Journal of Marine Science and Technology

Volume 23 | Issue 4 Article 11

# THE GEOMETRIC ALGORITHM OF INVERSE AND DIRECT PROBLEMS WITH AN AREA SOLUTION FOR THE GREAT ELLIPTIC ARCS

Wei-Kuo Tseng
*Department of Merchant Marine, National Taiwan Ocean University, Keelung, Taiwan*, wilco0917@yahoo.com.tw

Jiunn-Liang Guo
*Department of Merchant Marine, National Taiwan Ocean University, Keelung, Taiwan*

Chung-Ping Liu
*Department of Merchant Marine, National Taiwan Ocean University, Keelung, Taiwan*



Recommended Citation
Tseng, Wei-Kuo; Guo, Jiunn-Liang; and Liu, Chung-Ping (2015) "THE GEOMETRIC ALGORITHM OF INVERSE AND DIRECT PROBLEMS WITH AN AREA SOLUTION FOR THE GREAT ELLIPTIC ARCS," *Journal of Marine Science and Technology*: Vol. 23: Iss. 4, Article 11.
DOI: 10.6119/JMST-014-1120-1
Available at: https://jmstt.ntou.edu.tw/journal/vol23/iss4/11

# THE GEOMETRIC ALGORITHM OF INVERSE AND DIRECT PROBLEMS WITH AN AREA SOLUTION FOR THE GREAT ELLIPTIC ARCS


## Acknowledgements

This work was supported in part by the National Science Council of Taiwan, Republic of China, under grant NSC 98-2410-H-019-019-, NSC 99-2410-H-019-023-, and NSC 100-2410-H-019-018-, NSC 101-2410-H-019-025-, and NSC 102-2410-H-019-016-.

*Journal of Marine Science and Technology, Vol. 23, No. 4, pp. 481-490 (2015)*
DOI: 10.6119/JMST-014-1120-1 

# THE GEOMETRIC ALGORITHM OF INVERSE AND DIRECT PROBLEMS WITH AN AREA SOLUTION FOR THE GREAT ELLIPTIC ARCS

Wei-Kuo Tseng, Jiunn-Liang Guo, and Chung-Ping Liu

Key words: great ellipse, edge, geography type, computer algebra system.

## ABSTRACT

The definition of the connecting edge(s) between two vertices in a geography type is the great elliptic minor arc in Microsoft SQL Server's Geography Type. Similar to the way an edge is defined by Microsoft SQL Server, an edge in Geodyssey's Hipparchus library is also defined as a great circle arc on a reference sphere. Using Hipparchus for their computations, IBM's DB2 Geodetic Extender and Informix Geodetic Datablade share this definition. Compact formulae are given for the great elliptic sailing on a spheroid providing solutions to both the forward and inverse problems with exceptional accuracy, and latitude in terms of longitude. The solution incorporates a closed form for the azimuth and the derivation of the algorithm is presented and illustrated. In addition, the area of polygon bounded by the elliptic arcs is treated. This paper also shows that a computer algebra system is a powerful tool to solve mathematical derivations in navigation, geodesy, and cartography.

## I. INTRODUCTION

Lines, polygonal paths and polygons are widely used in the description of geospatial data, and they are usually defined in terms of their endpoints and vertices. The definition of the connecting edge between two vertices is the shorter great elliptic arc in Microsoft SQL Server's Geography Type (Kallay, 2007; Microsoft, 2013). Similar to the way an edge is defined by Microsoft SQL Server, an edge in Geodyssey's Hipparchus library is also defined as a great circle arc on a reference sphere (Geodyssey, 2013). Using Hipparchus for their computations, IBM's DB2 Geodetic Extender and Informix Geodetic Datablade share this definition (IBM, 2013).

*Paper submitted 03/05/14; revised 08/07/14; accepted 11/20/14. Author for correspondence: Wei-Kuo Tseng (e-mail: wilco0917@yahoo.com.tw).*
*[1]Department of Merchant Marine, National Taiwan Ocean University, Keelung, Taiwan, R.O.C.*

The result of any computation, e.g. the length of a path or the intersection of polygons, depends on the definition of the edges that connect these points. On a planar map, the edge between two points is obviously the line segment that connects them, but on an ellipsoidal earth model the choice is not obvious, and it varies between different software products. While differences in accuracy and performance are to be expected, it is a sad state of affairs when different software packages disagree on the theoretical results of their computations. The paper (Kallay, 2007) presents the definition of edges in Microsoft's SQL Server's Geography Type, proposing it as an industry standard. It stands to reason that a round earth edge should satisfy the following requirements:

1. Locally, an edge should be experienced as straight.
2. A pair of points should define a unique edge between them.
3. An edge should admit a differentiable parameterization, which assigns a point on the edge to every real number between 0 and 1.

The geodesic is the curve on the surface of an ellipsoid defining the shortest distance between two points. Kallay (2007) points out that geodesic curves score poorly on the requirement 2, 3, and even on 1 they are not the obvious choice:

1. While the geodesic curve is the shortest path that is confined to the surface, most human activities take us beyond the surface, for example, surveyors measure along straight lines of sight and airplanes fly miles above the surface of the globe.
2. There are numerous (not necessarily antipodal) pairs of points on an ellipsoid between which there are more than one short geodesic (Rapp, 1991).
3. Computing points along geodesic curves is notoriously difficult and expensive. An exact differentiable parameterization is not known, and approximate ones are also difficult and expensive to compute.

Classical surveying suggests the definition of an edge as a normal section (Rapp, 1991), which is a plane curve created by intersecting a plane containing the normal to the spheroid

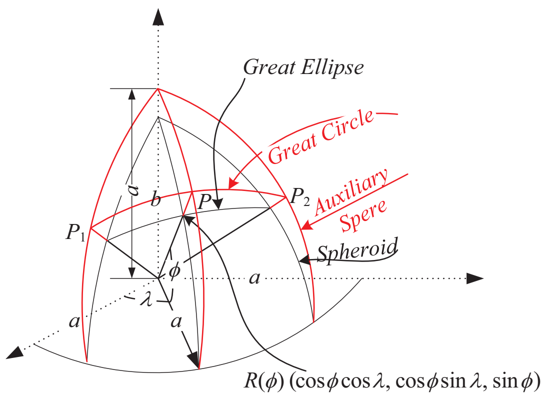


**Fig. 1. A great ellipse on a spheroid.**

with the surface of the spheroid. Alas, this definition is rarely unique. The surveyor's plane at the other endpoint may define a different normal section.

The paper (Kallay, 2007) evaluates the definition of the great elliptic arc against the above stated requirements:

1. Edges are experienced as straight or approximately straight in several senses: As a great elliptic arc, an edge is planar. The angular deviation of its plane from a surveyor's planes at either endpoint is no more than 12'. This translates to about 2.8 cm for an edge whose length is 10 km. An edge is approximately the shortest path between its endpoints – the length of no edge exceeds the geodesic distance by more than 0.02%. In the space of direction, the edge is a line segment, and so are its gnomonic projections.
2. Every pair of non-antipodal points defines a unique edge.
3. The parameterization is simple and differentiable. The parameterization as a quadratic rational Bezier curve may be slightly more expensive to set up but very efficient for generating multiple points along the edge.

The great elliptic arc on spheroid has been investigated in (Bowring, 1984; Walwyn, 1990; Williams, 1998; Earle, 2000, 2008; Kally, 2007; Tseng and Lee 2010), but is rarely mentioned elsewhere. The great elliptic arc between two points $P_1$ and P2 on a spheroid, centered at $O$, is the minor arc of the ellipse of the intersection between the spheroid and the plane OP1P2 (Fig. 1). If the two points are antipodal, the collinear points $P_1$, $O$, and $P_2$ do not determine a unique plane, in such a case it would be reasonable to choose the route passing through the two poles of the spheroid. The azimuth at the point $P_1$ is the angle that the tangent at $P_1$ to great ellipse $P_1P_2$ makes with the meridian through $P_1$, and is measured from the clockwise direction northerly. The azimuth at arbitrary points on the great ellipse would be similarly defined (Bowring, 1984).

Some approximate formulae, the great elliptic equation and great circle equations have been provided in a number of papers (Earle, 2000; Pallikaris and Latsas, 2009; Tseng and Lee 2007a, 2007b, 2010, 2012, 2013) that have studied this problem of the great ellipse sailing and achieved remarkable results. However, the existed formulae need cumbersome algorithms and their accuracies are not very high. In addition, the mathematical derivations in those literatures are a bit tedious, and abstruse, hardly suited for coding (Bowring, 1984; Pallikaris and Latsas, 2009). The direct solution was also not completely provided in those articles (Earle, 2011). For these reasons, in this paper we revisit the solution for the great elliptic arc and provide a more straightforward and compact mathematical derivation of the spherical trigonometric solutions. This paper also gives a general inverse and direct solution attaining any accuracy requirement for the calculation of the great ellipse sailing.

In the mathematical derivation, we consider the direct and inverse scenarios to produce solutions determining the great ellipse from one point and its azimuth or between two points. Furthermore, the interpolation for latitude in terms of longitude between end points of a great ellipse on the spheroid has not yet been found in the literature. As a consequence of these observations, the complete solution to the great ellipse presented here will include a method to determine latitude for any specified longitude along the ellipse. Because the calculation of the area of polygon bounded by the geodesics needs cumbersome algorithms (Sjöberg, 2006), the alternative calculation of the area of polygon bounded by the great elliptic arc is also provided here. The accuracies attained can satisfy the requirement of ECDIS and GIS environments. Finally, we give the full formulae of spherical trigonometry that can easily be coded in a programming language so that readers should comprehensively grasp the meaning of the geometry.

## II. PARAMETERS OF THE GREAT ELLIPSE

Using geodetic and geocentric latitudes, a point $P$ on the surface of a spheroid such as Earth can be represented as a vector function of longitude $\lambda$, geodetic latitude $\varphi$, or geocentric latitude $\phi$.

$$\vec{P}(\varphi,\lambda) = \begin{pmatrix} x & y & z \end{pmatrix} = N(\varphi)\begin{pmatrix} \cos\varphi\cos\lambda, & \cos\varphi\sin\lambda, & (1-e^2)\sin\varphi \end{pmatrix} \quad (1)$$

and

$$\vec{P}(\phi,\lambda) = \begin{pmatrix} x & y & z \end{pmatrix} = R(\phi)\begin{pmatrix} \cos\phi\cos\lambda, & \cos\phi\sin\lambda, & \sin\phi \end{pmatrix} \quad (2)$$

where $e$ is the eccentricity, and $N(\varphi) = a/(1-e^2\sin^2\varphi)^{1/2}$ is the radius of curvature of the prime vertical, and

$$R(\phi) = a\left[(1-e^2)/(1-e^2\cos^2\phi)\right]^{1/2}. \quad (3)$$

The following equation can transform Cartesian coordinates of a point on the spheroid to geodetic coordinates.

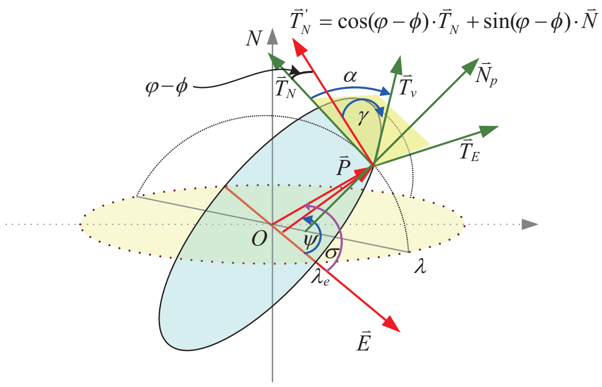


**Fig. 2. The azimuths along a great ellipse on spheroid.**

$$\phi(x, y, z) = \text{atan2}(z, (1-e^2)\sqrt{x^2+y^2})$$

$$\lambda(x, y, z) = \text{atan2}(y, x) \tag{4}$$

where atan2($y$, $x$) is the four quadrant arctangent of the elements of $x$ and $y$ supporting to the interval $[-\pi \quad \pi]$.

The geodetic and geocentric latitudes are related by

$$\phi = \text{atan2}\left[(1-e^2)\sin\varphi, \cos\varphi\right] \tag{5}$$

The azimuth at point $P$ is the angle that the tangent at $P$ to the great ellipse makes with the meridian through $P$, and it is measured in the clockwise direction northerly. The azimuth is also the angle between the meridian plane and the normal plane containing the velocity vector at point $P$. The normal plane usually is slightly different from the great elliptic plane at point $P$.

From the above definition, the azimuth can be obtained from the inner product of the velocity vector and the unit parallel tangent vector dividing by the inner product of the velocity vector and the unit meridian tangent vector at point $P$, as follows (Tseng and Lee, 2010).

$$\alpha = \text{atan2}(\vec{T}_V \bullet \vec{T}_E, \vec{T}_V \bullet \vec{T}_N), \tag{6}$$

where the vectors $\vec{T}_V$, $\vec{T}_N$, and $\vec{T}_E$ are the unit velocity vector, the unit meridian tangent vector, and the unit parallel tangent vector respectively (Fig. 2).

The auxiliary spherical azimuth also can be represented in the similar equation:

$$\gamma = \text{atan2}(\vec{T}_V \bullet \vec{T}_E, \vec{T}_V \bullet \vec{T}_N') \tag{7}$$

The spherical unit meridional tangent vector is the linear combination of the geodetic meridional tangent vector and the unit normal $\vec{N}_P$ to the spheroid at point $P$:

$$\vec{T}_N' = \cos(\varphi-\phi)\cdot\vec{T}_N + \sin(\varphi-\phi)\cdot\vec{N}_P \tag{8}$$

The normal to spheroid at point $P$ and the unit velocity vector at point $P$ are orthogonal, so the inner product of the two vectors equals to 0. Substitute Eq. (8) into Eq. (7) to obtain (Bowring, 1984; Earle, 2008):

$$\gamma = \text{atan2}\left[\vec{T}_V \bullet \vec{T}_E, \cos(\varphi-\phi)\cdot\vec{T}_V \bullet \vec{T}_N\right] \tag{9}$$

Therefore the following relations exist:

$$\alpha = \text{atan2}\left[\cos(\varphi-\phi)\cdot\sin\gamma, \cos\gamma\right] \tag{10}$$

and

$$\gamma = \text{atan2}\left[\sin\alpha, \cos(\varphi-\phi)\cdot\cos\alpha\right] \tag{11}$$

Application of the spherical trigonometric formula gives the azimuth at point $P_1$ on the sphere.

$$\gamma_1 = \text{atan2}(\cos\phi_2\sin\lambda_{12}, \cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2\cos\lambda_{12}) \tag{12}$$

Use Napier's mnemonic and spherical trigonometric formulae to find the longitude of the node which is the intersection between the great ellipse and the Equator.

$$\lambda_e = \lambda_1 + \text{atan2}(-\sin\phi_1\sin\gamma_1, \cos\gamma_1) \tag{13}$$

or

$$\lambda_e = \lambda_1 + \text{atan2}(\sin\phi_1\cos\phi_2\sin\lambda_{12}, \sin\phi_1\cos\phi_2\cos\lambda_{12} - \cos\phi_1\sin\phi_2)\,. \tag{14}$$

where $\lambda_{12} = \lambda_2 - \lambda_1$ is the longitude difference.

The above and following subscripts 1 and 2 denote the corresponding value of departure and destination and subscript 12 is the corresponding different value from destination to departure. The distance on the great elliptic arc is measured from the node because the node is the initial point in the distance integral. Using the spherical trigonometric formula gives the formulae of angle between two points (Fig. 3):

$$\cos\sigma_{12} = \sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2\cos\lambda_{12} \tag{15}$$

and

$$\sin\sigma_{12} = \sqrt{\cos^2\varphi_2\sin^2\lambda_{12} + (\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2\cos\lambda_{12})^2}\,. \tag{16}$$

This arccosine formula using Formula (15) in above equation has large rounding errors if the angle is small and the

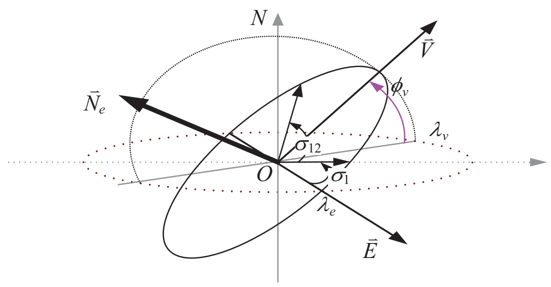


**Fig. 3. A great ellipse on a spheroid.**

value of arcsine formula is the interval $[-\pi/2 \quad \pi/2]$. The following formula is accurate for the most angles and has intervals between $[-\pi \quad \pi]$.

$$\sigma_{12} = \text{atan2}(\sin\sigma_{12}, \cos\sigma_{12})\,. \tag{17}$$

The latitude of the vertex along a great ellipse is:

$$\cos\phi_V = \cos\phi_1 \cos\phi_2 \sin\lambda_{12} / \sin\sigma_{12}\,, \tag{18}$$

or

$$\cos\phi_V = \cos\phi_1 \cdot \sin\gamma_1 \tag{19}$$

where $\gamma_1$ is the initial azimuth of departure on the auxiliary sphere.

The geocentric latitude of vertex on a great ellipse is the angle that the plane of the great ellipse makes with the plane of the Equator. The minor axis of a great ellipse is from the origin $O$ to the vertex, therefore the eccentricity of the great ellipse is:

$$\varepsilon = \sqrt{\frac{a^2 - R(\phi_v)^2}{a^2}} = \frac{e\sin\phi_v}{\sqrt{1-e^2\cos^2\phi_v}} \tag{20}$$

## III. THE GENERAL FORMULA OF THE LENGTH OF A GREAT ELLIPTIC ARC

Using Napier's rule gives the geocentric angle along a great ellipse from node.

$$\sigma = \text{atan2}\left[\sin(\lambda-\lambda_e)\sec\phi_V, \quad \cos(\lambda-\lambda_e)\right] \tag{21}$$

Using Eq. (5) gives the relationships between the geodetic and geocentric angles along a great ellipse.

$$\psi = \text{atan2}\left[\sin\sigma, (1-\varepsilon^2)\cos\sigma\right] \tag{22}$$

and

$$\sigma = \text{atan2}\left[(1-\varepsilon^2)\sin\psi, \cos\psi\right] \tag{23}$$

The distance from the node of the equator to one point $\bar{P}(\varphi, \lambda)$ on a great ellipse is given by standard oft-studied meridional arc-length formula (24).

$$L(\psi) = \int_0^{\psi} \rho(\theta)d\theta \tag{24}$$

where $\rho(\theta) = \dfrac{a(1-\varepsilon^2)}{(1-\varepsilon^2\sin^2\theta)^{3/2}}$ is the radius of curvature for the great ellipse.

The integral of arc length lacks convenient anti-derivatives. The binomial expansion series of integrant can discover the analytic solution term by term. The closed form of the general differential equation is usually unavailable. But the power series representation is always a welcome solution. Expanding the RHS of the Eq. (24) by binomial theorem as a rapidly convergent series yields Eq. (25).

$$L(\psi) = a(1-\varepsilon^2)\int_0^{\psi}\left[\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}-\frac{3}{2}\\ i\end{pmatrix}(\varepsilon^2\sin^2\theta)^i\right]d\theta\,. \tag{25}$$

Using power-reduction formulas (26) expands the powers of sine in RHS of Eq. (25).

$$\sin^n\theta = \frac{1}{2^n}\begin{pmatrix}n\\ n/2\end{pmatrix} + \frac{2}{2^n}\sum_{k=1}^{\frac{n}{2}}(-1)^k\begin{pmatrix}n\\ \frac{n}{2}-k\end{pmatrix}\cos(2k\theta)\,. \tag{26}$$

Evaluating the integral of powers of sine gives:

$$J_n = \int_0^{\psi}\sin^n\theta d\theta = \frac{1}{2^n}\begin{pmatrix}n\\ n/2\end{pmatrix}\psi + \sum_{k=1}^{\frac{n}{2}}\frac{(-1)^k\begin{pmatrix}n\\ \frac{n}{2}-k\end{pmatrix}\sin(2k\psi)}{2^n k}\,. \tag{27}$$

Substitute Eq. (27) into Eq. (25) and expand it to obtain the follows:

$$L(\psi) = a(1-\varepsilon^2)\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}-\frac{3}{2}\\ i\end{pmatrix}J_{2i}\varepsilon^{2i} \tag{28}$$

Expanding Eq. (28) yields Eq. (29). The general solution of integral (28) can be represented in terms of sine of multiple angles as the following.

$$L(\psi)=a(1-\varepsilon^2)\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}-\frac{3}{2}\\ i\end{pmatrix}\left[\frac{1}{2^{2i}}\binom{2i}{i}\psi+\sum_{k=1}^{i}\frac{(-1)^k\binom{2i}{i-k}\sin(2k\psi)}{2^{2i}k}\right]\varepsilon^{2i} \tag{29}$$

Rearrange (29) to obtain general solution (30).

$$L(\psi)=a(1-\varepsilon^2)\left[M_0\psi+\sum_{i=1}^{\infty}M_{2i}\sin(2i\psi)\right] \tag{30}$$

where

$$M_0=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}}\begin{pmatrix}-\frac{3}{2}\\ k\end{pmatrix}\binom{2k}{k}\varepsilon^{2k}$$

and

$$M_{2i}=\sum_{k=i}^{\infty}\frac{(-1)^{i+k}}{2^{2k}i}\begin{pmatrix}-\frac{3}{2}\\ k\end{pmatrix}\binom{2k}{k-i}\varepsilon^{2k}.$$

The following matrix (31) (Bian and Chen, 2006) tabulates the coefficients of sine function of Eq. (30) obtained by truncating the expansion at order $\varepsilon^{12}$ and $M_{12}$ up to $\varepsilon^{10}$ and $M_{10}$. The users can choose any expression of higher order to attain the high accuracy of calculation of the great elliptic arc lengths. Eq. (29) is the general geodetic formula for the accurate calculation of meridian arc length. Since the integral (30) is versatile, applying this integral can attain any accurate requirement for geodetic and sailing calculation.

$$M=\begin{bmatrix}M_0\\M_2\\M_4\\M_6\\M_8\\M_{10}\end{bmatrix}=\begin{bmatrix}1&\frac{3}{4}&\frac{45}{64}&\frac{175}{256}&\frac{11025}{16384}&\frac{43659}{65536}\\0&-\frac{3}{8}&-\frac{15}{32}&-\frac{525}{1024}&-\frac{2205}{4096}&-\frac{72765}{131072}\\0&0&\frac{15}{256}&\frac{105}{1024}&\frac{2205}{16384}&\frac{10395}{65536}\\0&0&0&-\frac{35}{3072}&-\frac{105}{4096}&-\frac{10395}{262144}\\0&0&0&0&\frac{315}{131072}&\frac{3465}{524288}\\0&0&0&0&0&-\frac{693}{1310720}\end{bmatrix}\begin{bmatrix}1\\\varepsilon^2\\\varepsilon^4\\\varepsilon^6\\\varepsilon^8\\\varepsilon^{10}\end{bmatrix} \tag{31}$$

If point $P$ is located on the same semi-sphere of the departure, then the length of elliptic arc can be computed as the following.

$$\mathrm{Dist}(\psi)=\left|L(\psi)-L(\psi_1)\right| \tag{32}$$

If the point $P$ is located on the opposite semi-sphere of the departure, then the distance can be computed by

$$\mathrm{Dist}(\psi)=\left|L(\psi)\right|+\left|L(\psi_1)\right| \tag{33}$$

There is an inversion series to Eq. (24), described by Synder (1987) and attributed to earlier work (Adam, 1921) that used the Lagrange Inversion Theorem to construct the inversion series of geodetic latitude in terms of elliptic arc length. Here, apply Hermite Interpolation Schemes (Bian and Chen, 2006) to derive a different kind of inversion series in terms of the first eccentricity. The rectifying latitude $\mu$ is the meridian distance scaled so that its value at the poles $\mu$ is equal 90 degrees or $\pi/2$ radians. It is denoted $\mu$ and is given by

$$\mu=\frac{\pi L(\psi)}{2L(\pi/2)} \tag{34}$$

where $L(\pi/2)=\frac{\pi}{2}a(1-e^2)M_0$.

Substituting Eq. (24) into the above Eq. (34) yields

$$\mu=\frac{\int_0^{\psi}\frac{d\theta}{(1-\varepsilon^2\sin^2\theta)^{3/2}}}{M_0} \tag{35}$$

Set up the expression of geodetic latitude in terms of rectifying latitude using up to $\sin(8\mu)$ terms.

$$\begin{aligned}\psi&=L^{-1}(\mu)=\mu+C_2\sin2\mu+\cdots\\&C_4\sin4\mu+C_6\sin6\mu+C_8\sin8\mu\end{aligned} \tag{36}$$

Applying Hermite Interpolation Schemes can derive the four coefficients of Eq. (36). We must impose 4 constraint equations by interpolation condition. Develop a data of points of Hermite interpolation function which passes through the function and its first, third, fifth, and seventh derivatives for the point 0.

From Eq. (35), the first derivative of geodetic latitude with respect to rectifying latitude can be yielded.

$$\psi^{(1)}=\frac{d\psi}{d\mu}=M_0(1-\varepsilon^2\sin^2\psi)^{3/2} \tag{37}$$

Making use of the chain rule of differentiation obtains the recursive relation of high order derivative.

$$\psi^{(n)}=\frac{d\psi^{(n-1)}}{d\psi}\frac{d\psi}{d\mu} \tag{38}$$

It is very difficult and even impossible to derive the high order derivatives in Eq. (38) by hand. Fortunately, these derivatives can be easily computed with any computer algebra system (such as Mathematica®, Maple®, MATLAB®, or Maxima®). If the rectifying latitude is equal to 0 ($\mu = 0$), then the geodetic latitude will be equal to 0 ($\psi = 0$). Differentiate continuously Eq. (35) with respect to rectifying latitude to get its first, third, fifth, and seventh derivatives (the second, fourth, and sixth derivatives cannot satisfy conditions of interpolation because their values are equal to 0). Constraint equations may be written in matrix form

$$AC = B \tag{39}$$

where

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ -8 & -64 & -216 & -512 \\ 32 & 1024 & 7776 & 32768 \\ -128 & -16384 & -279936 & -2097152 \end{bmatrix}, C = \begin{bmatrix} C_2 \\ C_4 \\ C_6 \\ C_8 \end{bmatrix} \tag{40}$$

and

$$B = \begin{bmatrix} \psi^1(0) - 1 \\ \psi^3(0) \\ \psi^5(0) \\ \psi^7(0) \end{bmatrix} = \begin{bmatrix} M_0 - 1 \\ M_0^3(-3\varepsilon^2) \\ M_0^5(45\varepsilon^4 + 12\varepsilon^2) \\ M_0^7(-1575\varepsilon^6 - 1116\varepsilon^4 - 48\varepsilon^2) \end{bmatrix}. \tag{41}$$

Apply the symbolic expression solves the inverse and solution of the symbolic system (39).

$$C = \begin{bmatrix} C_2 \\ C_4 \\ C_6 \\ C_8 \end{bmatrix} = A^{-1}B$$

$$= \begin{bmatrix} 3/8 & 3/16 & 213/2048 & 255/4096 \\ 0 & 21/256 & 21/256 & 533/8192 \\ 0 & 0 & 151/6144 & 151/4096 \\ 0 & 0 & 0 & 1097/131072 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon^2 \\ \varepsilon^4 \\ \varepsilon^6 \\ \varepsilon^8 \end{bmatrix}. \tag{42}$$

Rearrange a subset of the resulting equations into nested forms is more suitable for computation.

$$\begin{bmatrix} C_2 \\ C_4 \\ C_6 \\ C_8 \end{bmatrix} = \begin{bmatrix} 1/4096\left\{1536 + \left[768 + (426 + 255\varepsilon^2)\varepsilon^2\right]\varepsilon^2\right\}\varepsilon^2 \\ 1/8192\left[672 + (672 + 533\varepsilon^2)\varepsilon^2\right]\varepsilon^4 \\ 1/12288(302 + 453\varepsilon^2)\varepsilon^6 \\ 1097/131072\varepsilon^8 \end{bmatrix} \tag{43}$$

**Table 1. The inverse solution.**

| |
|---|
| Input: Two points, $P_i = (\varphi_i, \lambda_i)$, $i = 1, 2$ |
| Find $\phi_i$ from Eq. (5). |
| Find $\gamma_1$, $\lambda_e$ from Eqs. (12), (13), or (14). |
| Find $\phi_v$ from Eqs. (18) or (19). |
| Find $\varepsilon$, $\sigma_i$ from Eqs. (20) and (21). |
| Find $\psi_i$ from Eq. (22). |
| Output: GE arc-length and azimuths |
| Find $L = \begin{cases} \lvert L(\psi_2) - L(\psi_1)\rvert, & \mathrm{sgn}(\varphi_1) = \mathrm{sgn}(\varphi_2) \\ \lvert L(\psi_2)\rvert + \lvert L(\psi_1)\rvert, & \mathrm{sgn}(\varphi_1) \neq \mathrm{sgn}(\varphi_2) \end{cases}$ from Eq. (30). |
| Find $\alpha_1 = \mathrm{atan2}(\cos(\varphi_1 - \phi_1)\sin\gamma_1 \cos\gamma_1,$ from Eq. (10). |
| Find $\begin{matrix} \alpha_2 = \mathrm{atan2}(\cos(\varphi_2 - \phi_2)\cos\phi_2 \sin\lambda_{12}, \\ \ldots - \sin\phi_1 \cos\varphi_2 + \cos\phi_1 \sin\phi_2 \cos\lambda_{12}) \end{matrix}$ from Eq. (10) or using spherical trigonometric azimuth formula. |

**Table 2. The direct solution.**

| |
|---|
| Input: $P_1 = (\varphi_1, \lambda_1)$, initial azimuth ($\alpha_1$), distance to a second point (s) |
| Find $\phi_1$ from Eq. (5). |
| Find $\gamma_1$, $\lambda_e$ from Eqs. (11) and (13). |
| Find $\phi_v$ from Eq. (18). |
| Find $\varepsilon$, $\sigma_1$ from Eqs. (20) and (21). |
| Find $\mu_1$ from Eq. (35). |
| Let $\mu_2 = \mu_1 + \dfrac{sign(\alpha_1) \cdot s}{a(1-e^2)M_0}$. |
| Find $\psi_2$ from Eq. (36). |
| Find $\sigma_2$ from Eq. (23). |
| Set $\sigma_{12} = \sigma_2 - \sigma_1$ |
| Use spherical trigonometric function to find $\phi_2 = \mathrm{asin}(\sin\phi_1 \cos\sigma_{12} + \cos\phi_1 \sin\sigma_{12} \cos\gamma_1)$ |
| Output: the final position and azimuth. |
| Find $\varphi_2 = \mathrm{atan2}\left(\sin\phi_2, \ \cos\phi_2 \cdot (1 - e^2)\right)$ from Eq. (5). |
| Use spherical trigonometric function to find $\lambda_2 = \lambda_1 + \mathrm{atan2}(\sin\sigma_{12}\sin\gamma_1,$ $\cdots\cos\phi_1\cos\sigma_{12} - \sin\phi_1 \sin\sigma_{12}\cos\gamma_1)$ $\alpha_2 = \mathrm{atan2}(\cos(\varphi_2 - \phi_2)\cos\phi_1 \sin\gamma_1,$ $\cdots - \sin\phi_1 \sin\sigma_{12} + \cos\phi_1 \cos\sigma_{12}\cos\gamma_1)$ |

## IV. THE SOLUTIONS OF GREAT ELLIPTIC SAILING

Summarizing the above equations gives the inverse solution in Table 1.

Summarizing the above equations gives the direct solution in Table 2.

The solutions provided here give concise and logical procedures of calculation. The two-argument function atan2 gives a great advantage for calculating the azimuth and longitude which avoids the ambiguities between the first and third quadrants, and between the second and fourth quadrants. The

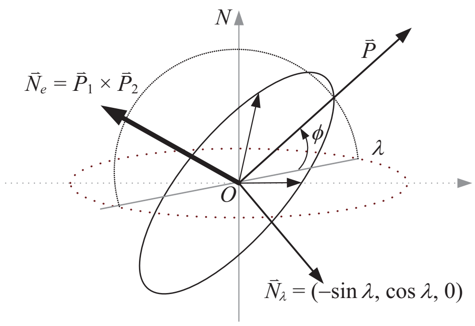


**Fig. 4. Great ellipse on spheroid.**

algorithms provided here are suitable for the programming implementation and can be applied in the areas of sailing line design and cartographical computation in GIS and ECDIS environments.

## V. LATITUDE IN TERMS OF LONGITUDE ALONG GREAT ELLIPSE.

The vector $P$ is the cross product of normal to a great ellipse and normal to meridian of known longitude (Fig. 4).

$$\vec{P} = \begin{pmatrix} x, & y, & z \end{pmatrix} = \vec{N}_e \times \vec{N}_\lambda \tag{44}$$

where $\vec{N}_e = \vec{P}_1 \times \vec{P}_2$ and $\vec{N}_\lambda = (-\sin\lambda, \cos\lambda, 0)$.

Substituting Formula (1) into the above equation and expanding the equation in terms of trigonometric function gives the latitude function of known longitude.

$$\tan\varphi = \tan(\varphi_1)\frac{\sin(\lambda_2-\lambda)}{\sin(\lambda_2-\lambda_1)} + \tan(\varphi_2)\frac{\sin(\lambda-\lambda_1)}{\sin(\lambda_2-\lambda_1)} \tag{45}$$

or

$$\tan\varphi = \tan(\varphi_V)\sin(\lambda-\lambda_e) \tag{46}$$

Eqs. (45) and (46) is not suitable for computing the distance along a great elliptic arc, nor it is suitable for computing the azimuth of the curve, but by certain re-arrangements it is possible to solve (directly) for the latitude of a point on the curve given a longitude somewhere between the longitudes of the terminal points of the curve. Or alternatively, using Eq. (47) solves for the longitude of a point given latitude.

$$\lambda = \lambda_e + \operatorname{asin}\left[\frac{\tan\varphi}{\tan\varphi_V}\right] \tag{47}$$

## VI. COMPUTING THE AREA OF A POLYGON

The area $dA$ of an infinitesimal slice bounded by two almost coinciding parallels from a meridian of longitude 0 and a meridian of longitude $\lambda$ can be written (Sjöberg, 2006; Deakin, 2010) (Fig. 5):

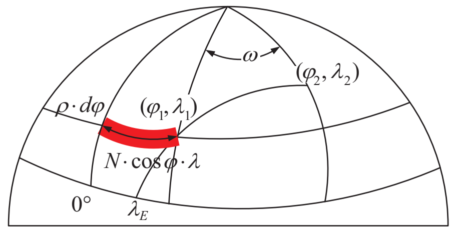


**Fig. 5. Area of an infinitesimal slice on the spheroid.**

$$dA = \lambda \cdot N(\varphi)\cdot\cos\varphi\cdot\rho(\varphi)d\varphi \tag{48}$$

where

$$\rho(\varphi) = a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2} \tag{49}$$

is the radius of curvature for meridian and

$$N(\varphi) = a/(1-e^2\sin^2\varphi)^{1/2} \tag{50}$$

is the radius of curvature of the prime vertical.

Differentiating Eq. (46) yields:

$$d\varphi = \frac{\tan(\varphi_V)}{\sec^2\varphi}\cos\omega d\omega \tag{51}$$

where $\omega = \lambda - \lambda_e$.

Substituting Eqs. (46) and (51) into Eq. (48) and rearranging the results gives:

$$A = a^2\gamma\tan\varphi_V\int_{\lambda_1-\lambda_e}^{\lambda_2-\lambda_e}\frac{\sqrt{1+\tan^2\varphi_V\sin^2\omega}}{(1+\gamma\tan^2\varphi_V\sin^2\omega)^2}\cos\omega(\omega+\lambda_e)d\omega \tag{52}$$

where $\gamma = 1 - e^2$.

If the longitudes of the two endpoints are on the identical meridian, the above equation can be transformed the following integral of latitudes.

$$A = \lambda a^2\gamma\int_{\varphi_1}^{\varphi_2}\frac{\cos\varphi}{(1-e^2\sin^2\varphi)^2}d\varphi \tag{53}$$

Separating the above integral into partial fractions and integrating the result gives the formula of the area bounded by two parallels from a meridian of longitude 0 and a meridian of longitude $\lambda$.

$$A = \lambda a^2\gamma\left[\frac{\sin\varphi}{2(1-e^2\sin^2\varphi)} + \frac{1}{4e}\ln\frac{1+e\sin\varphi}{1-e\sin\varphi}\right]\Bigg|_{\varphi_1}^{\varphi_2} \tag{54}$$

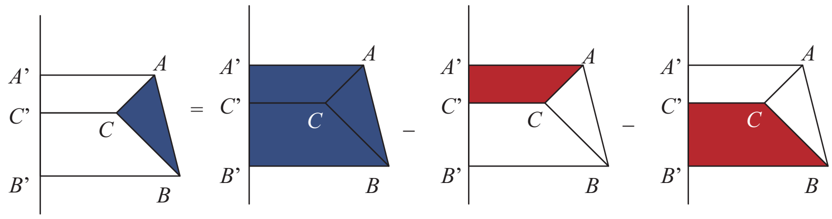


**Fig. 6. The signs of area on the spheroid bounded by a great elliptic and equator.**

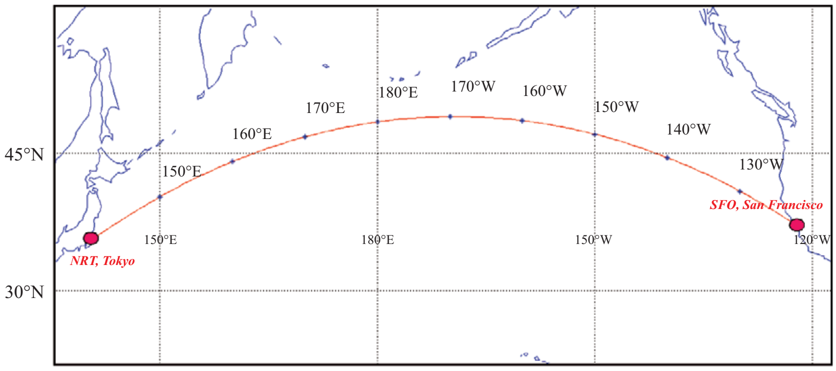


**Fig. 7. Great elliptic arc from Tokyo to San Francisco.**

The integral of the area is computed numerically with an adaptive Simpson's method. The area of a polygon is computed numerically as a sum of integral over edges. A general polygon can be partitioned into slices that are bounded by edges at their ends. The area of each slice is the difference between the area integrals (52) along these bounding edges. So the area is a sum of area integrals along edges with appropriate signs: negative when the edge goes south and positive when it goes north when the vertices are ordered to positive orientation (counterclockwise). The cosine of longitude in the RHS of Eq. (52) captures the needed sign, and the total area is therefore the sum of the integral (52) or (53) over all the polygon's edges.

$$A_t = \sum_i^n A_{i,i+1},\ A_{n,n+1} = A_{n,1} \qquad (55)$$

Fig. 6 and the following equation depict the computation of the area and the signs of area integral bounded by the three great elliptic arcs.

$$Area_{\triangle}ABC = \square A'ABB' - \square A'ACC' - \square C'CBB' \qquad (56)$$

## VII. NUMERICAL TESTS

An airplane flies from NRT Airport, Tokyo (35°45'55"N 140°23'08"E) to SFO Airport, San Francisco (37°37'08"N 122°22'30"W) along the great elliptic arc on the WGS84 Earth (Fig. 7). The latitudes of waypoints differing in longitude from F by 150°E, 160°E, … , 120°W , 130°W are found after using Eq. (45) and are shown in the following Table 3.

**Table 3. Latitude, distance, initial azimuth and final azimuth in terms of known longitude.**

| Lat | Long | Range | Az1 | Az2 |
|---|---|---|---|---|
| 35.765 | 140.386 | 536.967 | 54.952 | 60.880 |
| 40.537 | 150.000 | 493.915 | 60.880 | 67.602 |
| 44.127 | 160.000 | 446.981 | 67.602 | 74.702 |
| 46.541 | 170.000 | 417.094 | 74.702 | 82.031 |
| 47.947 | -180.000 | 402.240 | 82.031 | 89.471 |
| 48.445 | -170.000 | 401.272 | 89.471 | 96.919 |
| 48.071 | -160.000 | 414.111 | 96.919 | 104.270 |
| 46.799 | -150.000 | 441.761 | 104.270 | 111.410 |
| 44.536 | -140.000 | 486.137 | 111.410 | 118.196 |
| 41.126 | -130.000 | 412.159 | 118.196 | 123.021 |
| 37.619 | -122.375 | Null | Null | Null |

We use the eccentricity $e = 0.08181919$ and semi-major axis $a = 6378.137$ km same as the WGS84 model. Using the inverse solution in Table 1 obtains the initial and final azimuths, and distance between each segment (Table 3). Applying Vincenty's method (Vincenty, 1975) or other methods (Sjöberg, 2008) calculates the length and initial azimuth of the geodesic: azimuth 54.8178088° and distance 8246275.05775 m (4452.6323206 nm). Using the inverse solution provided here (Table 1) gives Distance: 8246282.09626 m (4452.63612109 nm). The difference between the great ellipse and geodesic is about 7.03851 m. It satisfies the requirements of accuracy for the most purposes of navigation and GIS Environments. The Az1 and Az2 express the initial and final azimuths.

**Table 4. Reverted dataset from Table 3.**

| Lat* | Error | Long* | Error | Az2* | Error |
|---|---|---|---|---|---|
| 35.765 | 0.00E+00 | 140.386 | 0.00E+00 | 60.880 | -2.09E-11 |
| 40.537 | -1.37E-11 | 150.000 | -3.20E-11 | 67.602 | -2.62E-11 |
| 44.127 | -1.12E-11 | 160.000 | -3.80E-11 | 74.702 | -2.75E-11 |
| 46.541 | -7.20E-12 | 170.000 | -3.80E-11 | 82.031 | -2.36E-11 |
| 47.947 | -3.00E-12 | -180.000 | -3.20E-11 | 89.471 | -1.51E-11 |
| 48.445 | -9.95E-14 | -170.000 | -2.00E-11 | 96.919 | -4.39E-12 |
| 48.071 | 4.97E-13 | -160.000 | -6.00E-12 | 104.270 | 6.01E-12 |
| 46.799 | -1.40E-12 | -150.000 | 7.99E-12 | 111.410 | 1.20E-11 |
| 44.536 | -5.00E-12 | -140.000 | 1.80E-11 | 118.196 | 1.40E-11 |
| 41.126 | -9.00E-12 | -130.000 | 2.20E-11 | 123.021 | 1.40E-11 |
| 37.619 | -1.21E-11 | -122.375 | 2.30E-11 | Null | Null |

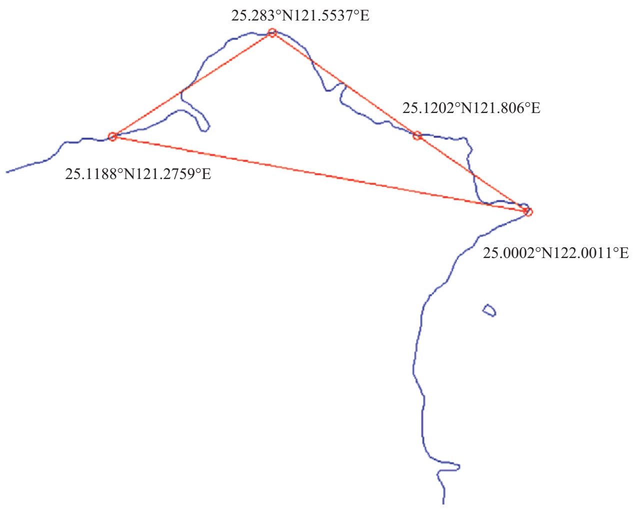


**Fig. 8. Polygon bounded by great elliptic arcs nearby north Taiwan.**

Use the direct solution as shown in Table 2 to restore the original latitude, longitude, and final azimuth from dataset in Table 3. The errors between original dataset and reverted dataset are very small and almost negligible Table 4.

The area of polygon bounded by four great elliptic arcs connected by the four polygon vertices is computed as the following table and displayed in Fig. 8 (in northern Taiwan). When eccentricity $e = 0$ and semi-major axis $a = 6378.137$ km are applied to calculate the area, lengths and azimuths of the polygon bounded four great circles on the spherical model.

A spherical polygon is a closed surface, whose sides are formed by great circles. By adding the surfaces of several spherical triangles, one obtains an area of polygon with $n$ edges:

$$A = a^2 \sum_{i=1}^{n} (\alpha_{i,i+1}^1 - \alpha_{i,i+1}^2),\ \alpha_{n,n+1}^j = \alpha_{n,1}^j,\ j = 1, 2\,, \qquad (57)$$

where $\alpha_{i,i+1}^1$ and $\alpha_{i,i+1}^2$ are the initial and the final azimuths from vertex $i$ to vertex $i + 1$.

**Table 5. The length and area of a polygon bounded by four great circles and great elliptic arcs. (Unit: km, km$^2$)**

| Lat | Long | Rang* | Area* | Range | Area |
|---|---|---|---|---|---|
| 25.1188 | 121.2759 | 33.4278 | 223644.30 | 33.393 | 222687.37 |
| 25.2830 | 121.5537 | 31.2123 | -222130.25 | 31.174 | -221179.82 |
| 25.1202 | 121.8060 | 23.7776 | -164186.57 | 23.751 | -163479.91 |
| 25.0002 | 122.0011 | 74.3074 | 161829.96 | 74.339 | 161133.41 |
| | Total | 162.7250 | -842.5600 | 162.657 | -838.94 |

*: great circle

**Table 6. The area of the polygon on a sphere given by the spherical excesses.**

| Lat | Long | Az1 | Az2 | (Az1-Az2)*$a^2$ |
|---|---|---|---|---|
| 25.1188 | 121.2759 | 56.7744 | 56.8927 | -0.1183 |
| 25.2830 | 121.5537 | 125.4442 | 125.5516 | -0.1074 |
| 25.1202 | 121.8060 | 124.1334 | 124.2160 | -0.0826 |
| 25.0002 | 122.0011 | -79.6197 | -79.9268 | 0.3072 |
| | Total | 226.7323 | 226.7335 | -842.5600 km$^2$ |

Applying Eq. (57) gives the area of polygon on a sphere which is identical with the area as calculated in Table 5.

The polygon vertices are ordered to negative orientations (clockwise orientation), so the signs of the area are negative respectively in Table 5 and Table 6. In Table 6, the area -842.5600 km$^2$ of the last entry in column 5 is obtained by the sum of radian angle (Az1-Az2) times the major axis radius km $a$ = 6378.137 km.

## VIII. CONCLUSIONS

The general direct and inverse solutions of the great ellipse have been described and demonstrated. It is found that the two solutions are useful alternatives to establish integral expansions for direct and inverse solutions of the great elliptic arc and provide latitude function in terms of specified longitude. The results of numerical tests also are very accurate. The solution of latitude in terms of longitude also has been described and demonstrated. The area of polygon bounded by elliptic arcs is more easily calculated than the calculation of area bounded by geodesics. The algorithms provided here are suitable for the programming implementation and can be applied in the area and cartographical computations in GIS and ECDIS environments.


## ACKNOWLEDGMENTS

This work was supported in part by the National Science Council of Taiwan, Republic of China, under grant NSC 98-2410-H-019-019-, NSC 99-2410-H-019-023-, and NSC 100-2410-H-019-018-, NSC 101-2410-H-019-025-, and NSC 102-2410-H-019-016-.

## REFERENCES

Adams, O. S. (1921). Latitude Developments Connected With Geodesy and Cartography, Special Publication 67, Department of Commerce, 123-125.

Bian, S. F. and Y. B. Chen (2006). Solving an inverse problem of a meridian arc in terms of computer algebra system. Journal of Survey Engineering 132(1) 7-10.

Bowring, B. R. (1984). The direct and inverse solutions for the great elliptic line on the reference ellipsoid. Bulletin Geodesique 58, 101-108.

Deakin, R. E. and M. N. Huntre (2010). Geometric Geodesy Part A. School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia.

Earle, M. A. (2000). A vector solution for navigation on a great ellipse. Journal of Navigation 53, 473-481.

Earle, M. A. (2008). Vector solutions for azimuth. Journal of Navigation 61, 537-545.

Earle, M. A. (2011). Accurate harmonic series for inverse and direct solutions for the great ellipse. Journal of Navigation 64, 557-570.

Geodyssey Limited Calgary, Alberta, Canada: Hipparchus® Tutorial and Programmer's Guide, Retrieved December 30, 2013 from http://www.geodyssey.com/tutorial/tutori al.html.

IBM Knowledge Center, Retrieved December 30, 2013 from http://publib.boulder.ibm.com/infocenter/db2luw/v9/index.jsp?topic=/com.ibm.db2.udb.spatial.doc/rsbgeo41.html.

Kallay, M. (2007). Defining edges on a round earth. Proceedings of the 15th annual ACM international symposium on Advances in geographic information systems (GIS '07), Article 63, 1-3.

Microsoft SQL Server: Spatial Data Types Overview, Retrieved December 30, 2013 from http://technet.microsoft.com/en-us/library/bb964711.aspx.

Pallikaris, A. and G. Latsas (2009). New algorithm for great elliptic sailing (GES). Journal of Navigation 62, 493-507.

Rapp, R. H. (1991). Geometric Geodesy Part I. The Ohio State University Department of Geodetic Science and Surveying.

Sjöberg, L. E. (2006). Determination of areas on the plane, sphere and ellipsoid. Survey Review 38(301), 583-593.

Sjöberg, L. E. (2008). New solutions to classical geodetic problems on the ellipsoid. International Association of Geodesy Symposia 133(4), 781-784.

Snyder, J. P. (1987). Map Projections - A Working Manual, USGS Professional Paper: 1395, U.S. Geological Survey.

Tseng, W. K. and H. S. Lee (2007a). The vector function of traveling distance for great circle navigation. Journal of Navigation 60, 158-164.

Tseng, W. K. and H. S. Lee (2007b). Building the latitude equation of the mid-longitude. Journal of Navigation 60, 164-170.

Tseng, W. K. and H. S. Lee (2010). Navigation on a great ellipse. Journal of Marine Science and Technology 18(3), 369-375.

Tseng, W. K., J. L. Guo, C. P. Liu (2013). A comparison of great circle, great ellipse, and geodesic sailing. Journal of Marine Science and Technology 21(3). 287-299.

Tseng, W. K., J. L. Guo, C. P. Liu and C. T. Wu (2012). The vector solutions for the great ellipse on the spheroid. Journal of Applied Geodesy 6(2), 103-109.

Vincenty, T. (1975). Direct and inverse solutions of geodesics on the ellipsoid with application of nested equations. Survey Review 23, 176, 88-93.

Walwyn, P. R. (1990). The great ellipse solution for distances and headings to steer between waypoints. Journal of Navigation 52, 421-424.

Williams, R. (1998). Geometry of Navigation. Horwood Publishing.