtz problem with boundary singularities is briefly described. The singularity subtraction technique is reviewed in Section 3. Followed by Section 4, we introduce the key idea of the BKM for Helmholtz problems with boundary singularities. Numerical results and conclusions are provided in Section 5 and Section 6, respectively.

## II. HELMHOLTZ PROBLEM WITH SINGULAR BOUNDARY

Without loss of generality, we consider the following Helmholtz problems in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ with a regular boundary $\Gamma$

$$\Delta u(\mathrm{x}) + k^2 u(\mathrm{x}) = 0, \quad \mathrm{x} \in \Omega \tag{1}$$

where $\Delta$ represents the Laplace operator, $k$ denotes the wavenumber, and $\mathrm{x} = (x_1, x_2)$. The polar coordinates $(r, \theta)$ used in this paper can be converted into the two-dimensional

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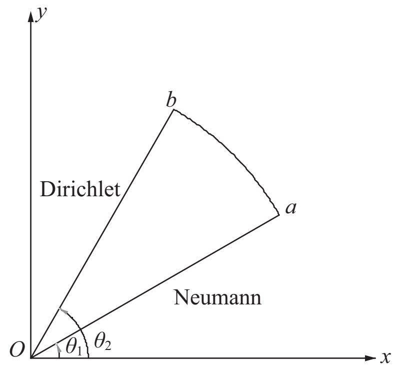


**Fig. 1. An example of wedge domain with interior angle $\theta_2 - \theta_1$.**

Cartesian coordinates $x_1$ and $x_2$ via the trigonometric functions transform $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. Supposing that the solution of Eq. (1) in the interest domain $\Omega$ can be evaluated by the separation of variables with respect to the polar coordinates, the general solution of Helmholtz equations can be obtained by

$$u(r,\theta) = [C_1 J_\lambda(kr) + C_2 N_\lambda(kr)][a\cos(\lambda\theta) + b\sin(\lambda\theta)], \qquad (2)$$

where $C_1$, $C_2$, $a$, $b$ are constants and will be determined by corresponding boundary conditions, and $J_\lambda$, $N_\lambda$ denote the $\lambda$ th-order Bessel function of the first and second kinds, respectively.

Considering an isotropic wedge domain of interior angle $(\theta_2 - \theta_1)$ with the tip at the origin $O$, $\Omega = \{x \in \mathbb{R}^2 \,|\, 0 < r < R(\theta), \theta_1 < \theta < \theta_2\}$, where $R(\theta)$ is either a bounded continuous function or infinity as shown in Fig. 1. This specified domain is governed by the homogeneous Helmholtz equation with Neumann or Dirichlet boundary conditions prescribed on the wedge edges. Assuming that the real part of $\lambda$ is positive, we obtain $C_2 = 0$ in Eq. (2). Hence expression (2) can be written as

$$u^{(s)}(r,\theta) = J_\lambda(kr)[a\cos(\lambda\theta) + b\sin(\lambda\theta)], \qquad (3)$$

where $a$ and $b$ are unknown singularity coefficients whilst $\lambda$ is referred as the eigenvalue. The singularity exponents as well as the corresponding singular coefficients can be determined by the geometry and the boundary conditions along boundaries containing the singular point.

The norm flux associated with the norm vector $n(\theta) = (-\sin(\theta), \cos(\theta))$ is given by

$$\frac{\partial u}{\partial n} = q^{(s)} = \frac{1}{r}\frac{\partial}{\partial \theta} u^{(s)}(r,\theta). \qquad (4)$$

Then, Eqs. (3) and (4) can be recast as

$$u^{(s)}(r,\theta) = J_\lambda(kr)\{a\cos[\lambda(\theta-\theta_1)] + b\sin[\lambda(\theta-\theta_1)]\},$$

$$q^{(s)}(r,\theta) = \frac{\lambda}{r} J_\lambda(kr)\{-a\sin[\lambda(\theta-\theta_1)] + b\cos[\lambda(\theta-\theta_1)]\}. \qquad (5)$$

This study considers four configurations of homogeneous Neumann and Dirichlet boundary conditions at the wedge edges applied to expression (5). In the end, we obtain general expressions of singular function in the case of a single wedge corresponding to homogeneous Neumann and Dirichlet boundary conditions, see Fig. 1 as a Neumann-Dirichlet example. More details can be found in [32].

Case 1: Neumann-Neumann wedge (N-N wedge)

$$\begin{aligned} u^{(s)}(r,\theta) &= \sum_{ns=0}^{\infty} a_{ns} u_{ns}^{NN}(r,\theta) \\ &= \sum_{ns=0}^{\infty} a_{ns} J_{\lambda_{ns}}(kr)\cos[\lambda_{ns}(\theta-\theta_1)], \\ \lambda_{ns} &= ns\frac{\pi}{\theta_2-\theta_1},\ ns \ge 0 \end{aligned} \qquad (6)$$

Case 2: Dirichlet-Dirichlet wedge (D-D wedge)

$$\begin{aligned} u^{(s)}(r,\theta) &= \sum_{ns=1}^{\infty} a_{ns} u_{ns}^{DD}(r,\theta) \\ &= \sum_{ns=1}^{\infty} a_{ns} J_{\lambda_{ns}}(kr)\sin[\lambda_{ns}(\theta-\theta_1)], \\ \lambda_{ns} &= ns\frac{\pi}{\theta_2-\theta_1},\ ns \ge 1 \end{aligned} \qquad (7)$$

Case 3: Neumann-Dirichlet wedge (N-D wedge)

$$\begin{aligned} u^{(s)}(r,\theta) &= \sum_{ns=1}^{\infty} a_{ns} u_{ns}^{ND}(r,\theta) \\ &= \sum_{ns=1}^{\infty} a_{ns} J_{\lambda_{ns}}(kr)\cos[\lambda_{ns}(\theta-\theta_1)], \\ \lambda_{ns} &= (ns-\frac{1}{2})\frac{\pi}{\theta_2-\theta_1},\ ns \ge 1 \end{aligned} \qquad (8)$$

Case 4: Dirichlet-Neumann wedge (D-N wedge)

$$\begin{aligned} u^{(s)}(r,\theta) &= \sum_{ns=1}^{\infty} a_{ns} u_{ns}^{DN}(r,\theta) \\ &= \sum_{ns=1}^{\infty} a_{ns} J_{\lambda_{ns}}(kr)\sin[\lambda_{ns}(\theta-\theta_1)], \\ \lambda_{ns} &= (ns-\frac{1}{2})\frac{\pi}{\theta_2-\theta_1},\ ns \ge 1 \end{aligned} \qquad (9)$$

It is noted that $ns$ is the number of singular function.

## III. SINGULARITY SUBTRACTION TECHNIQUE

We consider a two dimensional bounded piecewise smooth domain $\Omega$ with a singular point $O = (x_1^s, x_2^s)$. For the sake of simplifying expressions, it is assumed that the singular point is located at the intersection of Dirichlet and Neumann boundary parts. Hence the governing equation and the corresponding boundary conditions are recast as

$$\Delta u(\mathrm{x}) + k^2 u(\mathrm{x}) = 0, \qquad \mathrm{x} \in \Omega \tag{10}$$

$$u(\mathrm{x}) = \bar{u}(\mathrm{x}), \mathrm{x} \in \Gamma_D \tag{11}$$

$$\nabla u(\mathrm{x}) \bullet n(\mathrm{x}) = q(\mathrm{x}) = \bar{q}(\mathrm{x}), \mathrm{x} \in \Gamma_N \tag{12}$$

where $\Gamma_D \cap \Gamma_N = \varnothing$, $\Gamma_D \cup \Gamma_N = \Gamma$, $\bar{u}$ and $\bar{q}$ are prescribed boundary conditions.

It is a difficult task to obtain accurate solution of such problems in the presence of boundary singularity. In order to overcome these difficulties, we should reformulate the original singular boundary problem before applying BKM to solve it. Due to the linearity of Helmholtz operators and the boundary conditions, the superposition principle is valid and the solution $u$ and the normal flux $q$ can be re-written as

$$\begin{aligned} u(\mathrm{x}) &= (u(\mathrm{x}) - u^{(s)}(\mathrm{x})) + u^{(s)}(\mathrm{x}) \\ &= u^{(r)}(\mathrm{x}) + u^{(s)}(\mathrm{x}), \mathrm{x} \in \Omega \cup \Gamma, \end{aligned} \tag{13}$$

$$\begin{aligned} q(\mathrm{x}) &= (q(\mathrm{x}) - q^{(s)}(\mathrm{x})) + q^{(s)}(\mathrm{x}) \\ &= q^{(r)}(\mathrm{x}) + q^{(s)}(\mathrm{x}). \mathrm{x} \in \Gamma. \end{aligned} \tag{14}$$

where $u^{(s)}$ denotes a particular singular solution. $u^{(s)}$ satisfies both Eq. (10) and corresponding homogeneous boundary condition on the boundary containing the singular point $O$, and $q^{(s)}(\mathrm{x}) = \nabla u^{(s)} \bullet n(\mathrm{x})$ is its derivative. By adding and subtracting appropriate functions $u^{(s)}$, the singularity is removed. On the other hand, the regular solution satisfies

$$\Delta u^{(r)}(\mathrm{x}) + k^2 u^{(r)}(\mathrm{x}) = 0, \mathrm{x} \in \Omega \tag{15}$$

$$u^{(r)}(\mathrm{x}) = \bar{u}(\mathrm{x}) - u^{(s)}(\mathrm{x}), \mathrm{x} \in \Gamma_D \tag{16}$$

$$q^{(r)}(\mathrm{x}) = \bar{q}(\mathrm{x}) - q^{(s)}(\mathrm{x}), \mathrm{x} \in \Gamma_N \tag{17}$$

The regular solution can then be evaluated directly by the BKM. The key idea of the BKM will be described in the following section.

## IV. BOUNDARY KNOT METHOD

This study applies the BKM to obtain the regular solution of Eqs. (15)-(17). The BKM employs general solutions of the governing equation instead of the singular fundamental solution in the MFS and has particular advantages in the solution of such problems. The BKM is inherently meshfree, integration-free, highly accurate, fast convergent, mathematically simple, and easy-to-program. In the case of Helmholtz problems, the BKM approximated solutions can be represented as

$$u^{(r)}(\mathrm{x}_i) = \sum_{j=1}^{N} \alpha_j J_0(kr(\mathrm{x}_i, \mathrm{s}_j)), \tag{18}$$

where $J_0$ represents the Bessel function of the first kind, and $r(\mathrm{x}_i, \mathrm{s}_j)$ denotes the Euclidean distance between the collocation point $\mathrm{x}_i$ and source point $\mathrm{s}_j$, $k$ is the wave-number of the Helmholtz equation, and $\alpha_j$ presents the unknown coefficient.

Although the general solution in the approximation representation (18) satisfies governing equation, we have to force the regular solution to satisfy boundary conditions by means of collocation method. Let $\mathrm{x}_i$ be a set of points on the boundary, collocating (18) on boundary conditions, Eqs. (16) and (17) result in the following system of linear algebraic equations

$$u^{(r)}(\mathrm{x}_i) = \sum_{j=1}^{N} \alpha_j J_0(kr(\mathrm{x}_i, \mathrm{s}_j)), \quad \mathrm{x}_i \in \Gamma_D \tag{19}$$

$$q^{(r)}(\mathrm{x}_i) = \sum_{j=1}^{N} \alpha_j \frac{\partial J_0(kr(\mathrm{x}_i, \mathrm{s}_j))}{\partial n}, \quad \mathrm{x}_i \in \Gamma_N \tag{20}$$

It should be noted that the BKM employs the same set of boundary knots as the collocation and source points thanks to the smoothness of its kernel functions with no singularity at origin.

Eqs. (19)-(20) lead to the resulting matrix equation expression

$$A\alpha = b, \tag{21}$$

which can be used to determine the coefficients $\alpha$. In order to obtain a unique solution of Eq. (21), additional constraints should be specified as many as the number of the singular functions in the $u^s(\mathrm{x})$. This is achieved by constraining regular solution and/or its normal derivative in the neighborhood of the singular point $O$, as stated below

$$u^{(r)}(\mathrm{x}) = 0, \mathrm{x} \in \Gamma_N \cap B(O; \tau)$$

and/or

$$q^{(r)}(\mathrm{x}) = 0, \mathrm{x} \in \Gamma_D \cap B(O; \tau), \tag{22}$$

where $B(O; \tau) = \{\mathrm{x} \in \mathbb{R}^2 \mid r(\mathrm{x}, O) < \tau, \tau > 0\}$ is sufficiently small. Once the matrix equation is obtained, the unknown coefficients $\alpha$ can be calculated by Gaussian elimination or an

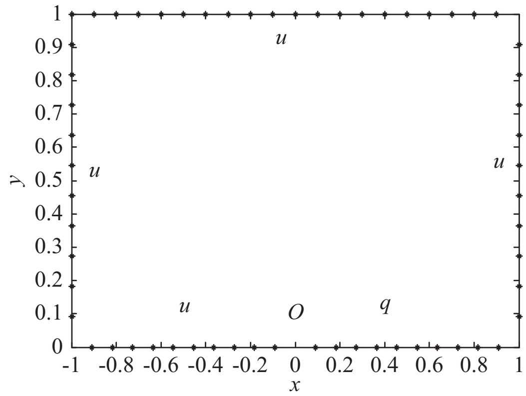

**Fig. 2. The profile of the square domain with singularity at the origin.**

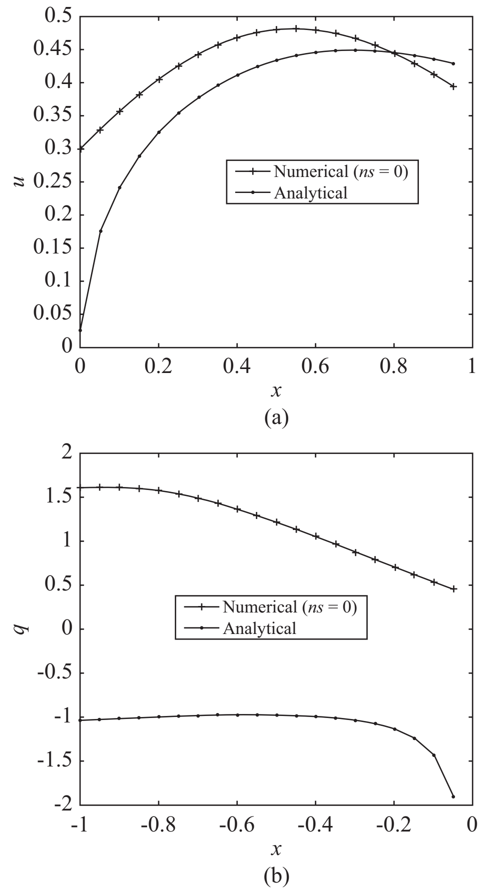

**Fig. 3. Analytical and numerical solutions of *u* (a) and flux *q* (b) without subtracting any singular functions (*ns* = 0) for Neumann-Dirichlet problem along *x*-axis with *y* = 0.**

iterative method. We notice that, due to the global interpolation scheme, the original BKM produces a highly ill-conditioned and dense matrix system when a large number of boundary knots are used. For more details, we refer the readers to [24, 41]. The ill-posed nature of this modified BKM will be discussed later by numerical examples.

## V. NUMERICAL RESULTS AND DISCUSSION

In this section, we demonstrate the accuracy and validity of the above-proposed methodology when there are boundary singularities. The numerical errors in the following examples are defined as follows

$$\mathrm{Err}(u(\mathrm{x}_j)) = |\, u(\mathrm{x}_j) - \tilde{u}(\mathrm{x}_j)\,|,$$

$$\mathrm{Err}(q(\mathrm{x}_j)) = |\, q(\mathrm{x}_j) - \tilde{q}(\mathrm{x}_j)\,|. \tag{23}$$

where $j$ is index of the test points, $u(\mathrm{x}_j)$ and $\tilde{u}(\mathrm{x}_j)$ denote the exact and BKM numerical solution, respectively. $q(\mathrm{x}_j)$ and $\tilde{q}(\mathrm{x}_j)$ represent the exact and numerical solutions of its derivative. This study considers three-types of singular boundary problems described in Section 2, i.e., Neumann-Dirichlet, Dirichlet-Dirichlet, and Neumann-Neumann singular boundary problems. Similar results can be obtained for Dirichlet-Neumann singular boundary problem.

**Example 1.** First, we consider Helmholtz equation containing the following Neumann-Dirichlet singular boundary with wave-number $k = 1$ in a rectangle domain with a singular point at the origin $O$ as shown in Fig. 2. For comparison, the exact solution is

$$u^{(an)} = u_1^{(ND)}(\mathrm{x}) - 1.3u_2^{(ND)}(\mathrm{x}) + 1.5u_3^{(ND)}(x) - 1.7u_4^{(ND)}(\mathrm{x}), \tag{24}$$

The domain and the boundary conditions of interest are defined by

$$\Omega = (-1,1)\times(0,1),$$

$$\Gamma_N = (0,1)\times\{0\},$$

$$\Gamma_D = \partial\Omega \setminus \Gamma_N,$$

where $\partial\Omega$ is the boundary of domain.

This example contains a singularity at the boundary point $O$ arising from both the abrupt change of the boundary conditions and the nature of the analytical solution. A typical distribution of the boundary knots is shown in Fig. 2.

Fig. 3 displays the numerical results of $u$ (a) on the boundary $(0,1) \times \{0\}$ and numerical flux $q$ (b) on the boundary $(-1,0) \times \{0\}$ by using the traditional BKM. It is observed that numerical solutions and the flux do not match well with their corresponding analytical solution when no singular functions are subtracted.

Fig. 4 illustrates a comprehensive comparison between the analytical and numerical solutions $u$ on the boundary $(0,1) \times \{0\}$ (c) and the flux $q$ on boundary $(-1,0) \times \{0\}$ (a) obtained by subtracting various numbers of singular functions. Compared with the analytical solution, we can see a significant

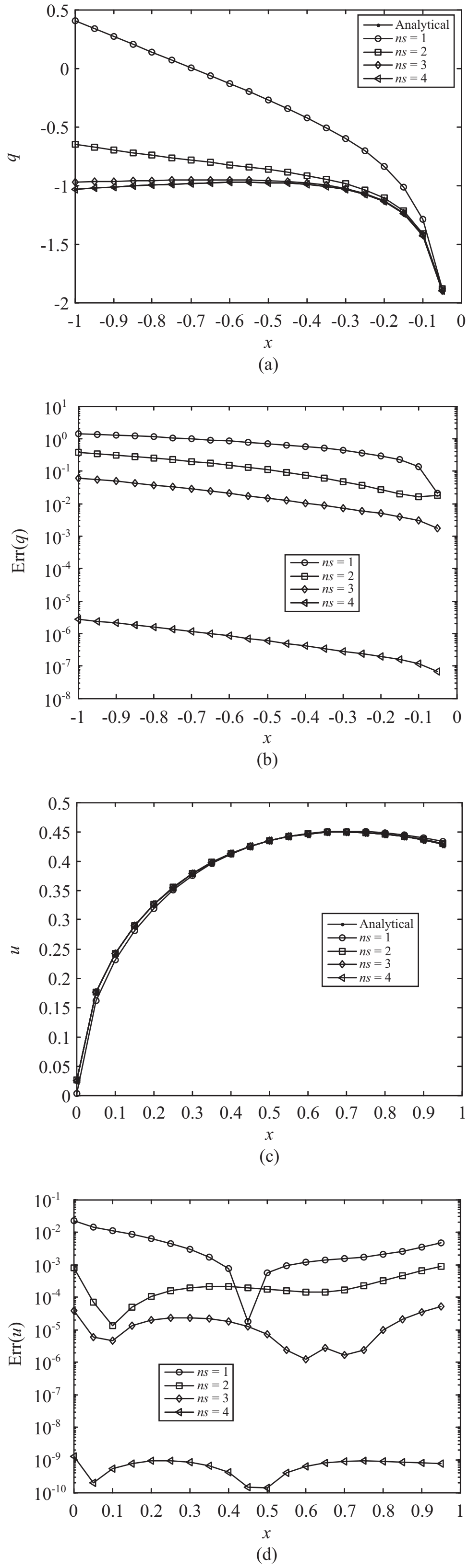


Fig. 4. **Analytical and numerical solutions and the corresponding absolute errors obtained by subtracting various numbers of singular functions, namely *ns* = {1, 2, 3, 4} for the Neumann-Dirichlet singular boundary problem along *x*-axis with *y* = 0.**

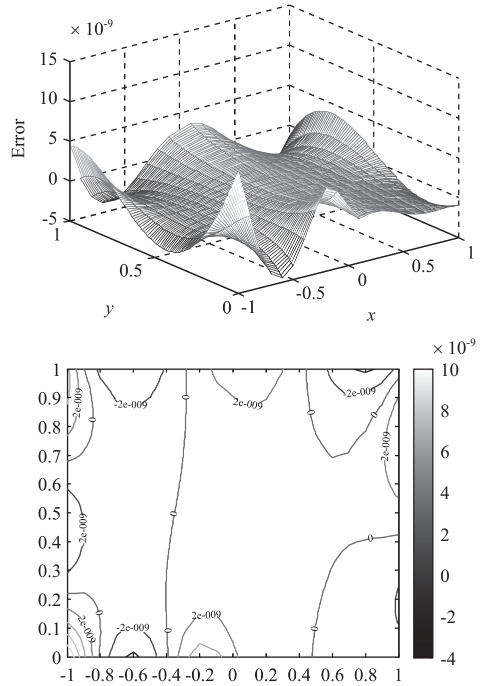


**Fig. 5. The errors in the computational domain and its contour line for Neumann-Dirichlet singular problem with the number of singular functions *ns* = 4.**

improvement in the absolute errors from $10^{-2}$ to $10^{-10}$ in term of Err($u$) (d) and $10^{0}$ to $10^{-8}$ in terms of Err($q$) (b). Generally speaking, the error decreases as the number of singular function increases. Only 200 boundary knots are used in the BKM for all results in Figs. 3 and 4. More boundary knots result in similar solutions and are not necessary. Fig. 5 plots the errors in terms of the absolute error $u - \tilde{u}$ and the corresponding contour lines of interest by the BKM subtracting four singular functions. We can see that the errors of numerical solution are below $10^{-9}$.

In order to highlight the effect of subtracting singular functions in the stability of modified BKM, the condition numbers versus the number of boundary collocation points in terms of varied singular functions are displayed in Fig. 6. We can see that the modified BKM scheme produces ill-conditioned coefficient matrix as in the original BKM.

**Example 2.** In this section, firstly, we consider Helmholtz problem with wave-number $k$ = 1. The exact solution and correspond boundary conditions are given by

$$u^{(an)} = u_1^{(DD)}(\mathrm{x}) + 1.3u_2^{(DD)}(\mathrm{x}) + 1.5u_4^{(DD)}(\mathrm{x}), \qquad (25)$$

where $u_1^{(DD)}$, $u_2^{(DD)}(\mathrm{x})$ and $u_4^{(DD)}(\mathrm{x})$ represent the singular function in Eq. (7) with $ns$ = 1, 2 and 4 respectively.

Fig. 7 displays the configuration of the $L$ shaped domain with the singularity at the point $O$ as well as boundary conditions and the sketch boundary knots.

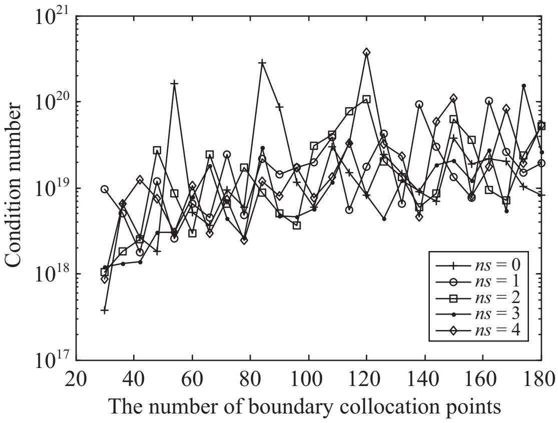


**Fig. 6. Condition number versus the number of collocation points with different number of singular functions.**

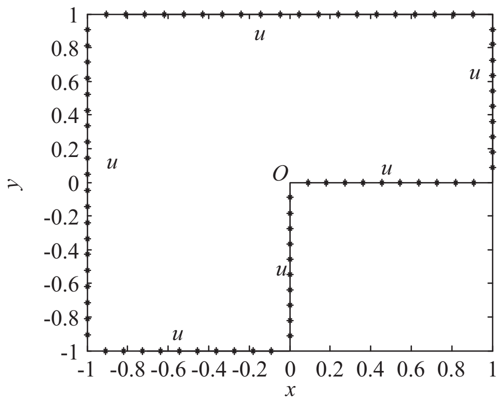


**Fig. 7. The configuration of *L* shaped domain with singularity at the origin *O* and the sketch boundary knots.**

Fig. 8 shows the absolute errors of the recovered $q$ along $x$ axis at $y$ = 0 (a) and $u$ along $y$ axis at $x$ = 0 (c) in terms of the number of singular functions. Similar observations are found as in the previous Neumann-Dirichlet problem that the absolute errors are considerably more sensitive to the number of singular functions $ns$. Compared with the analytical solution, we can see a significant reduction of the absolute errors of the numerical solutions $u$ and $q$, from $10^0$ to $10^{-8}$ for Err($u$) (d) and from $10^0$ to $10^{-10}$ for Err($q$) (b).

Fig. 9 illustrates the absolute errors in terms of $u - \tilde{u}$ and the corresponding contour lines in the interest domain for the $L$ shaped Dirichlet-Dirichlet singular problem by boundary knot method subtracting four singular functions. The errors of numerical solutions are of the order of $10^{-8}$. One can see from Fig. 9 that errors at the singular point appear on the same order as at the other non-singular points.

Finally, we consider the same example with much larger wave-number $k$ to verify the efficiency of present BKM scheme. In Fig. 10, we display the errors with wave-number $k$ = 10, and $k$ = 100 by subtracting four singular functions. And from Fig. 10, we can see that with the increasing of wave-number, the present scheme results in less accurate

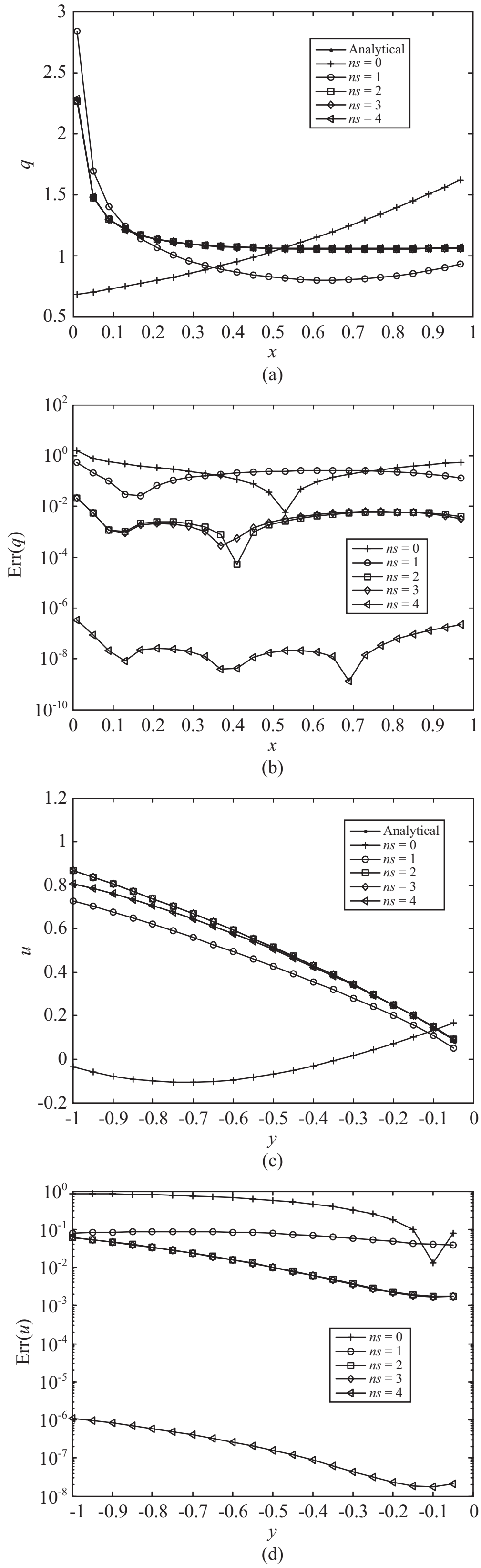


**Fig. 8. Analytical and numerical solutions and the corresponding absolute errors obtained by subtracting various numbers of singular functions, namely *ns* = {0, 1, 2, 3, 4} for the Dirichlet-Dirichlet singular problem along *x* axis with *y* = 0 and *y* axis with *x* = 0.**

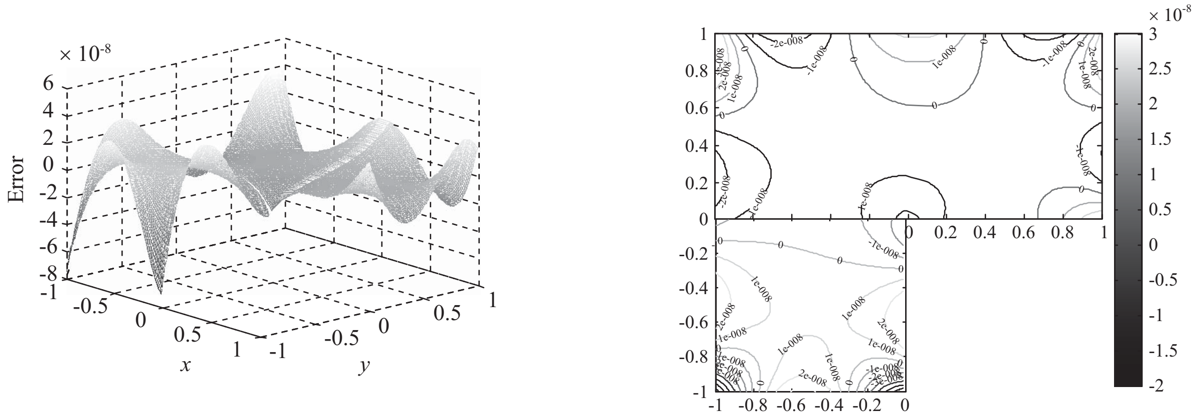


**Fig. 9. The domain errors and its contour for Dirichlet-Dirichlet singular problem with the number of singular function *ns* = 4.**

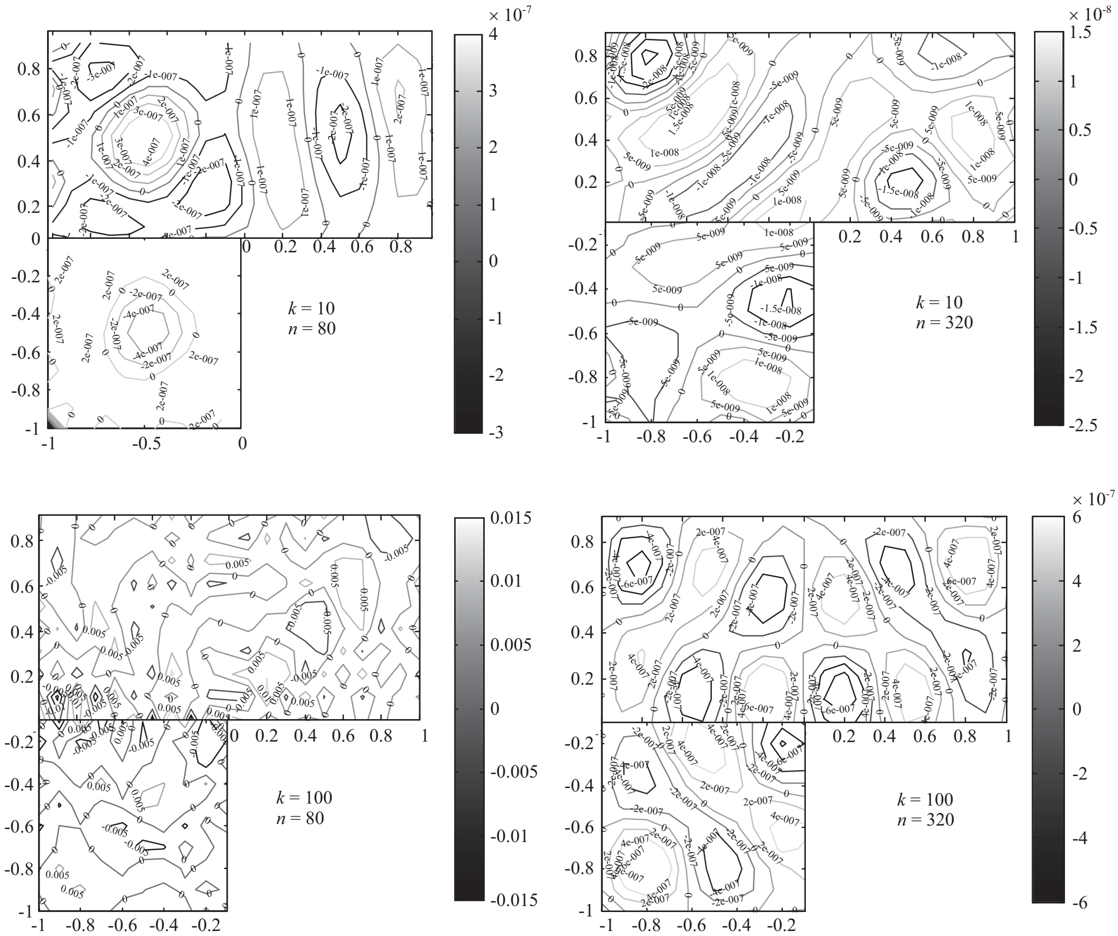


**Fig. 10. The errors contour for Example 2 of wave-number *k* = 10 and 100 by subtracted four singular functions with different number of collocation points.**

result from $10^{-7}$ to $10^{-2}$, and by increasing the number of boundary collocation points from $n$ = 80 to $n$ = 320, we observe the significant improvement in the accuracy from $10^{-7}$ to $10^{-8}$ for $k$ = 10 and from $10^{-2}$ to $10^{-7}$ for $k$ = 100.

**Example 3.** Here, we consider Neumann-Neumann singular boundary problem on the irregular domain. Fig. 11 displays a sketch distribution of the boundary knot and singular point $O$. The exact solution is expressed by

$$u^{(an)} = u_1^{(NN)}(\mathrm{x}) + 1.3u_2^{(NN)}(\mathrm{x}) + 1.5u_3^{(NN)}(\mathrm{x}), \tag{26}$$

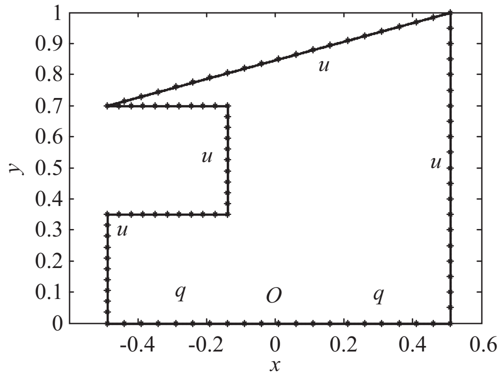


**Fig. 11. Typical distributions of boundary points of an irregular domain with singularity at the origin *O*.**

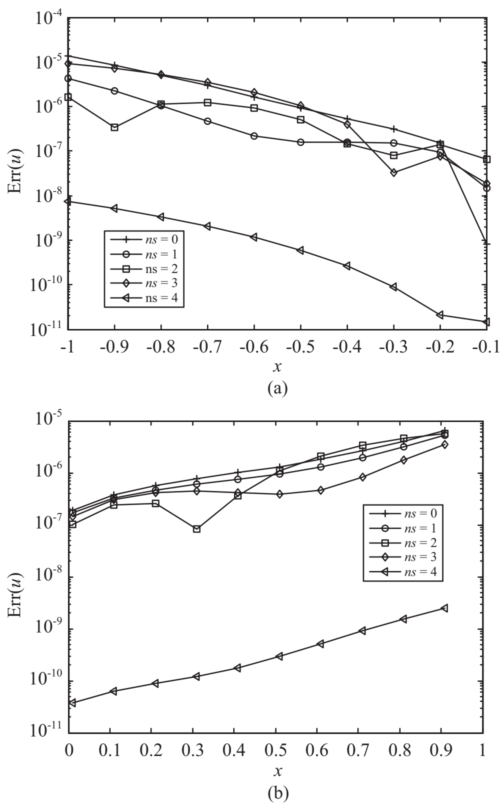


**Fig. 12. The absolute errors obtained by subtracting various number of singular functions, namely, *ns* = {1, 2, 3, 4} for the Neumann-Neumann singular problem along *x* axis with *y* = 0.01.**

where the wave number is 1 and the corresponding boundary conditions can be obtained from the above solution (26).

The absolute errors Err($u$) along $x$ axis at $y$ = 0.01 decreases from $10^{-4}$ for the singular function number $ns$ = 0 to $10^{-9}$ for $ns$ = 4 as verified by the results in Fig. 12. Fig. 13 shows the contour lines of the absolute error Err($u$) at internal points x located nearby the singular point $O$, namely $\Omega = (-10^{-2}, 10^{-2}) \times (10^{-3}, 2 \times 10^{-2})$ when four singular functions are subtracted from the BKM. Figs. 12 and 13 clearly demonstrate

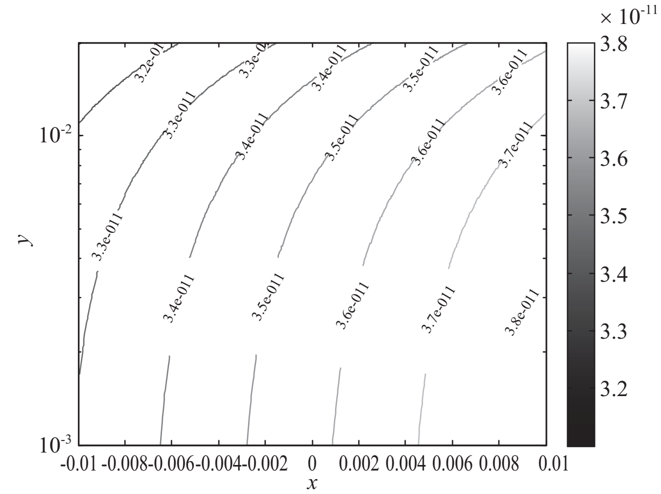


**Fig. 13. Contour plot of the absolute errors on the neighborhood of the singular point *O* obtained by subtracting 4 singular functions for Neumann-Neumann singular problem.**

the remarkable numerical solutions accuracy and efficiency of the BKM at the internal points near the singular point $O$ by employing the desingularization technique. It can be seen that the numerical solution accuracy is improved around $10^{-11}$ with the singularity subtraction by four singular functions in the revised BKM formulation.

**Example 4.** Finally, we consider the following problem in the rectangle $\Omega = (-1,1) \times (0,1)$, see Fig. 2. The same problem has been studied by the finite element method and boundary element method, see Refs. [31, 33].

$$\Delta u(\mathrm{x}) - u(\mathrm{x}) = 0, \qquad \mathrm{x} \in \Omega, \tag{27}$$

$$u(\mathrm{x}) = r^{-\frac{1}{2}} \sinh(r)\cos(\theta/2), \quad \mathrm{x} \in \partial\Omega / \Gamma_q, \tag{28}$$

$$q(\mathrm{x}) = \partial_n u(\mathrm{x}) = 0, \quad \mathrm{x} \in \Gamma_q = (0,1) \times \{0\}, \tag{29}$$

This problem has singularity at the origin (0,0), and its analytical solution is given by

$$u(r,\theta) = r^{-\frac{1}{2}} \sinh(r)\cos(\theta/2). \tag{30}$$

Fig. 14 presents the absolute errors when the standard BKM is used in comparison with the analytical values. From these figures, we can see that numerical solution does not approximate accurately the analytical solution in the neighborhood of singular point when no singular functions are subtracted. Once the modified BKM scheme is applied for this problem, numerical results are considerably improved as can be seen from Fig. 15, namely a significant improvement in the accuracy of the numerical solutions from $10^{-1}$ to $10^{-8}$, and the

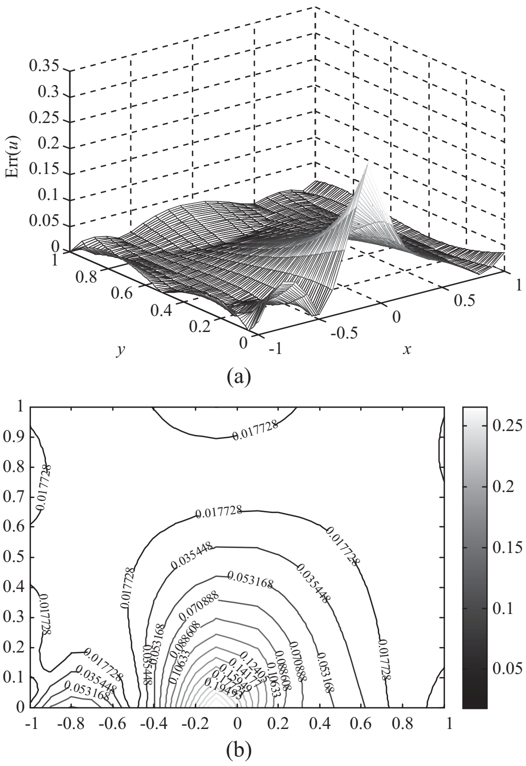


**Fig. 14. The absolute errors of traditional BKM without subtracted any singular functions.**

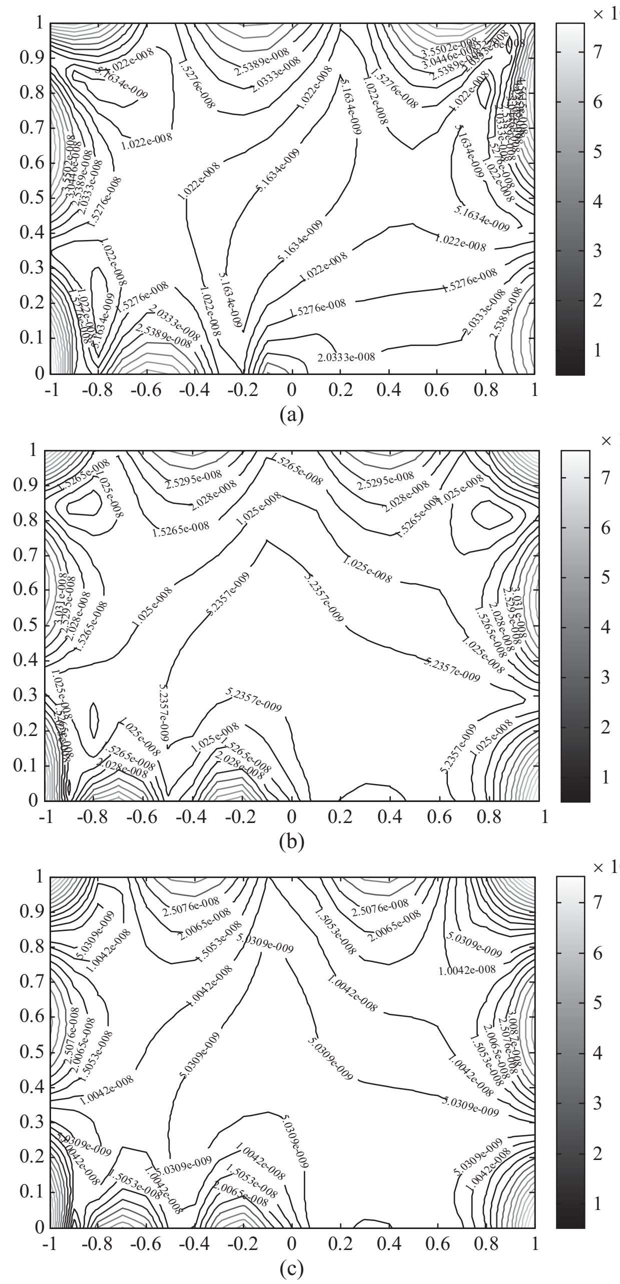


**Fig. 15. The absolute errors of modified BKM by subtracting different singular functions *ns* = 1, *ns* = 2 and *ns* = 3.**

same accurate results are observed if more singular functions are subtracted in the modified BKM.

## VI. CONCLUSION

This paper documents the first attempt to extend the BKM to Helmholtz problems with boundary singularities. A singularity subtraction technique was introduced to remove the severely adverse effect of the boundary singularities on the solution accuracy and stability. From the foregoing numerical results and discussions, we conclude that the proposed scheme is computationally efficient, robust and accurate for the numerical solution of Helmholtz problems containing boundary singularities. The proposed BKM scheme could be considered a competitive alternative to existing methods for Helmholtz problems with boundary singularities.

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