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# AN $O(T^2)$ BOUNDARY LAYER IN SLIDING MODE FOR TIME-DELAY SYSTEMS

Ming-Chang Pai

Key words: discrete-time, integral sliding mode control, time-delay, reaching phase, chattering phenomenon.

## ABSTRACT

A new discrete-time integral sliding mode control scheme is proposed for a class of linear multi-input systems with state delays. Based on the Lyapunov stability theory and one-step delayed disturbance approximation, a sliding mode controller not only drives the sliding mode into the  $O(T^2)$  boundary, but also achieves the  $O(T^2)$  boundary for state regulation. A novel integral sliding surface is introduced so that reaching phase is eliminated. Chattering phenomenon is eliminated and the knowledge of upper bound of external disturbances is not required. The validity of the proposed control methodology is demonstrated by simulation results.

## I. INTRODUCTION

Sliding mode control (SMC) has attractive features such as fast response, good transient performance, insensitiveness to the matching parameter uncertainties and external disturbances (Draženović, 1969; Hung et al., 1993; Young, 1999) so that SMC is an effective robust control approach for uncertain systems. In practice, using computers or digital signal processing (DSP) chips to implement the controller becomes more and more important nowadays, and discrete-time SMC has gained more and more attractive attention recently.

Several design methods for discrete-time SMC have been proposed in the literature. Sarpturk et al. (1987) proposed a new sufficient reaching condition for controlling discrete-time systems. Furuta (1990) used "sliding sector" concept to design sliding mode controllers for linear single-input discrete-time systems. Gao et al. (1995) strictly defined the quasi-sliding mode and quasi-sliding mode band, and established a new reaching condition for single-input discrete-time systems based on a reaching law approach. Young et al. (1999) and Su et al. (2000) developed discrete-time SMC schemes by using

one-step delayed disturbance approximation. Park (2000) developed a discrete-time sliding mode controller for linear time-varying systems with disturbances. Abidi et al. (2007) proposed a discrete-time integral SMC for sampled-data systems. Pai (2008) developed a discrete-time output feedback sliding mode control to stabilize a class of linear uncertain systems. Pai (2009) developed a discrete-time sliding mode controller for a class of uncertain linear systems to track dynamic outputs of a non-delay reference model. Pai (2012) developed a robust input shaping control for multi-mode flexible structures by using discrete-time neuro-sliding mode output feedback control. However, all these papers mostly focused on systems without time delays.

Yet, time delays are common in practical applications, and the existence of time delays is frequently a source of poor performance and instability. In this paper, we extend the idea of Young et al. (1999) and Su et al. (2000) from discrete-time systems to discrete time-delay systems. Based on Lyapunov stability theory and one-step delayed disturbance approximation (Young et al., 1999; Su et al., 2000), a discrete-time SMC scheme is developed for stabilizing a class of linear multi-input systems with state delays. The proposed method has the following attractive features: (1) the control design is rather straightforward and the stability of overall closed-loop time-delay systems is guaranteed without any state predictor. (2) the order of the motion equation in the quasi-sliding mode is equal to the order of the original system, rather than reduced by the number of dimension of the control input. The robustness of the system can be guaranteed throughout the entire response of the system starting from the initial time instance. (3) Chattering phenomenon will not occur. The switching control and the knowledge of upper bound of external disturbances are not required. (4) the proposed method can be easily extended to the case of multiple state delays.

The remainder of this paper is organized as follows. Section II briefly states problem formulation and assumptions. Section III provides the proposed discrete-time SMC scheme. The selection of sliding surface, the design of sliding mode controller, and the stability of the overall closed-loop system have been addressed. Section IV presents results from numerical simulations. Finally, a conclusion is provided in Section V.

## II. PROBLEM FORMULATION AND ASSUMPTIONS

Consider a class of linear continuous time-delay systems discretised at a small sampling time  $T > 0$ , and the resultant discrete-time systems are described by:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Phi_h x(k-h) + \Gamma u(k) + d(k) \\ x(k) &= x_o(k) \text{ for } k \in [-h \ 0] \end{aligned} \quad (1)$$

where  $x(k) \in R^n$  is the state,  $u(k) \in R^m$  is the control input,  $d(k) \in R^n$  is the bounded external disturbance.  $\Phi$ ,  $\Phi_h$  and  $\Gamma$  are constant matrices of appropriate dimensions, and  $h$  is a known positive integer for time delay. In this paper, the bounded external disturbance  $d(k)$  is assumed to satisfy the matched condition, i.e. there exists  $H(k) \in R^{m \times 1}$  such that  $d(k) = \Gamma H(k)$  with  $\|H(k)\| \leq \rho_H$ . To facilitate further development, the following definition and assumptions are made throughout this paper.

**Definition 1** The magnitude of a scalar  $v$  or a vector  $\mathbf{v}$  is said to be  $O(T^r)$  if and only if  $\lim_{T \rightarrow 0} \frac{v}{T^r} \neq 0$  and  $\lim_{T \rightarrow 0} \frac{v}{T^{r-1}} = 0$  for a scalar  $v$  or  $\lim_{T \rightarrow 0} \frac{\mathbf{v}}{T^r} \neq \mathbf{0}$  and  $\lim_{T \rightarrow 0} \frac{\mathbf{v}}{T^{r-1}} = \mathbf{0}$  for a vector  $\mathbf{v}$ , where  $r$  is an integer. Denote  $O(T^0) = O(1)$ .

**Assumption 1** The pair  $(\Phi, \Gamma)$  is controllable.

**Assumption 2** The sampling interval  $T$  is assumed to be sufficiently small such that the generalized disturbance  $d(k)$  does not vary too much between consecutive sampling instances. If the above assumption holds, one can obtain (Su et al., 2000):

$$d(k) - d(k-1) = O(T^2) \quad (2)$$

**Assumption 3** The sampling interval  $T$  is assumed to be sufficiently small such that  $\mathbf{v}_1, \mathbf{v}_1 \in O(T^r)$  and  $\mathbf{v}_2, \mathbf{v}_2 \in O(T^{r+1})$  gives  $\mathbf{v}_1 \gg \mathbf{v}_2$  or  $\|\mathbf{v}_1\| \gg \|\mathbf{v}_2\|$ . If the above assumption holds, the following relation exists about the effective approximation.

$$O(T^r) \cdot O(1) \approx O(T^r) \quad \forall r \in Z$$

where  $\approx$  stands for the effective approximation and  $Z$  is the set of integers.

The main objective of this paper is to design a discrete-time sliding mode controller such that discrete time-delay system (1) is stable. Also, it will be shown that the proposed scheme achieves a magnitude of the order  $O(T^2)$  for both the sliding

mode and state regulation.

**Remark 1** The sliding mode characteristics of discrete-time SMC systems are different from those of continuous-time SMC systems. It is noted that the motion of a discrete-time SMC system can approach the sliding surface but cannot stay on it in practice. Thus, only the quasi-sliding mode is ensured (Milosavljevic, 1985; Sarpurk et al., 1987; Gao et al., 1995).

## III. MAIN RESULTS

### 1. Sliding Surface Design

In this paper, we extend the concept of integral sliding functions for continuous-time SMC to discrete-time SMC. The integral sliding function is defined as

$$s(k) = Gx(k) - G \exp(-\eta k)x(0) - \xi(k), \quad \eta > 0 \quad (3.a)$$

$$\begin{aligned} \xi(k) &= \xi(k-1) + G(\Phi + \Gamma K)x(k-1) + G\Phi_h x(k-h-1), \\ \xi(0) &= 0 \end{aligned} \quad (3.b)$$

where  $G \in R^{m \times n}$  is chosen such that  $G\Gamma$  is invertible and  $K \in R^{m \times n}$  is synthesized later such that discrete time-delay system (1) in the quasi-sliding mode is stable. The exponential term  $G \exp(-\eta k)x(0)$  is used to eliminate the reaching phase, i.e.  $s(0) = 0$  means that the system is placed on the sliding surface initially no matter where  $x(0)$  is. In the following, the conditions are derived to evaluate the stability and robustness of discrete time-delay system (1) in the quasi-sliding mode.

### Theorem 1

The discrete time-delay system (1) with the sliding function (3) is stable in the quasi-sliding mode if there exist positive-definite symmetric matrices  $R \in R^{n \times n}$  and  $P \in R^{n \times n}$  such that the following inequality is satisfied

$$\Theta = \begin{bmatrix} \Phi_c^T P \Phi_c - P + R & \Phi_c^T P \Phi_h \\ \Phi_h^T P \Phi_c & \Phi_h^T P \Phi_h - R \end{bmatrix} < 0 \quad (4)$$

where  $\Phi_c = \Phi + \Gamma(G\Gamma)^{-1}G + \Gamma K$ .

### Proof:

Consider a forward expression of (3)

$$s(k+1) = Gx(k+1) - G \exp(-\eta(k+1))x(0) - \xi(k+1) \quad (5.a)$$

$$\xi(k+1) = \xi(k) + G(\Phi + \Gamma K)x(k) + G\Phi_h x(k-h) \quad (5.b)$$

Substituting (5.b) and (1) into (5.a), the equivalent control  $u_{eq}(k)$  can be found by solving for  $s(k+1) = 0$

$$u_{eq}(k) = Kx(k) + (G\Gamma)^{-1}G \exp(-\eta(k+1))x(0) \\ + (G\Gamma)^{-1}\xi(k) - (G\Gamma)^{-1}Gd(k) \quad (6)$$

where  $G$  is chosen such that  $G\Gamma$  is invertible.

From (3.a) and  $s(k) = 0$ , we have

$$\xi(k) = Gx(k) - G \exp(-\eta k)x(0) \quad (7)$$

Substituting (7) into (6), the equivalent control in (6) can be rewritten as

$$u_{eq}(k) = Kx(k) + (G\Gamma)^{-1}Gx(k) - (G\Gamma)^{-1}Gd(k) + L(k) \quad (8)$$

where  $L(k) = (G\Gamma)^{-1}G[\exp(-\eta(k+1)) - \exp(-\eta k)]x(0)$

Substituting (8) into (1), the dynamic equation of system (1) in the quasi-sliding mode can be obtained as

$$x(k+1) = \Phi_c x(k) + \Phi_h x(k-h) + \Gamma L(k) \quad (9)$$

where  $\Phi_c$  and  $L(k)$  are defined in (4) and (8), respectively.

It is noted that the exponential term  $\Gamma L(k)$  in (9) will decay to zero as  $k \rightarrow \infty$ . Hence, it will not affect the stability of the quasi-sliding mode dynamics (9). In order to examine the stability of the quasi-sliding mode dynamics (9) without the exponential term  $\Gamma L(k)$ , we choose the Lyapunov function candidate as

$$V(k) = x^T(k)Px(k) + \sum_{i=k-h}^{k-1} x^T(i)Rx(i) \quad (10)$$

which is positive definite. The corresponding Lyapunov difference along the trajectories of quasi-sliding mode dynamics (9) without the exponential term  $\Gamma L(k)$  is given by

$$\Delta V(k) = V(k+1) - V(k) \\ = x^T(k+1)Px(k+1) - x^T(k)Px(k) + x^T(k)Rx(k) \\ - x^T(k-h)Rx(k-h) \\ = \begin{bmatrix} x(k) \\ x(k-h) \end{bmatrix}^T \Theta \begin{bmatrix} x(k) \\ x(k-h) \end{bmatrix}$$

where  $\Theta$  is defined in (4). The requirement of negative-definiteness of  $\Delta V(k)$  for stability entails that  $\Theta < 0$  as required by (4). Therefore, the quasi-sliding mode dynamics of discrete time-delay system (1) is stable in the sense of Lyapunov stability. The proof is completed.

In the following, Theorem 1 will be formulated by LMI approach and the feedback controller gain  $K$  will be designed.

### Theorem 2

Consider the discrete time-delay system (1) with the sliding function (3). For a given matrix  $G$  such that  $G\Gamma$  is invertible, there exists a gain matrix  $K$  such that, under the integral sliding function (3), the discrete time-delay system (1) in the quasi-sliding mode is stable, if there exist a matrix  $W \in R^{m \times n}$  and symmetric positive-definite matrices  $X, M \in R^{n \times n}$  such that the following LMI is satisfied

$$\begin{bmatrix} -X & \tilde{\Phi}X + \Gamma W & \Phi_h M & 0 \\ X\tilde{\Phi}^T + W^T\Gamma^T & -X & 0 & X \\ M\Phi_h^T & 0 & -M & 0 \\ 0 & X & 0 & -M \end{bmatrix} < 0 \quad (11)$$

where  $\tilde{\Phi} = \Phi + \Gamma(G\Gamma)^{-1}G$ . Furthermore, the gain matrix  $K$  is given by

$$K = WX^{-1} \quad (12)$$

### Proof:

Matrix inequality (4) can be written as

$$\begin{bmatrix} \Phi_c^T \\ \Phi_h^T \end{bmatrix} P \begin{bmatrix} \Phi_c & \Phi_h \end{bmatrix} + \begin{bmatrix} -P + R & 0 \\ 0 & -R \end{bmatrix} < 0$$

Using the Schur complement (Boyd et al., 1994), the above inequality is equivalent to

$$\begin{bmatrix} -P^{-1} & \Phi_c & \Phi_h \\ \Phi_c^T & -P + R & 0 \\ \Phi_h^T & 0 & -R \end{bmatrix} < 0 \quad (13)$$

Pre- and post-multiplying both sides of (13) by  $\text{diag}(I, P^{-1}, R^{-1})$ , and letting  $X = P^{-1}$ ,  $W = KP^{-1}$ ,  $M = R^{-1}$ , it yields

$$\begin{bmatrix} -X & \tilde{\Phi}X + \Gamma W & \Phi_h M \\ X\tilde{\Phi}^T + W^T\Gamma^T & -X + XM^{-1}X & 0 \\ M\Phi_h^T & 0 & -M \end{bmatrix} < 0 \quad (14)$$

Again, using the Schur complement, it yields matrix inequality (11). Therefore, by Theorem 1, the existence of  $W, X$  and  $M$  satisfying (11) guarantees the quasi-sliding mode dynamics of discrete time-delay system (1) is stable. This proof is completed.

**Remark 2** Since equation (11) is a linear matrix inequality in matrices  $W, X$  and  $M$ , equation (11) defines a convex solution set of  $(W, X, M)$ , and therefore various efficient convex optimization algorithms can be used to test whether the LMI is solvable and to generate particular solution.

**2. Design of Discrete-Time Sliding Mode Controller**

After designing the sliding surface, the next phase is to design the control law such that quasi-sliding mode is reached and stayed thereafter. Ideally, the equivalent control (8) is a solution to the discrete-time SMC. However, under practical considerations, the equivalent control (8) can not be implemented because of the lack of prior knowledge regarding the generalized disturbance  $d(k)$ . To overcome this problem, one step delayed disturbance approximation (Young et al, 1999; Su et al., 2000) is applied under the Assumption 2, which implies  $d(k)$  can be estimated by its previous value  $d(k - 1)$ . Let

$$\hat{d}(k) = d(k - 1) = x(k) - \Phi_c x(k - 1) - \Phi_h x(k - h - 1) - \Gamma u(k - 1) \tag{15}$$

where  $\hat{d}(k)$  is the estimate of  $d(k)$ .

The control law for the discrete time-delay system (1) is proposed as follows:

$$u(k) = Kx(k) + (G\Gamma)^{-1} G \exp(-\eta(k + 1))x(0) + (G\Gamma)^{-1} \xi(k) - (G\Gamma)^{-1} G \hat{d}(k) \tag{16}$$

where the gain matrix  $K$  is given by (12).

**Theorem 3**

Consider the discrete time-delay system (1) with Assumptions 1-3. If the sliding function (3) and the proposed control law (16) are used, and there exist a matrix  $W \in R^{m \times n}$  and symmetric positive-definite matrices  $X, M \in R^{n \times n}$  such that LMI condition (11) is satisfied, then the control law (16) will drive the state to travel in the vicinity of the sliding surface at each sampling instant and guarantee the stability of overall closed-loop system. Furthermore,  $\lim_{k \rightarrow \infty} \|x(k)\| \leq O(T^2)$ .

**Proof:**

Using (5), (1), (15), (16) and Assumption 2, we have

$$s(k + 1) = G[d(k) - \hat{d}(k)] = G[d(k) - d(k - 1)] = O(T^2) \tag{17}$$

Hence, the control law (16) will drive the state to travel in the vicinity of the sliding surface at each sampling instant. Next, solving  $\xi(k)$  in (3.a) in terms of  $x(k)$  and  $s(k)$

$$\xi(k) = Gx(k) - G \exp(-\eta k)x(0) - s(k) \tag{18}$$

Substituting (16) and (18) into (1), the closed-loop dynamics becomes

$$x(k + 1) = \Phi_c x(k) + \Phi_h x(k - h) - \Gamma(G\Gamma)^{-1} s(k) + \Gamma L(k) + d(k) - d(k - 1) \tag{19}$$

where  $\Phi_c$  and  $L(k)$  are defined in (4) and (8), respectively.

Using (17) and Assumption 2, the closed-loop dynamics can be expressed

$$x(k + 1) = \Phi_c x(k) + \Phi_h x(k - h) + \delta(k) + \Gamma L(k) \tag{20}$$

where  $\delta(k) = [d(k) - d(k - 1)] - [d(k - 1) - d(k - 2)] = O(T^2)$ .

By introducing new state vectors,  $X(k) \in R^{n(h+1)}$  as  $X(k) = [x^T(k) \ x^T(k - 1) \ \dots \ x^T(k - h)]^T$  the closed-loop dynamics (20) without the exponential term  $\Gamma L(k)$  can be taken as the form

$$X(k + 1) = \Phi_{eq} X(k) + \Phi_\delta \delta(k) \tag{21}$$

where  $\Phi_{eq} = \begin{bmatrix} \Phi_c & 0 & \dots & 0 & \Phi_h \\ I_n & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \dots & 0 & 0 \\ \cdot & \cdot & \dots & 0 & 0 \\ 0 & 0 & \dots & I_n & 0 \end{bmatrix}$ ,  $\Phi_\delta = \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  and  $I_n$  is the

identity matrix of appropriate dimensions.

From Theorem 2, the existence of  $W, X$  and  $M$  satisfying (11) guarantees that the linear discrete time-delay system  $(\Phi_c, \Phi_h)$  is stable. Thus, all eigenvalues of the matrix  $\Phi_{eq}$  lie inside the unit circle of the  $z$ -plane, i.e.  $\|\lambda_j\| < 1 \ j = 1, \dots, n(h + 1)$ .

Using a similarity transformation matrix  $N$ , the matrix  $\Phi_{eq}$  can be expressed as  $\Phi_{eq} = N J N^{-1}$ , where the matrix  $J^{(n(h-1)) \times (n(h-1))}$  is the Jordan matrix of the eigenvalues of the matrix  $\Phi_{eq}$  and can be expressed as  $J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix}$  with

$J_1 \in R^{m \times m}$  being the Jordan matrix of  $m$  repeated eigenvalues and  $J_2^{(n(h-1)-m) \times (n(h-1)-m)}$  being a diagonal matrix of the distinct eigenvalues, i.e.  $J_2 = \text{diag}(\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_{n(h+1)})$ . Then, the solution of (21) can be obtained as

$$X(k) = N J^k N^{-1} X(0) + N \left[ \sum_{i=0}^{k-1} J^i N^{-1} \Phi_\delta \delta(k - i - 1) \right] \tag{22}$$

or

$$X(k) = N J^k N^{-1} X(0) + N \left[ \sum_{i=0}^{k-1} \begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} N^{-1} \Phi_\delta \delta(k - i - 1) \right] + N \left[ \sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i \end{bmatrix} N^{-1} \Phi_\delta \delta(k - i - 1) \right] \tag{23}$$

Let  $\lambda_m$  be the maximum norm of all distinct eigenvalues, i.e.  $\lambda_m = \max \{ \|\lambda_{m+1}\|, \|\lambda_{m+2}\|, \dots, \|\lambda_{n(h+1)}\| \}$  ( $\|\cdot\|$  indicates  $\|\cdot\|_2$ ) and  $\delta_m = \max \|\delta(i)\| \ i = 0, \dots, (k - 1)$ , respectively. Then, from (23)

$$\begin{aligned}
\lim_{k \rightarrow \infty} \|X(k)\| &\leq \lim_{k \rightarrow \infty} [ \|N\| ( \sum_{i=0}^{k-1} \|J_1^i\| \|N^{-1}\| \|\Phi_\delta\| \|\delta(k-i-1)\| ) \\
&\quad + \|N\| ( \sum_{i=0}^{k-1} \|J_2^i\| \|N^{-1}\| \|\Phi_\delta\| \|\delta(k-i-1)\| ) ] \\
&\leq \lim_{k \rightarrow \infty} [ \|N\| ( \sum_{i=0}^{k-1} \|J_1^i\| \|N^{-1}\| \delta_m ) \\
&\quad + \|N\| ( \sum_{i=0}^{k-1} \lambda_m^i \|N^{-1}\| \delta_m ) ] \quad (24)
\end{aligned}$$

Since  $\|\lambda_j\| < 1$   $j=1, \dots, n(h+1)$  for a stable system, it is easy to verify that

$$\sum_{i=0}^{\infty} \|J_1^i\| = \alpha_1 \sum_{i=0}^{\infty} \lambda_m^i = \frac{1}{1-\lambda_m} = \alpha_2 \quad (25)$$

where  $\alpha_1$  and  $\alpha_2$  are constants.

Substituting (25) into (24)

$$\lim_{k \rightarrow \infty} \|X(k)\| \leq \beta \delta_m \quad (26)$$

where  $\beta = \|N\|(\alpha_1 + \alpha_2) \|N^{-1}\|$  is a constant.

Since  $\lim_{k \rightarrow \infty} \|x(k)\| < \lim_{k \rightarrow \infty} \|X(k)\|$  and  $\delta_m \in O(T^2)$ , it can be derived from (26) and Assumption 3 that

$$\lim_{k \rightarrow \infty} \|x(k)\| \leq O(1) \cdot O(T^2) = O(T^2) \quad (27)$$

From (17), (27) and Theorem 2, we concluded that the control law (16) will drive the state to travel in the vicinity of the sliding surface at each sampling instant and guarantee the stability of overall closed-loop system. Also, it achieves a magnitude of the order  $O(T^2)$  for state regulation. The proof is completed.

**Remark 3** Based on one-step delayed disturbance approximation, the proposed SMC method does not only drive the sliding mode into  $O(T^2)$  boundary, but also achieve a magnitude of the order  $O(T^2)$  for state regulation without the switching control and the chattering. Hence, the proposed SMC method is better than the conventional one in the implementation and satisfactory performance.

The design procedures of the proposed discrete-time SMC for systems with state delays are summarized as follows:

Step 1 Choose the sliding surface matrix  $G$  such that  $GT$  is invertible.

Step 2 Calculate the gain matrix  $K$  in (12) by solving LMI (11) in Theorem 2.

Step 3 Estimate the generalized disturbance  $\hat{d}(k)$  by using (15).

Step 4 Construct the sliding mode controller (16) with (3.b).

#### IV. ILLUSTRATIVE EXAMPLE

Consider an uncertain time-delay system given by

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 1 & 0.1 \\ 0 & -1 & 1 \\ -1 & 0 & 0.1 \end{bmatrix} x(t-0.1) \\
&\quad + \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix} (u(t) + 0.2 \sin(t)) \quad (28)
\end{aligned}$$

The initial conditions for system states are set as  $x(t) = [0.1 \ -0.1 \ 0.1]^T$  for  $t \in [-0.1 \ 0]$ . The open-loop system (28) is unstable since the eigenvalues of system (28) are  $-0.4329$ ,  $0.7164 + 2.0266i$ , and  $0.7164 - 2.0266i$ , respectively. To illustrate the utilization of this approach, the system (28) are sampled with a sampling time  $T = 0.01$  second. Then, the discrete-time system of (28) for each sampling time can be obtained in (29) using the Matlab program function `c2d` (Grace, 1993).

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 1 & 0.01 & 0.0001 \\ -0.0001 & 1.0098 & 0.0201 \\ -0.01 & -0.0201 & 0.9998 \end{bmatrix} x(k) \\
&\quad + \begin{bmatrix} 0.2 & 1 & 0.1 \\ 0 & -1 & 1 \\ -1 & 0 & 0.1 \end{bmatrix} x(k-10) \\
&\quad + \begin{bmatrix} 0 \\ 0.001 \\ 0.01 \end{bmatrix} (u(k) + 0.2 \sin(k)) \quad (29)
\end{aligned}$$

where  $H(k) = 0.2 \sin(k)$ .

Following the design procedures in the above section, the sliding mode controller is given by the following steps. Step 1: the sliding surface matrix is chosen as  $G = [1 \ 1 \ 1]$  such that  $GT = 0.011$  is invertible. Step 2: Solve LMI (11) using the LMI toolbox in Matlab (Gahinet et al., 1995). All solutions are obtained at a time as follows:

$$\begin{aligned}
X &= \begin{bmatrix} 2.3593 & -1.7425 & 0.6369 \\ -1.7425 & 5.4908 & 13.4426 \\ 0.6369 & 13.4426 & 300.006 \end{bmatrix}, \\
W &= [-104 \ -4040 \ -54936],
\end{aligned}$$

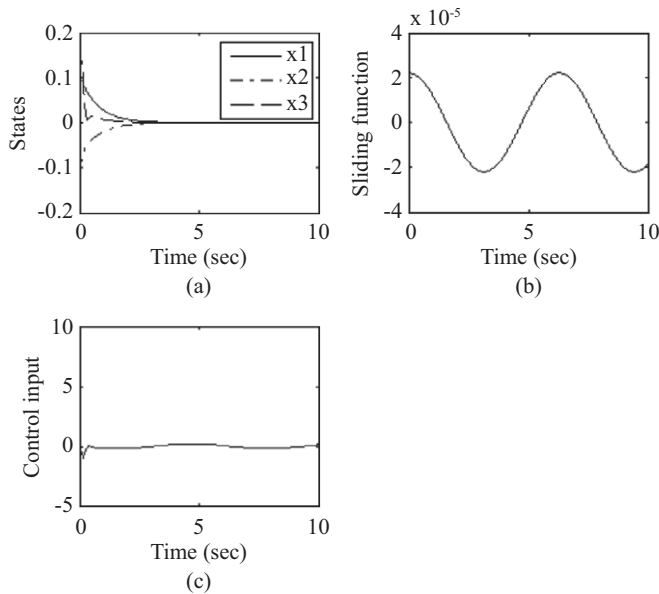


Fig. 1. (a) System states (b) sliding function (c) control input.

$$M = \begin{bmatrix} 508.5606 & -161.4357 & 79.8602 \\ -161.4357 & 215.3507 & -112.6309 \\ 79.8602 & -112.6309 & 772.6903 \end{bmatrix} \quad (30)$$

Then, the state feedback gain matrix from (12) can be obtained as  $K = [-347.0055 \quad -469.0987 \quad -161.3603]$ . Steps 3-4: Using (15) and (16), the controller is designed as

$$\begin{aligned} u(k) = & [-347.0055 \quad -469.0987 \quad -161.3603]x(k) \\ & + [90.9091 \quad 90.9091 \quad 90.9091]\exp(-2(k+1))x(0) \\ & + 90.9091\xi(k) - [90.9091 \quad 90.9091 \quad 90.9091]\hat{d}(k) \end{aligned} \quad (31)$$

where  $\eta = 2$ .

With the designed parameter setting and initial condition  $x(k) = [0.1 \quad -0.1 \quad 0.1]^T$  for  $k \in [-10 \quad 0]$ , the closed-loop dynamic responses of simulation are shown in Fig. 1. Fig. 1(a) shows the trajectories of system states. It is clearly shown that the system states approach to zero. Fig. 1(b) shows the sliding function. It clearly shows that the reaching phase is eliminated due to the novel integral sliding surface (3) being introduced. Fig. 1(c) shows the control input. Since the proposed method needs not a switching type of control law, it can be seen that no chattering phenomenon would occur. Note that the controllers proposed in the literature (Sarturk et al., 1987; Furuta, 1990; Gao et al., 1995; Young et al., 1999; Park, 2000; Su et al., 2000; Abidi et al., 2007; Pai, 2008, 2009, 2012) can be used for the plant without time-delays. Therefore, the controllers proposed in these papers cannot be used directly for this case either.

## V. CONCLUSION

In this paper, a discrete-time sliding mode controller has been successfully proposed for a class of multi-input systems with state delays. The advantage of the proposed method is that the design technique is simple and computationally efficient, and stability of closed-loop system is guaranteed without any state predictor. Furthermore, the proposed method achieves accurate control performance for both sliding mode and state regulation, meanwhile eliminates the reaching phase and avoids occurring chattering phenomenon. The simulation results verify the theoretical analysis and show that the proposed method effectively controls linear multi-input systems with state delays.

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