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DYNAMIC OUTPUT FEEDBACK SLIDING MODE CONTROLLER DESIGN FOR CHATTERING AVOIDANCE

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Key words: second order, dynamic output feedback, sliding mode, chattering avoidance.

ABSTRACT

For a class of linear MIMO uncertain systems, a dynamic output feedback sliding mode control algorithm that avoids the chattering problem and high gain control is proposed in this paper. Without using any differentiator, we develop a modified asymptotically stable second order sliding mode control law in which the developed controller can guarantee the finite time convergence to the sliding mode and the system states asymptotically approach to zero. Finally, a numerical example is explained for demonstrating the applicability of the proposed scheme.

I. INTRODUCTION

Sliding mode control (SMC) has been successfully used in controlling many uncertain systems [10, 19]. For a system with the matched disturbance, SMC can obtain the perfect disturbance rejection during the sliding mode. In many practical systems, only the system output is available and therefore, researchers [1, 8, 10, 12, 20] have designed the output feedback controllers via the sliding mode technique to stabilize multivariable plants with matched uncertainties. The control objectives are attained by constraining the system dynamics on a properly chosen sliding variable by means of discontinuous control laws. In theory, SMC offers robust stability to systems through high gain control with the infinite fast switching action. However, high-gain control designs suffer from the drawback of peaking phenomenon, in which the control input peaks to an extremely large value during the transient stage. The peaking phenomenon can easily violate the control saturation constraint. Moreover, SMC using the discontinuous high speed switching action results in the chattering problem. The chattering action may excite the unmodeled high order dynamics and even cause instability [2, 10, 19]. There are two major approaches reported to cope with the chattering problem. The first approach is the boundary layer control [2, 6, 18] which uses the saturation function instead of the signum function inserts a boundary layer around the sliding variable. However, the boundary layer control cannot guarantee the property of perfect disturbance rejection and hence reduces control accuracy.

Another approach to eliminate the control chattering is to derive a dynamic sliding mode controller [17] or design high order sliding mode control [3-5]. High order sliding mode control not only removes some of the fundamental limitations of the traditional approach but also provides improved tracking accuracy under sliding mode. For example, the case of second order sliding mode corresponds to the control acting on the second derivative of the sliding variable. The main problem in implementation of high order sliding modes is the increasing information demand. Several such second order sliding mode algorithms have been presented in the literatures [7, 9, 13-16]. Levant [13, 14] presented the twisting algorithm to stabilize second order nonlinear systems but required the knowledge of the derivative of the sliding variable. Bartolini [3, 4] presented a sub-optimal version of the twisting algorithm to cope with the chattering problem. However, this method requires at least the knowledge of the sign of the derivative of the sliding variable. The super twisting algorithm [14, 15] does not require the output derivative to be measured but it has been originally developed and analyzed for system with relative degree one. A robust exact finite time convergence differentiator is proposed in Levant [15] and utilized to estimate the derivative of the sliding variable. For these abovementioned approaches, estimation of extend sliding variable becomes the main control difficulty.

An alternative dynamic second order sliding mode controller for avoiding the chattering problem and high gain control is proposed in this paper. A dynamic output feedback sliding mode controller is developed based on modified second-order sliding mode techniques and the resulting control

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forces are chattering-free. Introducing a reduced-order state estimator into the controller, the proposed control law can provided theoretically the finite time convergence to the sliding mode. We show that the system states asymptotically approach to zero. As a result, the control accuracy is better than those performed by the conventional boundary layer control. A sufficient condition for the closed-loop stability is given and the implementation of the proposed control algorithm is simple. Finally, the feasibility of the proposed method is illustrated by a numerical example.

The work of this paper is organized as follows. Section 2 describes a class of uncertain MIMO linear systems and Section 3 presents the dynamic output feedback sliding mode controller design. The simulation result is included in Section 4. Section 5 offers a brief conclusion.

II. PROBLEM FORMULATION

Consider an uncertain system satisfying the matched condition of the form

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\big(\boldsymbol{u}(t) + \boldsymbol{d}(t,\boldsymbol{x})\big) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \Re^n$ is the state vector, $\boldsymbol{u} \in \Re^m$ is the control input vector, $\boldsymbol{y} \in \Re^p$ is the output vector, and $\boldsymbol{d} \in \Re^m$ is the unknown matched disturbance vector with the known upper bounds $\|\boldsymbol{d}(t,\boldsymbol{x})\| \le a_1$ and $\|\dot{\boldsymbol{d}}(t,\boldsymbol{x})\| \le a_2$. Suppose that the system output in system (1) is only measurable. Let the sliding variable be chosen as

$$\boldsymbol{s}(t) = \boldsymbol{G}\boldsymbol{y}(t) \tag{2}$$

where the matrix $\boldsymbol{G} \in \Re^{m\times p}$ is designed to stabilize the reduced-order system. To satisfy the reaching and sliding condition, the control input for the conventional sliding mode controller is designed as [10, 19]

$$\boldsymbol{u}(t) = -\left(\boldsymbol{GCB}\right)^{-1}\left(\boldsymbol{S}\boldsymbol{y}(t) + \gamma sign\left(\boldsymbol{s}(t)\right)\right) \tag{3}$$

where $\gamma > 0$ is a high gain to design such that the system reaches and slides on the sliding surface in finite time. The main disadvantages of (3) are that it produces the chattering phenomena and requires high gain control. However, undesired chattering effect produced by the high switching action of the control input is the main implementation problem of SMC. The continuous approximation techniques [2, 6, 18] have been presented to reduce the chattering. A drawback of continuous approximation methods is the reduction of the control accuracy.

For a linear MIMO system with the matched disturbance, in this paper we develop a dynamic output feedback sliding mode control algorithm in which the proposed procedure can effectively reduce the chattering effect. Introducing a PI-type control input into the controller, the finite time convergence to the sliding mode is guaranteed. Moreover, the system states asymptotically approach to zero once the system is in the sliding mode. Before introducing the proposed method, the following assumptions are made throughout this paper.

Assumption 1. System (1) is minimum phase and $rank(\boldsymbol{CB}) = m$.

Assumption 2. The pairs $(\boldsymbol{A}, \boldsymbol{B})$ and $(\boldsymbol{C}, \boldsymbol{A})$ are stabilizable and detectable, respectively.

III. DYNAMIC OUTPUT FEEDBACK SLIDING MODE CONTROLLER DESIGN

In this section, we propose a dynamic output feedback sliding mode control algorithm which can successfully avoid the chattering. A modified second-order sliding mode control algorithm that does not require the derivative of the sliding variable is presented. Introducing a reduced-order state estimator into the controller, the proposed control law can guarantee the finite time convergence to the sliding mode and stabilize the reduced-order system in which the system states asymptotically approach to zero.

Since system (1) is minimum phase and $rank(\boldsymbol{CB}) = m$, Edward and Spurgeon [10] have shown that a matrix $\boldsymbol{F} \in \Re^{m\times p}$ can be found such that the matrix $(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC})\boldsymbol{A}$ has $n-m$ non-zero eigenvalues λ_i, $i = 1, 2, \cdots, n-m$, satisfying $Re\{\lambda_i\} < 0$. Then we define a new measurable state $\boldsymbol{s} \in \Re^m$ of the form

$$\boldsymbol{s}(t) = \left(\boldsymbol{FCB}\right)^{-1}\boldsymbol{F}\boldsymbol{y}(t) = \left(\boldsymbol{FCB}\right)^{-1}\boldsymbol{FC}\boldsymbol{x}(t) \tag{4}$$

There exists a full rank matrix $\boldsymbol{W} \in C^{n\times(n-m)}$ such that

$$\left(\boldsymbol{I}_n - \boldsymbol{B}\left(\boldsymbol{FCB}\right)^{-1}\boldsymbol{FC}\right)\boldsymbol{AW} = \boldsymbol{W\Lambda} \tag{5}$$

where $\boldsymbol{\Lambda}$ is the Jordan form of eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$ and $\boldsymbol{W}$ contains the right eigenvectors corresponding to $\boldsymbol{\Lambda}$. As for the other m eigenvalues of $(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC})\boldsymbol{A}$, they are all zeros. This can be easily seen from (5) that $\boldsymbol{FC}(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC})\boldsymbol{A} = \boldsymbol{0}$ and $rank(\boldsymbol{FC}) = m$. Also, premultiplying $\boldsymbol{C}$ into (5) yields

$$\boldsymbol{FC}\left(\boldsymbol{I}_n - \boldsymbol{B}\left(\boldsymbol{FCB}\right)^{-1}\boldsymbol{FC}\right)\boldsymbol{AW} = \boldsymbol{FCW}\ \boldsymbol{\Lambda} = \boldsymbol{0} \tag{6}$$

Because $\boldsymbol{\Lambda}$ is of full rank, we have

$$\boldsymbol{FCW} = \boldsymbol{0} \tag{7}$$

Now, let's focus on $\boldsymbol{W} \in C^{n\times(n-m)}$, a complex matrix possessing $n-m$ complex eigenvectors in accordance with λ_i, i = 1, 2, $\cdots$, $n-m$, satisfying $Re\{\lambda_i\} < 0$. It is known if a complex eigenvalue λ belongs to $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$, so is its conjugate $\bar{\lambda}$; therefore, $\boldsymbol{W}\boldsymbol{W}^H \in \Re^{n\times n}$ where $\boldsymbol{W}^H = \bar{\boldsymbol{W}}^T$, the conjugate and transpose matrix of $\boldsymbol{W}$. Therefore, applying the singular-value decomposition technique to $\boldsymbol{W}$ leads to

$$\boldsymbol{W} = \boldsymbol{U}\begin{bmatrix}\boldsymbol{Q}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{V}^H \tag{8}$$

where $\boldsymbol{U} \in \Re^{n\times n}$ is orthogonal, i.e. $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_n$, $\boldsymbol{V} \in C^{(n-m)\times(n-m)}$ is unitary, i.e. $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}_{n-m}$, and $\boldsymbol{Q} \in \Re^{(n-m)\times(n-m)}$ is invertible. Let $\boldsymbol{U} = [\boldsymbol{U}_1 \quad \boldsymbol{U}_2]$ where $\boldsymbol{U}_1 \in \Re^{n\times(n-m)}$, then from (8) we obtain

$$\boldsymbol{W} = \boldsymbol{U}_1\boldsymbol{Q}\boldsymbol{V}^H \tag{9}$$

Note that $\boldsymbol{U}_1^T\boldsymbol{U}_1 = \boldsymbol{I}_{n-m}$ since $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_n$. Substituting (9) into (7) leads to $\boldsymbol{C}\boldsymbol{U}_1\boldsymbol{Q}\boldsymbol{V}^H = \boldsymbol{0}$. Obviously,

$$\boldsymbol{C}\boldsymbol{U}_1 = \boldsymbol{0} \tag{10}$$

since $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}_{n-m}$ and $\boldsymbol{Q}$ is invertible. Based on (7), (10) and $\boldsymbol{U}_1^T\boldsymbol{U}_1 = \boldsymbol{I}_{n-m}$, we have

$$\begin{bmatrix}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\\ \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\end{bmatrix}[\boldsymbol{B} \quad \boldsymbol{U}_1] = \boldsymbol{I}_n \tag{11}$$

Define

$$\boldsymbol{M} = \begin{bmatrix}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\\ \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\end{bmatrix}$$

and

$$\boldsymbol{N} = [\boldsymbol{B} \quad \boldsymbol{U}_1]$$

then both $n\times n$ square matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ are real and $\boldsymbol{M}$ = $\boldsymbol{N}^{-1}$. Most significantly, $\boldsymbol{M}$ can be used as a transformation matrix. Let $\boldsymbol{z} = \boldsymbol{U}_1^T(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC})\boldsymbol{x}$, which is not measurable, then

$$\boldsymbol{Mx} = \begin{bmatrix}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\\ \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}\boldsymbol{s}\\ \boldsymbol{z}\end{bmatrix} \tag{12}$$

or

$$\boldsymbol{x} = \boldsymbol{M}^{-1}\begin{bmatrix}\boldsymbol{s}\\ \boldsymbol{z}\end{bmatrix} = \boldsymbol{N}\begin{bmatrix}\boldsymbol{s}\\ \boldsymbol{z}\end{bmatrix} = \boldsymbol{Bs} + \boldsymbol{U}_1\boldsymbol{z} \tag{13}$$

Pre-multiplying $\boldsymbol{M}$ into (1) becomes

$$\begin{aligned}\dot{\boldsymbol{s}}(t) &= (\boldsymbol{FCB})^{-1}\boldsymbol{FCABs}(t) + (\boldsymbol{FCB})^{-1}\boldsymbol{FCAU}_1\boldsymbol{z}(t)\\ &\quad + \boldsymbol{u}(t) + \boldsymbol{d}(t, \boldsymbol{x})\end{aligned} \tag{14}$$

and

$$\begin{aligned}\dot{\boldsymbol{z}}(t) &= \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1\boldsymbol{z}(t)\\ &\quad + \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{ABs}(t)\end{aligned} \tag{15}$$

From (9), $\boldsymbol{V}^H\boldsymbol{V} = \boldsymbol{I}_{n-m}$, and $\boldsymbol{U}_1^T\boldsymbol{U}_1 = \boldsymbol{I}_{n-m}$, we have

$$\boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1 = \boldsymbol{QV}^{-1}\boldsymbol{\Lambda}\left(\boldsymbol{QV}^{-1}\right)^{-1} \tag{16}$$

Evidently, $\boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1$ has eigenvalues related to $\boldsymbol{\Lambda}$, i.e., $\{\lambda_1, \lambda_2, \cdots, \lambda_{n-m}\}$, all located in the left-half complex plane. Viewing from (15), an estimator for $\boldsymbol{z}$ can be built up as

$$\begin{aligned}\dot{\hat{\boldsymbol{z}}}(t) &= \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1\hat{\boldsymbol{z}}(t)\\ &\quad + \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{ABs}(t)\end{aligned} \tag{17}$$

Let $\tilde{\boldsymbol{z}} = \boldsymbol{z} - \hat{\boldsymbol{z}}$. From (15) and (17), we have

$$\dot{\tilde{\boldsymbol{z}}}(t) = \boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1\tilde{\boldsymbol{z}}(t) \tag{18}$$

where all the eigenvalues of $\boldsymbol{U}_1^T\left(\boldsymbol{I}_n - \boldsymbol{B}(\boldsymbol{FCB})^{-1}\boldsymbol{FC}\right)\boldsymbol{AU}_1$ possess negative real part. As a result, we can conclude

$$\hat{\boldsymbol{z}}(t) \to \boldsymbol{z}(t) \qquad \text{for } t \to \infty \tag{19}$$

This completes the design of state-estimator (15) for $\boldsymbol{z}$.

With the use of the state-estimator (17), the total system is rewritten as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{Ax}(t) + \boldsymbol{B}(\boldsymbol{u}(t) + \boldsymbol{d}(t, \boldsymbol{x}))\\ \boldsymbol{y}(t) &= \boldsymbol{Cx}(t)\end{aligned}$$

$$\dot{\hat{z}}(t) = U_1^T\left(I_n - B\left(FCB\right)^{-1}FC\right)AU_1\hat{z}(t)$$
$$+U_1^T\left(I_n - B\left(FCB\right)^{-1}FC\right)ABs(t) \quad (20)$$

Now choose the sliding variable as $s = \left(FCB\right)^{-1}FCx = \left(FCB\right)^{-1}Fy$ in (4), then from (14) we obtain

$$\dot{s}(t) = \left(FCB\right)^{-1}FCABs(t) + \left(FCB\right)^{-1}FCAU_1z(t)$$
$$+u(t) + d\left(t, x\right) \quad (21)$$

Based on the state estimator (17), we design the PI-type dynamic sliding mode controller as

$$u(t) = -\left(FCB\right)^{-1}\left(FCABs(t) + FCAU_1\hat{z}(t)\right)$$
$$-L_1s(t) - L_2\int_0^t s(\tau)d\tau - K\int_0^t sign\left(s(\tau)\right)d\tau \quad (22)$$

where $L_1 \in \Re^{m\times m}$, $L_2 \in \Re^{m\times m}$ and $K \in \Re^{m\times m}$ are the positive definite diagonal matrix given by

$$L_1 = diag\left(l_{11}, \cdots l_{1m}\right),\ L_2 = diag\left(l_{21}, \cdots l_{2m}\right)$$

and

$$K = diag\left(k_1, \cdots k_m\right)$$

Moreover, these parameters in the matrices are designed in the latter. Substituting the control input (22) into (21) can obtain

$$\dot{s}(t) = \left(FCB\right)^{-1}FCAU_1\tilde{z}(t) + d(x,t)$$
$$-L_1s(t) - L_2\int_0^t s(\tau)d\tau - K\int_0^t sign\left(s(\tau)\right)d\tau \quad (23)$$

Further differentiating (23) yields the following dynamics

$$\ddot{s}(t) + L_1\dot{s}(t) + L_2s(t) = -Ksign\left(s(t)\right) + f(t) \quad (24)$$

where

$$f(t) = \left(FCB\right)^{-1}FCAU_1U_1^T\left(I_n - B\left(FCB\right)^{-1}FC\right)AU_1\tilde{z}(t)$$
$$+\dot{d}(t, x)$$

It follows that $f = \left[f_1 \quad \cdots \quad f_m\right]^T \in \Re^m$. Before giving the main result, we have the following lemmas.

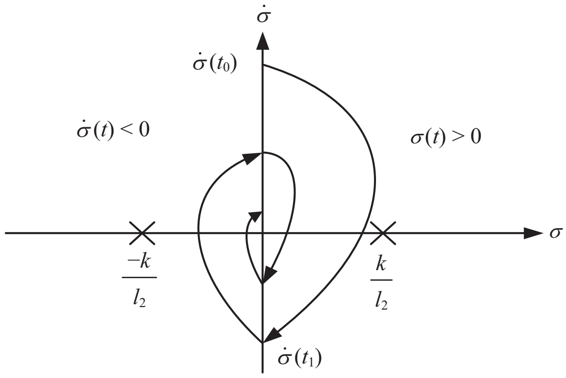

Fig. 1. Phase paths of the second-order system.

Lemma 1

Consider the unperturbed system as

$$\ddot{\sigma}(t) + l_1\dot{\sigma}(t) + l_2\sigma(t) = -ksign\left(\sigma(t)\right) \quad (25)$$

If the roots of the characteristic equation $s^2 + l_1s + l_2 = 0$ are located in the left-half plane, then $\sigma(t)$ and $\dot{\sigma}(t)$ asymptotically converge to zero in finite time for a sufficiently large value of $k > 0$.

Proof:

First, we choose the parameters l_1 and l_2 such that the roots of the characteristic equation, $s^2 + l_1s + l_2 = 0$, are located in the left-half plane. Assume now for simplicity that the initial conditions are $\sigma(t_0) = 0$ and $\dot{\sigma}(t_0) > 0$. Thus the trajectory enters the half-plane $\sigma(t) > 0$ (quadrant I), given in Fig. 1. When $\sigma(t) > 0$, we have $\ddot{\sigma}(t) + l_1\dot{\sigma}(t) + l_2\sigma(t) = -k$ and obtain its equivalent point as $\left(\sigma, \dot{\sigma}\right) = \left(\frac{-k}{l_2}, 0\right)$. Let the function $g(t)$ be generated by

$$g(t) = \dot{\sigma}(t) + r\sigma(t) + \frac{k}{\mu}sign\left(\sigma(t)\right)$$

where $\mu + r + l_1$ and $\mu r + \gamma = l_2$. The parameters $\mu > 0$, $r > 0$ and $\gamma \geq 0$ are real constants and are chosen such that the equation, $s^2 + l_1s + (l_2 - \gamma) = 0$, has real distinct roots. Since $\sigma(t) > 0$, it follows from $\ddot{\sigma}(t) + l_1\dot{\sigma}(t) + l_2\sigma(t) + k = 0$ that

$$\frac{dg(t)}{dt} + \mu g(t) = \ddot{\sigma}(t) + \left(\mu + r\right)\dot{\sigma}(t) + \mu r\sigma(t) + k$$
$$= \ddot{\sigma}(t) + l_1\dot{\sigma}(t) + \left(l_2 - \gamma\right)\sigma(t) + k$$
$$= -\gamma\sigma(t)$$

The solution to $g(t)$ is then given by

$$g(t) = g(0)e^{-\mu t} - \gamma\int_0^t e^{-\mu\tau}\sigma(t-\tau)d\tau$$

Since $\mu > 0$ and the characteristic equation $s^2 + l_1 s + l_2 = 0$ has the stable roots, we have $|\sigma(0)| \geq |\sigma(t)|$ for $t > 0$ and then obtain

$$|g(t)| \leq Ce^{-\mu t} + \gamma\int_0^t e^{-\mu\tau}|\sigma(t-\tau)|d\tau$$
$$\leq Ce^{-\mu t} + \frac{\gamma|\sigma(0)|}{\mu}\left(1 - e^{-\mu t}\right)$$

where $C > 0$ is a constant. Choose a Lyapunov function as

$$V(t) = |\sigma(t)|$$

and then obtain its time derivative as

$$\dot{V}(t) = sign\left(\sigma(t)\right)\left(g(t) - \frac{k}{\mu}sign\left(\sigma(t)\right) - r\sigma(t)\right)$$
$$\leq Ce^{-\mu t} + \frac{\gamma|\sigma(0)|}{\mu}\left(1 - e^{-\mu t}\right) - \frac{k}{\mu} - r|\sigma(t)|$$

Since $\mu > 0$, there exists a finite time, $T_1 > 0$, such that $Ce^{-\mu t} + \frac{\gamma|\sigma(0)|}{\mu} < \frac{k}{\mu} - \rho$ for a sufficiently large k and $t > T_1$, where $\rho > 0$ is a constant. Hence

$$\dot{V}(t) \leq -\rho - r|\sigma(t)| \text{ for } t > T_1$$

The above equation implies that the function $\sigma(t)$ converges to zero in finite time. Let the trajectory of (25) intersect next time with the axis $\sigma(t) = 0$ at the point $\dot{\sigma}(t_1)$. Since the roots of the characteristic equation, $s^2 + l_1 s + l_2 = 0$, are stable, we know that the spiral trajectories converge to the equivalent point and the behavior of $\dot{\sigma}(t)$ changes monotonously. Hence,

$$\frac{|\dot{\sigma}(t_1)|}{|\dot{\sigma}(t_0)|} = q < 1$$

Extending the trajectory into the half plane $\sigma(t) < 0$ after a similar reasoning achieves that successive crossing the axis $\sigma(t) = 0$ satisfies the inequality $\frac{|\dot{\sigma}(t_{i+1})|}{|\dot{\sigma}(t_i)|} = q < 1$. Therefore, its solutions cross the axis $\sigma(t) = 0$ from quadrant II to quadrant I, and from quadrant IV to quadrant III. Every trajectory of the system crosses the axis $\sigma(t) = 0$ in finite time. After gluing these paths along the line $\sigma(t) = 0$, we obtain the phase portrait of the system, as shown in Fig. 1. This algorithm features a twisting of the phase portrait around the origin and an infinite number encircling the origin occurs. According to Lavent's papers [13, 14], the total convergence time is estimated as

$$T \leq \sum|\dot{\sigma}(t_i)| \leq |\dot{\sigma}(t_0)|\left(1 + q + q^2 + \cdots\right) = \frac{|\dot{\sigma}(t_0)|}{1-q}$$

As a result, we show that the trajectories perform rotations around the origin while converging in finite time to the origin of the phase plane. The finite time convergence to the origin is due to switching between two different control amplitudes as the trajectory comes nearer to the origin. The proof of the lemma is finished.

Lemma 2

Consider the following system

$$\ddot{\sigma}(t) + l_1\dot{\sigma}(t) + l_2\sigma(t) = -ksign\left(\sigma(t)\right) + f(t) \tag{26}$$

where the function $f(t)$ has the upper bound $|f(t)| \leq \eta$ and $\eta > 0$ is a known constant. If the parameters l_1 and l_2, and the gain k are chosen to satisfy the following condition:

$$l_2 < \frac{l_1^2}{4} \text{ and } k > \eta \tag{27}$$

then $\sigma(t)$ and $\dot{\sigma}(t)$ converge to zero in finite time.

Proof:

When $\sigma(t) > 0$, Eq. (26) becomes

$$\ddot{\sigma}(t) + l_1\dot{\sigma}(t) + l_2\sigma(t) = -k + f(t)$$

Let $v_1 = \sigma + \frac{k}{l_2}$ and $v_2 = \dot{v}_1 = \dot{\sigma}$. It follows that

$$\dot{\boldsymbol{v}}(t) = \begin{bmatrix} 0 & 1 \\ -l_2 & -l_1 \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} = \boldsymbol{\Phi}\boldsymbol{v}(t) + \boldsymbol{b}f(t),$$

where $\boldsymbol{v} = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T \in \Re^2$, $\boldsymbol{\Phi} = \begin{bmatrix} 0 & 1 \\ -l_2 & -l_1 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Write the above dynamic equation as its solution in an explicit form

$$\boldsymbol{v}(t) = \boldsymbol{e}^{\boldsymbol{\Phi}t}\boldsymbol{v}(0) + \int_0^t \boldsymbol{e}^{\boldsymbol{\Phi}\tau}\boldsymbol{b}f(t-\tau)d\tau$$

where

$$\boldsymbol{e}^{\Phi t}=\begin{bmatrix}\frac{1}{\beta-\alpha}\left(\beta e^{-\alpha t}-\alpha e^{-\beta t}\right) & \frac{1}{\beta-\alpha}\left(e^{-\alpha t}-e^{-\beta t}\right)\\ \frac{\alpha\beta}{\alpha-\beta}\left(e^{-\alpha t}-e^{-\beta t}\right) & \frac{1}{\alpha-\beta}\left(\alpha e^{-\alpha t}-\beta e^{-\beta t}\right)\end{bmatrix}.$$

We take the parameters l_1 and l_2 to satisfy $l_2>\frac{l_1^2}{4}$ and then obtain the characteristic polynomial of system (26) having two distinct real roots $\lambda_{1,2}=-\alpha, -\beta$ where $\alpha>0$, $\beta>0$, $\beta>\alpha$, $l_1=\alpha+\beta$ and $l_2=\alpha\beta$. The upper bounds of $v_1(t)$ and $v_2(t)$ can be constructed as

$$\begin{aligned}\left|v_1(t)\right| &\le C_1e^{-\alpha t}+\frac{1}{\beta-\alpha}\int_0^t\left|e^{-\alpha\tau}-e^{-\beta\tau}\right|\left|f(t-\tau)\right|d\tau\\ &\le C_1e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\int_0^t\left|e^{-\alpha\tau}-e^{-\beta\tau}\right|d\tau\\ &\le C_1e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\int_0^\infty\left(e^{-\alpha\tau}-e^{-\beta\tau}\right)d\tau\\ &= C_1e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\left(\frac{\beta-\alpha}{\alpha\beta}\right)=C_1e^{-\alpha t}+\frac{\eta}{l_2}\end{aligned}$$

and

$$\begin{aligned}\left|v_2(t)\right| &\le C_2e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\left(\int_0^t\left|\alpha e^{-\alpha\tau}-\beta e^{-\beta\tau}\right|d\tau\right)\\ &\le C_2e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\left(\int_0^\infty\left|\alpha e^{-\alpha\tau}-\alpha e^{-\beta\tau}+(\alpha-\beta)e^{-\beta\tau}\right|d\tau\right)\\ &\le C_2e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\left(\int_0^\infty\left|\alpha e^{-\alpha\tau}-\alpha e^{-\beta\tau}+(\alpha-\beta)e^{-\beta\tau}\right|d\tau\right)\\ &\le C_2e^{-\alpha t}+\frac{\eta}{\beta-\alpha}\left(\alpha\frac{\beta-\alpha}{\alpha\beta}+\frac{\beta-\alpha}{\beta}\right)=C_2e^{-\alpha t}+\frac{2\alpha\eta}{l_2}\end{aligned}$$

where $C_1>0$ and $C_2>0$ are constants. It follows that

$$\left|v_1(t)\right|\le C_1e^{-\alpha t}+\frac{\eta}{l_2}\ \text{ and }\ \left|v_2(t)\right|\le C_2e^{-\alpha t}+\frac{2\eta\alpha}{l_2} \tag{28}$$

Since $v_1=\sigma+\frac{k}{l_2}$, we can from (28) obtain

$$\left|\sigma(t)+\frac{k}{l_2}\right|\le C_1e^{-\alpha t}+\frac{\eta}{l_2}$$

This part shows that the ball of radius $r=\frac{\eta}{l_2}$ with center located at $\left(\frac{-k}{l_2},0\right)$ is an attractor B_{s1}. Similar to the work, we have, when $\sigma(t)<0$, the ball of radius r, with center located at $\left(\frac{k}{l_2},0\right)$ is another attractor B_{s2}. Choose the gain k to satisfy the inequality $k>\eta$ and then we have

$$\left(\frac{k}{l_2}-r\right)=\frac{k}{l_2}-\frac{\eta}{l_2}>0\ \text{ and }\ \left(-\frac{k}{l_2}+r\right)=\frac{-k}{l_2}+\frac{\eta}{l_2}<0$$

It follows from the above two inequalities that the two attractors B_{s1} and B_{s2} do not intersect each other, and the behavior of the perturbed system (26) will be qualitatively similar to the behavior of the nominal system. Therefore, the perturbed system converges to the origin in the same way of the nominal system and the condition that $\sigma(t)$ and $\dot{\sigma}(t)$ converge to zero in finite time can be guaranteed. We complete the proof of the lemma.

Since $\left\|\dot{\boldsymbol{d}}(t,\boldsymbol{x})\right\|\le a_2$ and $\left\|\tilde{\boldsymbol{z}}(t)\right\|\to 0$ as $t\to\infty$, we know $\left\|\boldsymbol{f}(t)\right\|\le\left\|\dot{\boldsymbol{d}}(\boldsymbol{x},t)\right\|+\varphi=a_2+\varphi$ where $\varphi>0$ is a given constant. Based on linear algebraic theory, we have

$$\left|f_i(t)\right|<\left\|\boldsymbol{f}(t)\right\|\le a_2+\varphi$$

Theorem 1

Consider system (1) with the sliding variable (4) and the control input (22). Let $\eta=a_2+\varphi$ where $\varphi>0$ is a known constant. If the elements of these matrices $\boldsymbol{L}_1$, $\boldsymbol{L}_2$, and $\boldsymbol{K}$ in the controller satisfy the following conditions:

$$l_{2i}<\frac{l_{1i}^2}{4}\ \text{ and }\ k_i>\eta,\ \text{ for }\ i=1,\cdots,q \tag{29}$$

then the system states $\boldsymbol{x}$ asymptotically approach to zero.

Proof:

We first express system (24) as a set of second-order systems with the form

$$\ddot{s}_i(t)+l_{1i}\dot{s}(t)+l_{2i}s(t)=-k_i sign\left(s_i(t)\right)+f_i(t) \tag{30}$$

where $i=1,\cdots,q$ and $\left|f_i(t)\right|\le\eta$. Applying the result of Lemma 2 into (30), if the parameters l_{1i}, l_{i2} and k_i can satisfy

$$l_{2i}<\frac{l_{1i}^2}{4}\ \text{ and }\ k_i>\eta,\ \text{ for }\ i=1,\cdots,q$$

then $s_i(t)$ asymptotically converges to zero in finite time according to Lemma 2. It follows that the sliding variable converges to zero in finite time. When the condition $\boldsymbol{s}(t)=\boldsymbol{0}$ is guaranteed, it follows from the concept of the equivalent control [10, 19] that the system dynamics in the sliding mode is

$$\dot{x}(t) = \left(A - B\left(FCB\right)^{-1} FCA\right) x(t).$$

As a result, we can from the above equation conclude that the system states asymptotically approach to zero and finish the proof of the theorem.

IV. SIMULATION RESULTS

To demonstrate the effectiveness of the proposed method, we consider an unstable batch reactor where the matched disturbance is introduced into the system as

$$\dot{x}(t) = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix} \left(u(t) + \begin{bmatrix} 1.5\sin(0.5t) \\ 1\cos(\pi t/2) \end{bmatrix} \right)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)$$

The sliding variable is chosen as

$$s(t) = \begin{bmatrix} 0 & 0.1761 \\ -0.3179 & 0 \end{bmatrix} y(t)$$

where the nonzero eigenvalues of the system in the sliding mode are assigned as $\{-5.0394, -1.1916\}$. The estimator is designed as

$$\dot{\hat{z}}(t) = \begin{bmatrix} -5.0394 & 0.0006 \\ 0.0027 & -1.1916 \end{bmatrix} z(t) + \begin{bmatrix} -9.6035 & -27.2719 \\ 38.5981 & 2.4637 \end{bmatrix} s(t)$$

The conventional boundary layer controller is designed as

$$u(t) = \begin{bmatrix} 4.1550 & 0 \\ 0 & 1.2820 \end{bmatrix} s(t) + \begin{bmatrix} 0.0543 & -0.0731 \\ 1.0015 & 0.4894 \end{bmatrix} \hat{z}(t)$$
$$-3sat\left(s(t), 0.05\right)$$

Based on the proposed algorithm, we design the control input as

$$u(t) = \begin{bmatrix} 4.1550 & 0 \\ 0 & 1.2820 \end{bmatrix} s(t) + \begin{bmatrix} 0.0543 & -0.0731 \\ 1.0015 & 0.4894 \end{bmatrix} \hat{z}(t)$$
$$-10s(t) - 2\int_0^t s(\tau)d\tau - 6\int_0^t sign\left(s(\tau)\right)d\tau$$

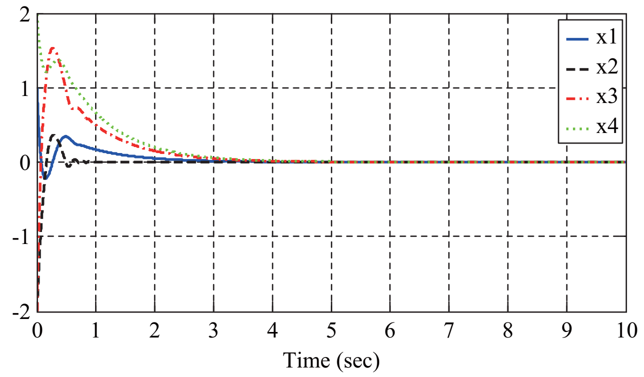

Fig. 2. System states using the proposed method.

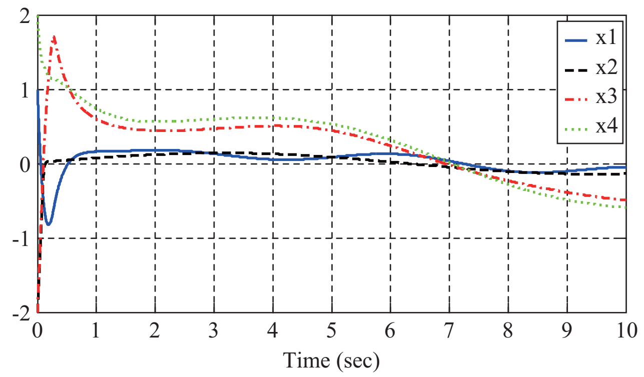

Fig. 3. System states using the saturation function.

Two cases are simultaneously simulated under the initial condition $x(0) = [1 \quad -2 \quad -2 \quad 2]^T$ and the simulation is carried out at a fixed step size of 1 milliseconds. The time responses of the system states in the two cases are shown in Fig. 2 and Fig. 3, respectively. The proposed method can guarantee that the responses of the system states asymptotically approach to zero. Figs. 4 and 5 are the responses of the sliding variable and Figs. 6 and 7 depict the control inputs of two cases. It is clear from Figs. 6 and 7 that the input gain in our method is smaller than the conventional boundary layer controller. As can be seen from these figures, the proposed method can produce the finite time convergence to the sliding mode and obtain the desired performance.

V. CONCLUSION

In this paper we have proposed a modified second-order sliding mode control algorithm to avoid the chattering problem for a MIMO uncertain system. The algorithm does not require the derivative of the sliding variable, thus eliminating the requirement of designing a differentiator. Under the developed dynamic output feedback sliding mode controller, we show that the finite time convergence to the sliding mode is guaranteed and the system states can asymptotically approach to zero. Simulation results demonstrate that the proposed control scheme exhibits reasonably good performance.

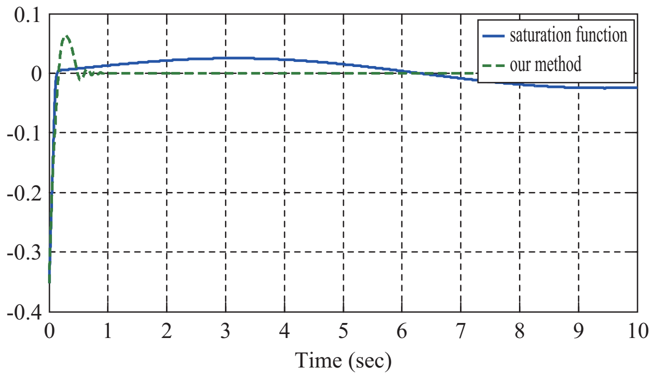

Fig. 4. Responses of $s_1(t)$ using the two methods.

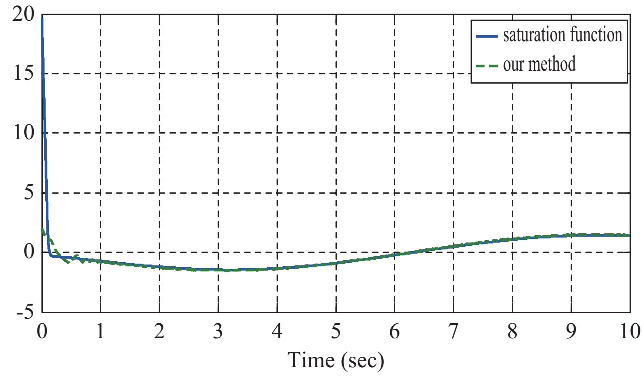

Fig. 6. Responses of $u_1(t)$ using the two methods.

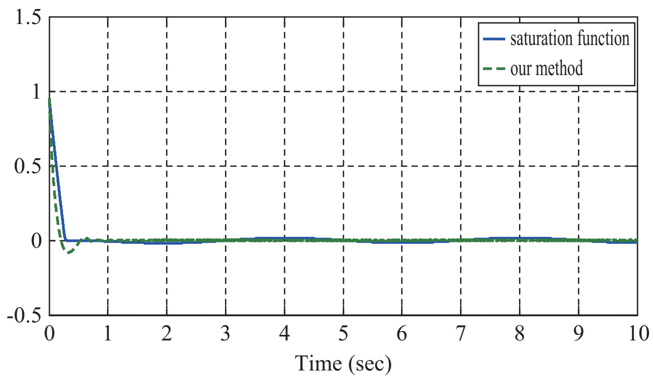

Fig. 5. Responses of $s_2(t)$ using the two methods.

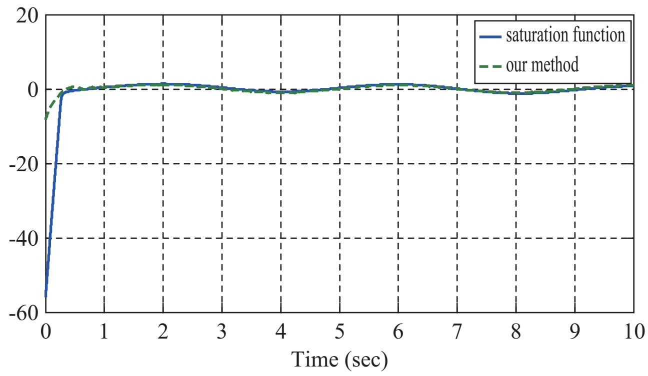

Fig. 7. Responses of $u_2(t)$ using the two methods.

REFERENCES

1. Andrade-Da Silva, J. M., Edwards, C., and Spurgeon, S. K., "Sliding-mode output feedback control based on LMIs for plants with mismatched uncertainties," *IEEE Transactions on Industrial Electronics*, Vol. 56, pp. 3675-3683 (2009).
2. Bartolini, G., "Chattering phenomena in discontinuous control systems," *International Journal of Systems Science*, Vol. 20, pp. 2471-2481 (1989).
3. Bartolini, G., Ferrara, A., and Usai, E., "Chattering avoidance by second-order sliding mode control," *IEEE Transactions on Automatic Control*, Vol. 43, pp. 241-246 (1998).
4. Bartolini, G., Ferrara, A., Usai, E., and Utkin, V. I., "On multi-input chattering-free second-order sliding mode control," *IEEE Transactions on Automatic Control*, Vol. 45, pp. 1711-1717 (2000).
5. Bartolini, G. and Pydynowski, P., "An improved, chattering free, VSC scheme for uncertain dynamics systems," *IEEE Transactions on Automatic Control*, Vol. 41, pp. 1220-1226 (1996).
6. Burton, J. A. and Zinober, A. S. I., "Continuous approximation of variable structure control," *International Journal of Systems Science*, Vol. 17, pp. 875-885 (1986).
7. Chen, M. S., Chen, C. H., and Yang, F. Y., "An LTR-observer-based dynamic sliding mode control for chattering reduction," *Automatica*, Vol. 453, pp. 1111-1116 (2007).
8. Choi, H. H., "Variable structure output feedback control design for a class of uncertain dynamic systems," *Automatica*, Vol. 38, pp. 335-341 (2002).
9. Davila, J., Fridman, L., and Levant, A., "Second-order sliding-mode observer for mechanical systems," *IEEE Transactions on Automatic Control*, Vol. 50, pp. 1785-1789 (2005).
10. Edwards, C. and Spurgeon, S. K., *Sliding Mode Control Theory and Application*, Taylor & Francis, London (1998).
11. Floquet, T., Spurgeon, S. K., and Edwards, C., "An output feedback sliding mode control strategy for MIMO systems of arbitrary relative degree," *International Journal of Robust and Nonlinear Control*, Vol. 21, pp. 119-133 (2011).
12. Kwan C. M., "On variable structure output feedback Controller," *IEEE Transactions on Automatic Control*, Vol. 41, pp. 1691-1693 (1996).
13. Levant, A., "Sliding order and sliding accuracy in the sliding mode control," *International Journal of Control*, Vol. 58, pp. 1247-1263 (1993).
14. Levant, A., "High-order sliding mode, differentiation and output-feedback control," *International Journal of Control*, Vol. 76, pp. 924-941 (2003).
15. Levant, A., "Robust exact differentiation via sliding mode technique," *Automatica*, Vol. 34, pp. 379-384 (1998).
16. Rosas Almeida, D. I., Alvarez, J., and Fridman, L., "Robust observation and identification of nDOF Lagrangian systems," *International Journal of Robust and Nonlinear Control*, Vol. 17, pp. 842-861 (2007).
17. Sira-Ramirez, H., "On the dynamical sliding mode control of nonlinear systems," *International Journal of Control*, Vol. 57, pp. 1039-1061 (1993).
18. Slotine J. J. E. and Sastry, S., "Tracking control of nonlinear systems using sliding surface with application to robot manipulator," *International Journal of Control*, Vol. 38, pp. 931-938 (1983).
19. Utkin, V. I., Guldner, J., and Shi, J., *Sliding Mode Control in Electro-Mechanical Systems*, CRC Press (2009).
20. Zak, S. H. and Hui, S., "Output feedback variable structure controllers and state estimators for uncertain/nonlinear dynamic systems," *IEE Proceedings D, Control Theory and Applications*, Vol. 140, pp. 41-49 (1993).