



## SOLVING THE INVERSE CAUCHY PROBLEM OF THE LAPLACE EQUATION USING THE METHOD OF FUNDAMENTAL SOLUTIONS AND THE EXPONENTIALLY CONVERGENT SCALAR HOMOTOPY ALGORITHM (ECSHA)

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# SOLVING THE INVERSE CAUCHY PROBLEM OF THE LAPLACE EQUATION USING THE METHOD OF FUNDAMENTAL SOLUTIONS AND THE EXPONENTIALLY CONVERGENT SCALAR HOMOTOPY ALGORITHM (ECSHA)

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**Key words:** inverse Cauchy problem, method of fundamental solutions, exponentially convergent scalar homotopy algorithm (ECSHA), ill-posed.

## ABSTRACT

In this paper, the inverse Cauchy problem of the Laplace equation is considered. Using the method of fundamental solutions, a system of linear algebraic equations can be obtained by satisfying the Cauchy boundary conditions on the overprescribe boundary points. The resulting linear algebraic equation is ill-posed and is treated by the exponentially convergent scalar homotopy algorithm (ECSHA). Four examples are adopted to show the validity of the proposed numerical scheme and it is concluded that the current approach can successfully resolve the ill-posedness of the inverse Cauchy problem even when the noise exists.

## I. INTRODUCTION

The inverse Cauchy problem is a very important problem for the non-destructive testing. Unlike the standard boundary value problem, the inverse Cauchy problem has both the Dirichlet and Neumann boundary conditions on part of the boundary and has no information on the remaining boundary. It is well known that the inverse Cauchy problem is an ill-posed problem in nature. It means that a small disturbance in given data may result in great errors in solution. Since in the realistic cases boundary data come from measurements which contain noise, it then becomes very important to find a stable and accurate numerical algorithm to solve the resulting ill-

posed algebraic equations.

However, Many research works have been conducted to seek for a stable and efficient method to deal with the ill-posed behaviors of Cauchy problem. We cannot list all of them, and the following mentioned methods are some well-known and popular methods. The most popular one is the Tikhonov's regularization method (Tikhonov and Arsenin, 1977), which transforms the original incorrectly posed problem into a correctly posed problem by a minimization of the  $L_2$ -norm of the solution subjected to the constraint equations. To determine the optimal regularization parameter, Hansen (1992) proposed the so-called L-curve concept, in which the optimal parameter is to seek for the best balance between the distortion of the original equations and the norm of solution. Another well known method is the truncated singular value decomposition method (Chang et al., 2001). This method discards some small singular values below the threshold, such that the amplification of error of data for these small singular values would not appear. Other methods have been proposed to regularize the Cauchy problem, for example, an energy-minimizing approach (Andrieux et al., 2006), methods using quasi-reversibility (Klibanov and Santosa, 1991; Bourgeois, 2005), and methods of alternating Dirichlet and Neumann problems, with regularizing properties (Kozlov et al., 1992; Belgacem and Fekih, 2005). For more references, the following listed papers are some related inverse problems appeared recently (Koya et al., 1993; Yeih et al., 1993; Hào and Lesnic, 2000; Leitao, 2000; Berntsson and Eldén, 2001; Cheng et al., 2001; Engl and Leitao, 2001; Hon and Wei, 2001; Aliev and Hosseini, 2002; Marin et al., 2002; Marin and Lesnic, 2002; Li, 2004; Cheng and Cabral, 2005; Mera et al., 2006; Ling and Takeuchi, 2008; Liu, 2008a, 2008b).

The method of fundamental solutions (MFS) has been developed for a long time. For readers who are interested in the development of MFS, the following two classic papers are suggested references (Mathon and Johnston, 1977; Bogomolny, 1985). A review paper on the application of MFS on

elliptic boundary value problem is found in Fairweather and Karageorghis (1998). Alves and Chen (2005) applied the MFS to the nonhomogeneous elliptic problems. Poullikkas et al. (1998) applied MFS for both harmonic and biharmonic problems. The MFS although has its advantage on collocating boundary conditions only, it has been reported that MFS may be sensitive to source points since it may result in an ill-conditioned matrix (Ramachandran, 2002). Golberg (1995) used the MFS to solve the Poisson equation. Fairweather et al. (2003) applied the MFS to solve the scattering and radiation problems.

Marin (2005a) has applied MFS to solve the inverse Cauchy problem of a Helmholtz equation and the Tikhonov's regularization technique and the L-curve concept. Marin (2005b) also used MFS to solve the inverse Cauchy problem of the elastostatics. Chen et al. (2005) used MFS to solve the inverse 2D Stokes flow problem with overprescribed boundary conditions on part of boundary and underdetermined boundary conditions on the remaining. Young et al. (2008) have adopted MFS to solve the inverse Cauchy problem of Laplace equation and they found that for some cases this method does not require regularization or iteration at all. Marin and Lesnic (2005) has applied MFS to solve the inverse Cauchy problem associated with two-dimensional Helmholtz-type equation and in that paper the Tikhonov's regularization and L-curve concept were used. Hon and Wei (2004) used the MFS to solve the inverse heat conduction problem in which they applied the Tikhonov's regularization and L-curve concept. Jin et al. (2006) used MFS to solve the inverse boundary value problems associated with the steady-state heat conduction with anisotropic media. Marin and Lesnic (2004) used MFS to solve the inverse Cauchy problem in linear elasticity. Wei et al. (2007) used the MFS with regularization techniques to resolve the Cauchy problems of elliptic operators. Among these papers which used MFS to solve the inverse problems, most of them adopted regularization technique to tackle with the ill-posed problem arising in the linear algebraic equations. In this paper, we will use a newly-developed exponentially convergent scalar homotopy algorithm (ECSHA) to tackle with the ill-posed problem. In such a technique, no need of regularization parameters is required and thus it does not distort the original equation at all. The remaining parts of this paper contain the following arrangements. In section 2 we will give a brief introduction of MFS, inverse Cauchy problem and the ECSHA. In the section 3, four numerical examples are shown to check the validity of the proposed method. Final section contains conclusions.

## II. MATHEMATICAL FORMULATIONS

The governing equation is the Laplace equation:

$$\nabla^2 u = 0 \quad (1)$$

where  $u$  denotes the potential and  $\nabla^2$  is the Laplacian operator.

There are many numerical methods to solve the Laplace equation. Here, we adopt the MFS. The MFS assumes the solution can be written as the linear combination of fundamental solutions measuring from various source points as:

$$u(\mathbf{x}_i) = \sum_j C_j \log r_{ij}, \quad r_{ij} \equiv \sqrt{(\mathbf{x}_i - \mathbf{s}_j) \cdot (\mathbf{x}_i - \mathbf{s}_j)} \quad (2)$$

where  $\mathbf{x}_i$  is the observation point,  $\mathbf{s}_j$  is the source point and  $C_j$  is the source density. Notice that the fundamental solution becomes singular when the observation point coincides with the source point; therefore, we arrange the source points outside the domain of interest. In addition, we should stress that for a multiply connected domain with genus  $p$  we need to arrange source inside each hole. Therefore, the source points usually are arranged on an artificial source point surface which is outside the domain but similar shape as the physical boundary.

Since the fundamental solution already satisfies the governing equation, we only need to satisfy the boundary conditions on the boundary collocation points. There are several types of boundary conditions listed as:

$$u = \bar{f} \quad (\text{Dirichlet boundary condition, 1st kind B.C.}), \quad (3)$$

$$\frac{\partial u}{\partial n} = \bar{g} \quad (\text{Neumann boundary condition, 2nd kind B.C.}), \quad (4)$$

$$\alpha u + \beta \frac{\partial u}{\partial n} = \bar{h} \quad (\text{mixed type boundary condition, 3rd kind B.C.}), \quad (5)$$

where  $\alpha$  and  $\beta$  are constants,  $n$  denotes the outward normal direction and  $\bar{f}$ ,  $\bar{g}$  and  $\bar{h}$  are known functions. To understand the influence of noise, we assign the absolute random error as:

$$b = \bar{b} + sn \times \text{random} \quad (6)$$

where  $b$  is the noisy data,  $\bar{b}$  is the noise-free data,  $sn$  denotes the noise level and  $\text{random}$  is a random number in the range of  $[-1, 1]$ .

The standard boundary value problem is defined as: for each boundary point only one kind of boundary condition is given. The inverse Cauchy problem is different. For the inverse Cauchy problem, on part of the boundary both the potential and the normal derivative of the potential is known while on the remaining part of boundary no information is given. Unlike the standard boundary value problem, the inverse Cauchy problem encounters great difficulty in numerical calculation because it is an ill-posed problem. More specifically speaking, a small disturbance in data may result in great deviation in solution. This is called the numerical instability,

or the solution is not continuously dependent on data in Hadamard sense. To deal with this problem, the conventional approach try to formulate a well-posed problem from the original ill-posed one by slightly distorting the original system (the Tikhonov's regularization) or design a filter to discard some sensitive singular values (the truncated singular value decomposition method). These methods all require us to determine some regularization parameter and the process becomes very complex. In addition, for some extreme cases the above-mentioned methods even cannot obtain acceptable results. Recently, the fictitious time integration method has been used to deal with the inverse Cauchy problem successfully (Chi et al., 2009) and it has a better noise resistance than the conventional Tikhonov's regularization method. In a recent paper by Liu and Atluri (2009), they used the filter theory to explain why the fictitious time integration method is a better filter than the conventional Tikhonov's regularization. The fictitious time integration method although shows its successfulness in many difficult problems, it has a main drawback that it requires the number of equation should be equal to the number of unknown. To relax this restriction, a scalar formulation of the fictitious time integration method has been developed and named as the manifold-based exponentially convergent algorithm (MBECA) in a previous researches (Ku et al., 2010). Actually, the MBECA can be viewed as an exponentially convergent scalar homotopy algorithm by considering a change of variables. Therefore, we give a new name for this method as the exponentially convergent scalar homotopy algorithm (ECSHA). In the followings, the ECSHA (or MBECA) will be briefly reviewed. The ECSHA has been used for solving the nonlinear plate problem (Dai et al., 2011) and other problems (Chan et al., 2011; Fan and Chan, 2011; Fan et al., 2012; Fan et al., 2013) involving ill-posed systems.

Assuming there are  $m$ -algebraic equations in  $n$  variables to be solved as:

$$\mathbf{F}(\mathbf{x}) = 0 \tag{7}$$

where  $\mathbf{F} \in R^m$  and  $\mathbf{x} \in R^n$ .

Consider a space-time manifold as:

$$h(\mathbf{x}, t) := \frac{1}{2}Q(t)\|\mathbf{F}\|^2 = C$$

where  $Q(t) > 0$  and is a monotonically increasing function with  $Q(0) = 1$  and  $C$  can be determined from the initial guess  $\mathbf{x}_0$  as  $C = \frac{1}{2}\|\mathbf{F}(\mathbf{x}_0)\|^2$ . By requiring that the trajectory of  $\mathbf{x}$  remains on the manifold and assign appropriate function for  $Q(t)$  we can obtain the revolution ODE for  $\mathbf{x}$  as:

$$\dot{\mathbf{x}} = \frac{-\nu}{2(1+t)^\gamma} \frac{\|\mathbf{F}\|^2}{\|\mathbf{B}^T \mathbf{F}\|} \mathbf{B}^T \mathbf{F} \tag{8}$$

where  $\nu$  and  $\gamma$  are parameters that control the convergence speed (Ku et al., 2009),  $\mathbf{B}$  is the Jacobian matrix with its components as  $B_{ij} = \frac{\partial F_i}{\partial x_j}$ . If one takes a look at Eq. (8), it

seems that this equation is a scalar formulation of the fictitious time integration method. If one introduces a variable  $\tau$  such that when  $t = 0$  then  $\tau = 0$  and when  $t \rightarrow \infty$  then  $\tau = 1$ , then equation can be rewritten and the new formulation for  $\mathbf{x}$  and  $\tau$  is a evolution equation derived from the scalar homotopy method (Liu et al., 2009).

It is noticed here the direction of evolution in Eq. (8) is  $\mathbf{B}^T \mathbf{F}$  which is the same as that used in the Landsweber iteration method. If one takes the time step in Eq. (8) as 1 and discarding the modification factor in front of  $\mathbf{B}^T \mathbf{F}$ , that is

$$\frac{\nu}{2(1+t)^\gamma} \frac{\|\mathbf{F}\|^2}{\|\mathbf{B}^T \mathbf{F}\|},$$

then Eq. (8) is the same as the Landsweber iteration method. The modification factor really can help convergence and this declaration can be verified in the next section.

In addition, the formula in Eq. (8) is easily implemented for parallel computation in large-scale calculation. Moreover, the method proposed here although is used for an ill-posed linear algebraic system in this article it can be applied to an ill-posed nonlinear algebraic system as well.

We can use Eq. (8) and employ numerical integration technique to obtain numerical solution. In this paper, a constant time step size Euler forward integration scheme is used. It should be mentioned here that one needs to use adaptive time step size if we want the trajectory of  $\mathbf{x}$  always on the manifold. However, we seldom use adaptive time step size in the real calculation. If the time step size exceeds the limit, it results in the trajectory of  $\mathbf{x}$  deviating from the manifold and we will not have decreasing  $\|\mathbf{F}\|^2$  as we expect but have a sharp jump in  $\|\mathbf{F}\|^2$ . Nevertheless, the trajectory will still go back and locate on the manifold after several time steps since the constraint of manifold forces such a motion. In reality, we may see a fluctuating curve of  $\|\mathbf{F}\|^2$  versus the time (or number of time steps) but the trend of  $\|\mathbf{F}\|^2$  keeps decreasing as shown in Fig. 1. Therefore, we have to record the relatively smallest value of  $\|\mathbf{F}\|^2$  and its corresponding time. The relatively smallest value of  $\|\mathbf{F}\|^2$  is defined as the local minimum of  $\|\mathbf{F}\|^2$ .

The following convergence criterions are used, when any of them is satisfied the procedure terminates:

- (1) when the root mean square error (RMSE), defined as  $RMSE := \sqrt{\frac{\|\mathbf{F}\|^2}{m}}$  is small enough, i.e.,  $RMSE \leq err1$  where  $err1$  is the assigned tolerance;
- (2) when the number of iteration exceeds the maximum iteration number (maxiter);

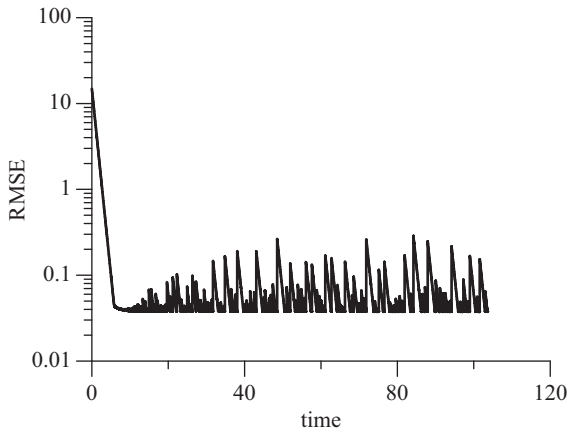


Fig. 1. A typical plot of RMSE versus fictitious time using ECSHA.

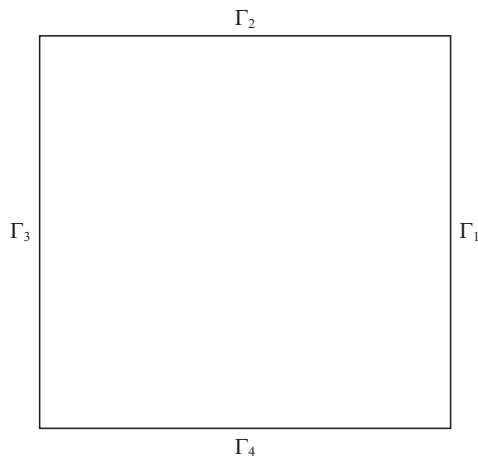


Fig. 2. A square domain considered in Example 2.

(3) when the slope of RMSE is small enough, i.e.,

$$\frac{\left\| \log_{10}(RMSE_{new}) - \log_{10}(RMSE_{old}) \right\|}{t_{new} - t_{old}} \leq err2$$

where  $RMSE_{new}$  is the newest relatively minimum RMSE value with its corresponding time denoted as  $t_{new}$ ,  $RMSE_{old}$  is the previous relatively minimum RMSE value with its corresponding time denoted as  $t_{old}$  and  $err2$  is another user defined tolerance value.

### III. NUMERICAL EXAMPLES

#### 1. Example 1

In this example, a simple square region with edge length equal to 1.0 is considered as shown in Fig. 2. The designated analytical solution is given as

$$u(x, y) = 2xy.$$

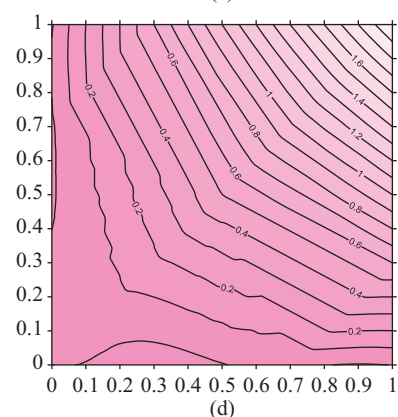
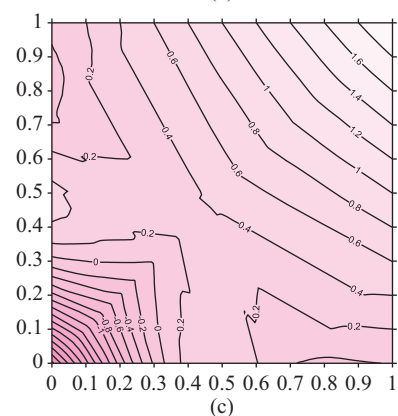
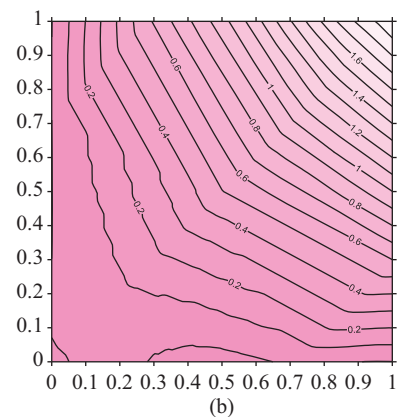
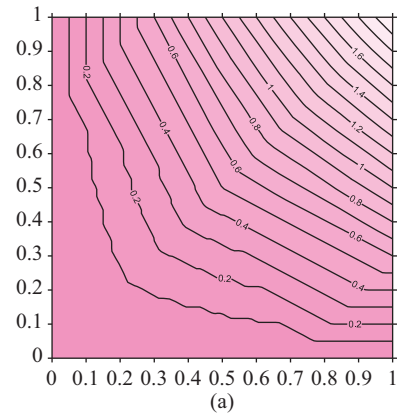


Fig. 3. (a) Exact solution; (b) Numerical solution of case 1; (c) Numerical solution of case 2; (d) Numerical solution of case 2 with aid by adding Cauchy data at (0,0).

We have 100 collocation points on the boundary and 60 source points on a fictitious surface of source, which is also a square with edge length equal to 1.2. Inside the domain, we place 400 interior points to represent the solution. We assign half of the boundary has the Cauchy boundary data and half of the boundary has no information. Two cases are studied, the first case we give Cauchy boundary data on the sub-boundaries  $\Gamma_1$  and  $\Gamma_3$  and for the second case we give Cauchy data on the sub-boundaries  $\Gamma_1$  and  $\Gamma_2$ . The following parameters for ECSHA are used:  $\nu = 10.0$ ,  $\gamma = 0.001$ ,  $\Delta t = 0.001$ ,  $\text{maxiter} = 1000000$ ,  $\text{err1} = 1.0 \times 10^{-3}$ ,  $\text{err2} = 1.0 \times 10^{-7}$ . The initial guess is that all the source strengths are set to be 1.

For case 1, the inversion is shown in Fig. 3(b) and it is acceptably close to the analytical solution in Fig. 3(a). However, for case 2 the inversion is shown in Fig. 3(c). It can be seen that only in the region for  $x + y > 1$  the contour lines of solution is close to the analytical one. For the region  $x + y < 1$ , the solution deviates from the analytical solution and is not acceptable. It shows that how Cauchy data is given influence the result. And similar to previous research the inversion becomes better when the Cauchy data is prescribed more diversely (Chang et al., 2001; Wei et al., 2007). Since we do not have good result in case 2, we add one point prescribing Cauchy data to the system and (0,0) is selected. After adding additional point with Cauchy data, the result is much better as shown in Fig. 3(d). In reality, the Cauchy problem usually relates to the non-destructive test. From this example, we can claim that if one can arrange the Cauchy data point more diversely a better result can be obtained.

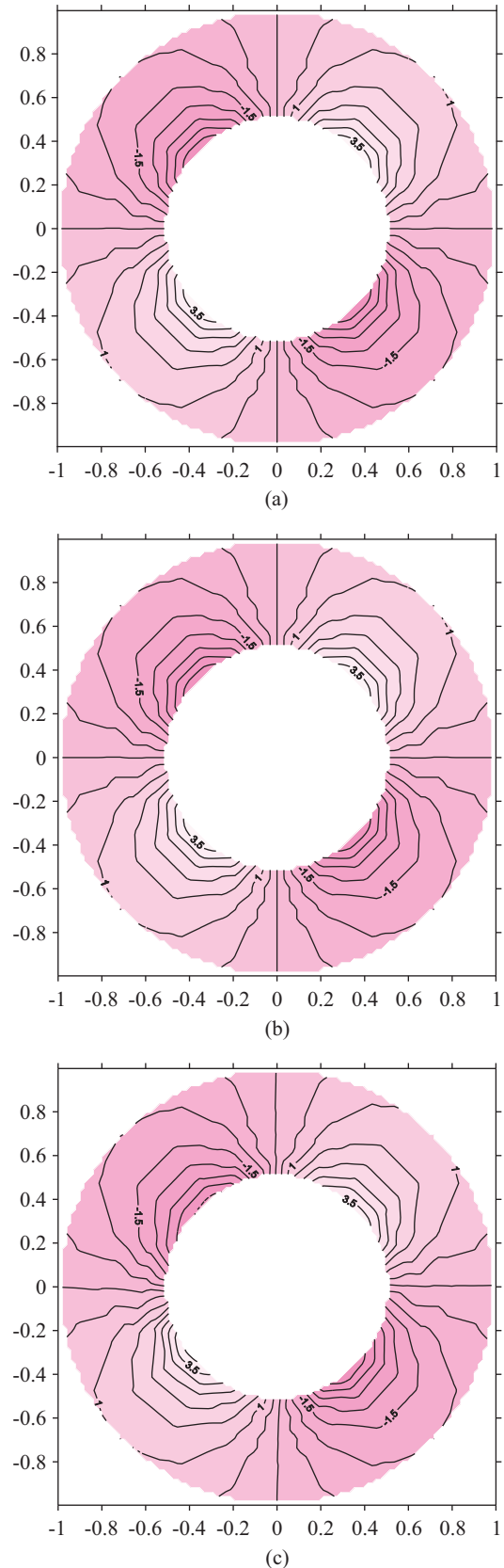
**2. Example 2**

In this example, an annular region with outer radius equal to 1 and inner radius equal to 0.5 is considered. The designated analytical solution is given as

$$u(r, \theta) = \frac{1}{r^2} \sin 2\theta$$

where  $(r, \theta)$  is the polar coordinates measuring from the origin. On the outer and inner radius, we both place 50 collocation points. To place the source points of MFS, we select two circles having radius 1.1 and 0.3 and 30 source points are arranged on each circle. To represent the solution, except the boundary points we place totally 80 interior points inside the domain by equally dividing the distance in radial direction with 4 layers and 20 points equally distributes in the  $\theta$ -direction. The following parameters for ECSHA are used:  $\nu = 10.0$ ,  $\gamma = 0.001$ ,  $\Delta t = 0.001$ ,  $\text{maxiter} = 1000000$ ,  $\text{err1} = 1.0 \times 10^{-3}$ ,  $\text{err2} = 1.0 \times 10^{-7}$ . The initial guess is that all the source strengths are set to be 1.

We first consider the Cauchy problem with Cauchy data on the outer radius and no information on the inner radius. The analytical result, numerical result for noise level  $sn = 0.0$  and numerical result with noise level  $sn = 0.1$  are illustrated in Fig. 4. It can be seen that even under absolute random error



**Fig. 4. Inverse Cauchy problem of an annular region for prescribing Cauchy data on the outer radius: (a) exact solution; (b) numerical solution with  $sn = 0.0$ ; (c) numerical solution with  $sn = 0.1$ .**

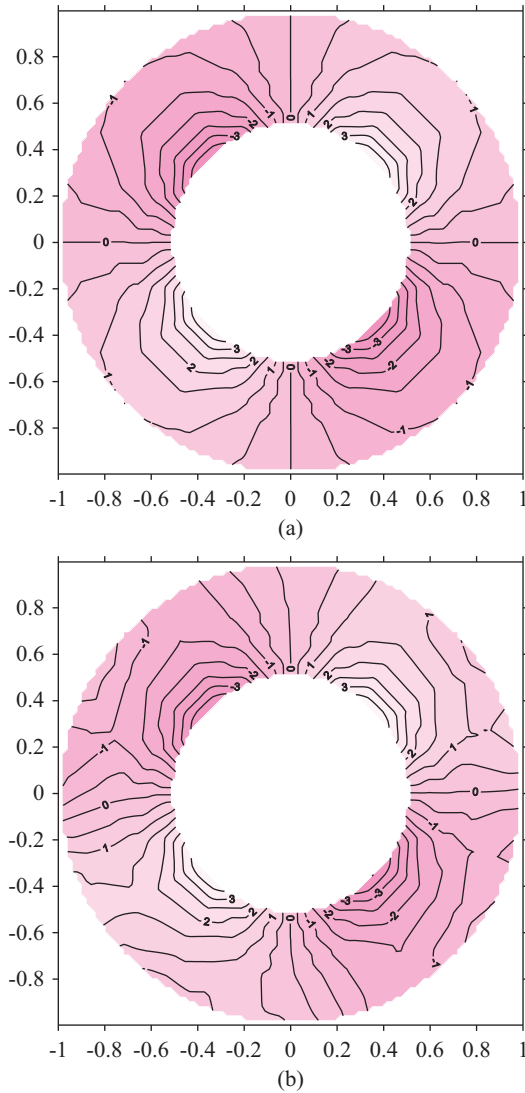


Fig. 5. Inverse Cauchy problem of an annular region for prescribing Cauchy data on the inner radius: (a) numerical solution with  $sn = 0.0$ ; (b) numerical solution with  $sn = 0.1$ .

of 0.1, the current approach still can obtain good results. In addition, we observe that for the case of no noise totally 1619 steps are used and for the case of noise level of  $sn = 0.1$  totally 145210 steps are used. They all terminate the process because the slope of RMSE tends to zero.

Next, we consider the Cauchy problem with Cauchy data given on the inner radius and no information on the outer radius. The numerical result for noise level  $sn = 0.0$  and numerical result for noise level  $sn = 0.1$  are shown in Fig. 5. Comparing with the previous case, one can find out that to do the inverse Cauchy problem for the Cauchy data on the inner radius is more difficult than that for the Cauchy data on the outer radius under the same noise level in this example. For noise level  $sn = 0.1$ , the basic shape of the solution contour is recovered but with a little distortion. However, for an ill-posed inverse problem like this case such result is acceptable.

In addition, it is observed that for the case of no noise only 1985 steps are required to achieve the convergence but for the case of noise level of 0.1, totally 93886 steps are required.

### 3. Example 3

In this example, a peanut shape domain with its boundary curve written as:

$$r(\theta) = 0.3\sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}} \quad 0 \leq \theta \leq 2\pi$$

where  $(r, \theta)$  is the common polar coordinate system.

The source points are placed on a enlarged peanut shape with its curve written as:

$$r(\theta) = 0.36\sqrt{\cos 2\theta + \sqrt{1.1 - \sin^2 2\theta}} \quad 0 \leq \theta \leq 2\pi$$

We have two cases studied in this example: for the first case we assign Cauchy boundary data on  $0 \leq \theta \leq \pi$  and for the second case we assign Cauchy boundary data on  $0 \leq \theta \leq \frac{3\pi}{2}$ .

The designated analytical solution is written as

$$u(x, y) = e^x \cos y.$$

On the boundary, we arrange 120 points and distances between any two adjacent points are equal. On the fictitious surface of source point, we arrange 100 source points. In the domain, we uniformly distribute 58 interior points. The following parameters for ECSHA are used:  $\nu = 1.0$ ,  $\gamma = 0.001$ ,  $\Delta t = 0.001$ ,  $\text{maxiter} = 1000000$ ,  $\text{err1} = 1.0 \times 10^{-3}$ ,  $\text{err2} = 1.0 \times 10^{-7}$ . The initial guess is that all the source strengths are set to be 1.

For each case, we study the case of no noise in boundary data and that of absolute random error of the level  $sn = 0.05$ . The analytical result, numerical result of Cauchy data on 1/2 boundary with no noise in data, numerical result of Cauchy data on 1/2 boundary with noise level  $sn = 0.05$  in data, numerical result of Cauchy data on 3/4 boundary with no noise in data and numerical result of Cauchy data on 3/4 boundary with noise level  $sn = 0.05$  in data are illustrated in Fig. 6(a) to 6(e), respectively. It can be found that more portion of the boundary has given Cauchy data more accurate result we can obtain. For 1/2 boundary with Cauchy data, the result in the region  $x > 0$  is better while the result in the region  $x < 0$  is worse. It is because that the potential value in the region  $x > 0$  is bigger and under the same error level the deviation of contour lines will not so apparent as that for the region with smaller value potential. In Table 1, we also tabulated the required iteration steps for all cases. We should mention here for the noisy case it is not necessary to have more steps to convergent.

### 4. Example 4

In the last example, in stead of giving the Cauchy data derived from an analytical solution we will give the Cauchy data

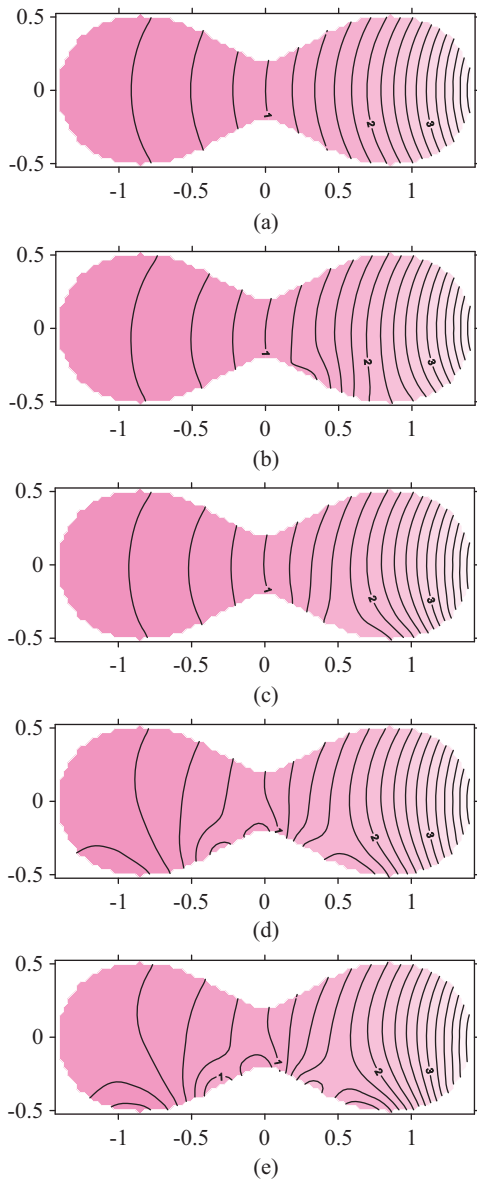


Fig. 6. Inverse Cauchy problem for a peanut shape: (a) exact solution; (b) numerical solution with Cauchy data on 3/4 boundary and  $sn = 0.0$ ; (c) numerical solution with Cauchy data on 3/4 boundary and  $sn = 0.05$ ; (d) numerical solution with Cauchy data on 1/2 boundary and  $sn = 0.0$ ; (e) numerical solution with Cauchy data on 1/2 boundary and  $sn = 0.05$ .

from a forward standard boundary value problem. The interested domain is a circular region with its radius equal to 2. The source points are arranged on a fictitious circular boundary with a radius of 2.1. The forward problem is designed to solve the following boundary value problem:

$$u(r = 2.0, \theta) = 10, 0 \leq \theta < \frac{\pi}{2}$$

$$\frac{\partial u}{\partial n}(r = 2.0, \theta) = 5, \frac{\pi}{2} \leq \theta < \pi$$

$$u(r = 2.0, \theta) = 5, \pi \leq \theta < \frac{3\pi}{2}$$

$$\frac{\partial u}{\partial n}(r = 2.0, \theta) = 0, \frac{3\pi}{2} \leq \theta < 2\pi$$

Table 1. Iteration steps for Example 3.

		Number of Steps
1/2 boundary with Cauchy data	$sn = 0.0$	70137
	$sn = 0.05$	82907
3/4 boundary with Cauchy data	$sn = 0.0$	103662
	$sn = 0.05$	63738

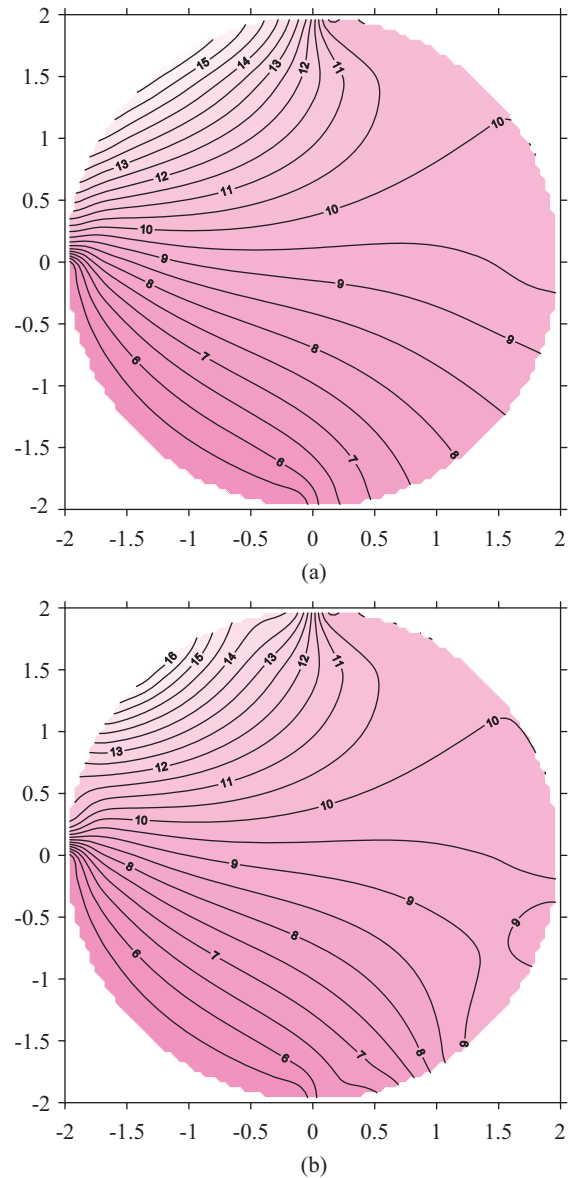


Fig. 7. (a) Numerical solution for a forward boundary value problem; (b) numerical solution for the inverse Cauchy boundary value problem.



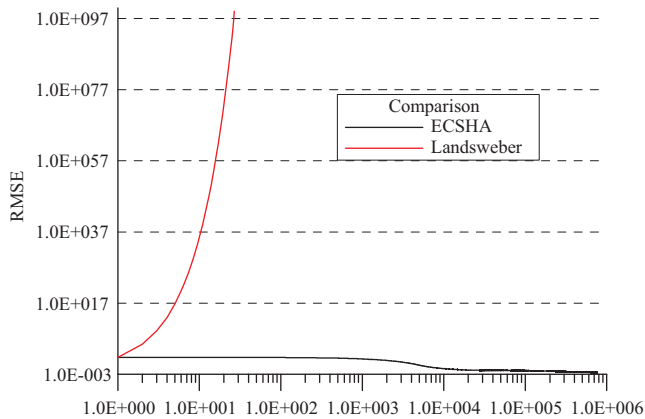


Fig. 8. Comparison of the performances between the proposed method (ECSHA) and Landsweber iteration.

The numerical result for the forward problem using MFS is plotted in Fig. 7(a). After we solve the forward problem, we assign numerical Cauchy data on the boundary:  $r = 2, 0 \leq \theta < \frac{\pi}{2}$  and  $r = 2, \pi \leq \theta < \frac{3\pi}{2}$ . We assign 120 boundary points, 120 source points and 179 interior nodes. The parameters used for ECSHA are:  $\nu = 1.0, \gamma = 0.001, \Delta t = 0.001, \text{maxiter} = 1000000, \text{err1} = 1.0 \times 10^{-3}, \text{err2} = 1.0 \times 10^{-7}$ . The initial guess is that all the source strengths are set to be 1.

The numerical result of this inverse Cauchy problem is plotted in Fig. 7(b). Comparing these two figures, one can find out that the current approach can obtain acceptable results. The errors in contours appear in the area with smaller values and the reason is given in the previous examples.

Next, we will compare the performances of the proposed method with Landsweber iteration method and the Tikhonov's regularization method. In Fig. 8, the RMSEs for our proposed method (ECSHA) and the Landsweber iteration method are plotted. It can be found that although the directions of the evolution for both methods are the same, the RMSE obtained from Landsweber iteration method increases as the number of iteration steps increases while the RMSE obtained from ECSHA decreases gradually. It can be concluded that the factor before the evolution direction in ECSHA, which is

$$\frac{\nu}{2(1+t)^\gamma} \frac{\|\mathbf{F}\|^2}{\|\mathbf{B}^T \mathbf{F}\|}, \text{ helps the convergence.}$$

To compare with the Tikhonov's regularization method, the best regularization parameter is determined from the L-curve concept [22]. The best regularization parameter is  $1.0\text{e-}5$ . Substitute this parameter into the regularized equation obtained from the Tikhonov's regularization method, the numerical solution then can be obtained. The result is shown in Fig. 9, and one can find that the numerical solution obtained from the Tikhonov's regularization method is far from the solution obtained from the forward problem (see Fig. 7(a)). Therefore, among three compared methods, the method proposed here is the best method.

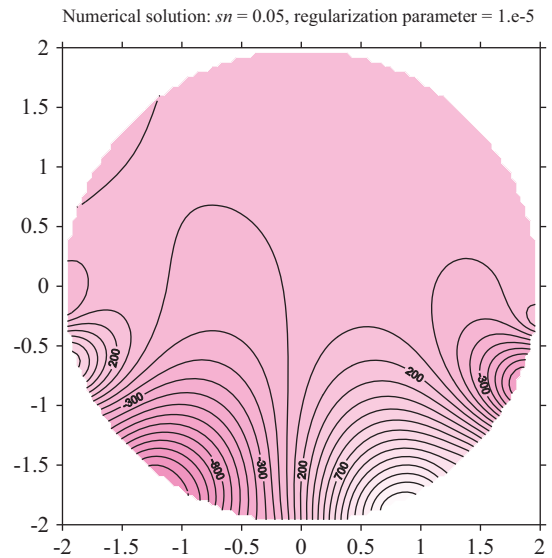


Fig. 9. Numerical solution obtained from the Tikhonov's regularization method with the noise ratio  $sn = 0.05$  and regularization parameter of  $1\text{e-}5$ .

#### IV. CONCLUSIONS

In this paper, we solve the inverse Cauchy problems associated with Laplace equation by using the MFS and ECSHA. It is found that due to the merit of ECSHA, no regularization is necessary and thus this method does not solve the distort equation but directly solve the original equation. It is found that when the Cauchy data information can be given more diversely on the boundary the inversion results are better. Even when noise exists in data, the current approach still can obtain reasonable results. Since the MFS is a meshfree method and collocation only needs to be done on the boundary, it is concluded that the current approach is an easy way to solve the inverse Cauchy problem of the Laplace equation.

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