Journal of Marine Science and Technology

Volume 23 | Issue 1 Article 3

A NOVEL APPROACH TO SOLVE THE GREAT CIRCLE TRACK BASED ON ROTATION TRANSFORMATION

Chih-Li Chen
Merchant Marine Department, National Taiwan Ocean University, Keelung, Taiwan, R.O.C., clchen@mail.ntou.edu.tw

Tsung-Hsuan Hsieh
Department of Civil Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.

Tien-Pen Hsu
Department of Civil Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.

Recommended Citation

Chen, Chih-Li; Hsieh, Tsung-Hsuan; and Hsu, Tien-Pen (2015) "A NOVEL APPROACH TO SOLVE THE GREAT CIRCLE TRACK BASED ON ROTATION TRANSFORMATION," *Journal of Marine Science and Technology*: Vol. 23: Iss. 1, Article 3.
DOI: 10.6119/JMST-013-1212-1
Available at: https://jmstt.ntou.edu.tw/journal/vol23/iss1/3

A NOVEL APPROACH TO SOLVE THE GREAT CIRCLE TRACK BASED ON ROTATION TRANSFORMATION

Acknowledgements

The first author would like to express the gratitude to the National Science Council of Taiwan for financial support under contract number: NSC 95-2221-E-019-092- and NSC 96-2628-E-019-022-MY3.

Journal of Marine Science and Technology, Vol. 23, No. 1, pp. 13-20 (2015)
DOI: 10.6119/JMST-013-1212-1

A NOVEL APPROACH TO SOLVE THE GREAT CIRCLE TRACK BASED ON ROTATION TRANSFORMATION

Chih-Li Chen[1], Tsung-Hsuan Hsieh[2], and Tien-Pen Hsu[2]

Key words: great circle track, rotation transformation, plane of the celestial meridian.

ABSTRACT

The great circle track (GCT) is composed of several rhumb lines in practice. Given the initial conditions, the waypoints along the GCT can be determined by the navigator. To reduce the complex calculation in a space system, transforming the system into a planar coordinate one is considered. Thus, an idea of the plane of the celestial meridian in celestial navigation is proposed to determine the waypoints, which are presented in equator coordinate system and horizontal one. For locating the waypoints on the GCT, the rotation transformation is introduced to derive the governing equations for the computation procedures with respect to different given initial conditions. Due to the rotation transformation, all initial conditions to obtain the waypoints can be dealt with simultaneously. A program, GCTPro_RT, is developed to solve the GCT problems for any smart devices. Its effectiveness and real applications are presented through demonstrated examples.

I. INTRODUCTION

Marine navigation is a process of directing a moving vessel from one port to another in a safe and economical way. When the Earth is treated as a sphere and the wind and tide effects are not considered, the shortest distance is the great circle track (GCT). To follow a GCT exactly would involve continuous course changes. In practice, the GCT is divided into a series of rhumb lines (RLs), approximating the great circle (GC) (Royal Navy, 2008). In this regard, the navigator has to provide initial conditions for determining the waypoints along the GCT. Once all the waypoints on the GCT are available, the navigator can treat the Earth as an oblate spheroid. For more accurate results, the course and distance of the rhumb line (RL) between two adjacent waypoints can be computed by using the spheroidal RL sailing (Royal Navy, 2008).

The GCT is composed of several RLs with given initial conditions in practice. Theoretically, these given initial conditions include: (1) giving the GC distances to reach the longitudes and latitudes of the waypoints (hereafter is called "condition 1"); (2) giving the longitudes of the waypoints to obtain its latitudes (hereafter is called "condition 2"); (3) giving latitudes of the waypoints to yield its longitudes. Note that the third condition is impractical and seldom used because it might results in two waypoints for an additional judgment and leading to tedious calculation. Therefore only conditions 1 and 2 are considered in this article.

From the viewpoint of a navigator, the direct method (DM) is taking the departure point as the reference point to obtain the waypoints; while the indirect method (IM) is taking the vertex or equator crossing point as the reference point to yield the waypoints. In previous research works, some took the vertex as the reference point (called IM-V) to reach the waypoints because this method uses the characteristic of the right-angled spherical triangles. In contrast, if the equator crossing point is taken as the reference point (called IM-E), characteristic of quadrantal spherical triangles is used.

The IM-V has been developed for many years and it can yield the waypoints for problems of conditions 1 and 2. The advantage of this method comes from adopting the Napier's rules of the right-angled spherical triangles but its disadvantage results from needing more computational steps (Holm, 1972; Bowditch, 1981, 2002; Chen, 2003; Chen et al., 2004; Cutler, 2004; Royal Navy, 2008). As for the IM-E, it adopts the Napier's rules of the quadrantal spherical triangles (Chen, 2003; Chen et al., 2014). The advantage and disadvantage of IM-E are the same as those of IM-V. In contrast to the IM, the DM is more simple and direct to reach the waypoints because it is unnecessary to compute the vertex or equator crossing point. However, the DM is only valid for single given initial condition, for example, some used the DM to obtain the waypoints of given condition 1, see (Miller et al., 1991; Nastro and Tancredi, 2010; Chen et al., 2014); while some adopted

Paper submitted 06/18/13; accepted 12/12/13. Author for correspondence: Chih-Li Chen (e-mail: clchen@mail.ntou.edu.tw).
[1] *Merchant Marine Department, National Taiwan Ocean University, Keelung, Taiwan, R.O.C.*
[2] *Department of Civil Engineering, National Taiwan University, Taipei, Taiwan, R.O.C.*

Table 1. A comparison of different methods for solving the GCT.

Authors	Category	Condition 1	Condition 2
Holm, 1972	IM-V	NA	available
	DM (Meridian Method)	NA	available
Jofeh, 1981	DM (Linear Equation)	NA	available
Bowditch, 1981 and 2002	IM-V	available	available
Miller et al., 1991	DM (Linear Combination)	available	NA
Chen, 2003	IM-V, IM-E	available	available
	DM (GCEM)	NA	available
Chen et al., 2004	IM-V	available	available
	DM (GCEM)	NA	available
Cutler, 2004	IM-V	available	available
Royal Navy, 2008	IM-V	NA	available
	DM (Meridian Method)	NA	available
Nastro and Tancredi, 2010	DM (Linear Combination)	available	NA
Chen et al., 2014	IM-E	available	available
	DM (COFI)	available	NA
Chen et al., 2015 (the current paper)	DM (Rotation Transformation)	available	available

the DM to yield the waypoints for problem of given condition 2, see (Holm, 1972; Jofeh, 1981; Chen, 2003; Chen et al., 2004; Royal Navy, 2008). A comparison of the aforementioned methods are listed in Table 1 as a quick reference.

To improve the methods mentioned above, the proposed approach first takes the departure point as the reference point. Then, two practical initial conditions, conditions 1 and 2, which are usually encountered, are respectively considered in this article. To yield the waypoints on the GCT and obtain additional necessary information, an idea of the plane of the celestial meridian in celestial navigation is proposed to determine the same waypoints presented in equator coordinate system and horizontal one. In addition, the rotation transformation is introduced to bridge the two systems and thus, the governing equations used in the computation procedures with respect to different initial conditions can be easily derived. Due to the rotation transformation, two initial conditions of obtaining waypoints arising in the practical navigation can be dealt with by using this proposed approach.

Apart from the current section, Section 2 represents the theoretical backgrounds. Computation procedures of the given initial conditions and its numerical program are illustrated in Section 3. Several demonstrated examples with remarks are given in Section 4. Finally, Section 5 draws concrete conclusions after recasting this research work.

II. THEORETICAL BACKGROUNDS

First of all, we treat the Earth as a unitary sphere. To simplify the space as a planar coordinate system, the idea of the plane of the celestial meridian in celestial navigation is proposed to construct a set of combined systems including the equator system of the Earth and the horizontal system of departure point. Accordingly, representations of the waypoints in the equator system and the horizontal system, respectively, can be determined. Then, based on the rotation transformation technique, the related equations can be formulated in the combined systems. Thereafter, aimed at different given initial conditions, the governing equations for yielding the waypoints can be obtained. Finally, replacing the waypoints by the destination, equator crossing point or the vertex, we can derive more additional necessary equations. Note that without a judgment of sign convention, the concept of the fixed coordinate system is needed to adopt, that is, the north latitude is treated as a positive value and the south latitude is treated as a negative one. For decreasing the number of the variables, the relative longitude concept that replacing the Greenwich meridian by the meridian of the departure point is introduced. All symbols used in this paper are listed in the Appendix for quick reference.

1. A Diagram on the Plane of the Combined Systems

As shown in Fig. 1, since $\vec{\mathrm{N}}$ and $\vec{\mathrm{F}}$ are a set of orthonormal vectors (Spiegel et al., 2009), the linear combination of position vectors in horizontal coordinates can be expressed as

$$\vec{R}_{\mathrm{H}} = (\sin D_{\mathrm{F}X})\vec{\mathrm{N}} + (\cos D_{\mathrm{F}X})\vec{\mathrm{F}}, \tag{1a}$$

$$\vec{r}_{\mathrm{H}} = (\sin D_{\mathrm{F}X} \cos C)\vec{\mathrm{N}} + (\sin D_{\mathrm{F}X} \sin C)\vec{\mathrm{F}}, \tag{1b}$$

$$\vec{X}_{\mathrm{H}} = (\vec{r}_{\mathrm{H}} \cdot \vec{\mathrm{N}})\vec{\mathrm{N}} + (\vec{R}_{\mathrm{H}} \cdot \vec{\mathrm{F}})\vec{\mathrm{F}} = (\sin D_{\mathrm{F}X} \cos C)\vec{\mathrm{N}} + (\cos D_{\mathrm{F}X})\vec{\mathrm{F}}. \tag{1c}$$

Similarly, as shown in Fig. 2, since $\overline{\mathrm{P}}$ and $\overline{\mathrm{Q}}$ are also a set of orthonormal vectors (Spiegel et al., 2009), the linear combination of position vectors in equator coordinates can be expressed as

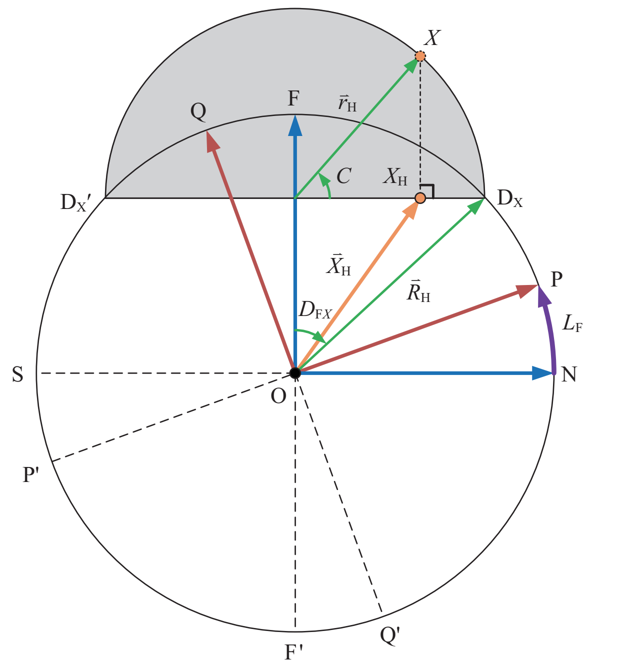


Fig. 1. Locating the waypoints in the horizontal coordinate system of a diagram on the plane of the combined systems.

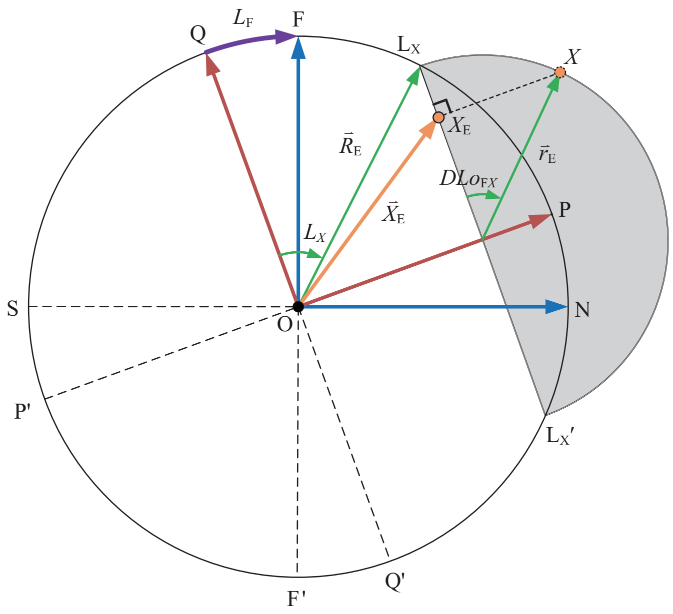


Fig. 2. Locating the waypoints in the equator coordinate system of a diagram on the plane of the combined systems.

$$\vec{R}_E = (\sin L_X)\vec{P} + (\cos L_X)\vec{Q}\,, \tag{2a}$$

$$\vec{r}_E = (\cos L_X \sin DLo_{FX})\vec{P} + (\cos L_X \cos DLo_{FX})\vec{Q}\,, \tag{2b}$$

$$\vec{X}_E = (\vec{R}_E \cdot \vec{P})\vec{P} + (\vec{r}_E \cdot \vec{Q})\vec{Q} = (\sin L_X)\vec{P} + (\cos L_X \cos DLo_{FX})\vec{Q}. \tag{2c}$$

As shown in Figs. 1 and 2, the two vertical distances from the waypoints on the small circles to the planes in the two coordinate systems are equal, that is, $\overline{XX_H} = \overline{XX_E}$. Thus, its equation can be expressed as

$$\sin D_{FX} \sin C = \cos L_X \sin DLo_{FX}\,. \tag{3}$$

2. Rotation Transformation

When the rotation transformation is introduced, the equations formulated by related variables in different systems, such as the GC arcs and dihedral angles, are yielded.

(1) Rotating counterclockwise angle, L_F, from horizontal system to equator system, as shown in Fig. 1, one can have

$$\begin{bmatrix} \sin L_X \\ \cos L_X \cos DLo_{FX} \end{bmatrix} = \begin{bmatrix} \cos L_F & \sin L_F \\ -\sin L_F & \cos L_F \end{bmatrix} \begin{bmatrix} \sin D_{FX} \cos C \\ \cos D_{FX} \end{bmatrix}.$$

By expanding the above equation, we have

$$\sin L_X = \cos L_F \sin D_{FX} \cos C + \sin L_F \cos D_{FX}\,, \tag{4}$$

$$\cos L_X \cos DLo_{FX} = -\sin L_F \sin D_{FX} \cos C + \cos L_F \cos D_{FX}\,. \tag{5}$$

(2) Rotating clockwise angle, L_F, from equator system to horizontal system, as shown in Fig. 2, one can have

$$\begin{bmatrix} \sin D_{FX} \cos C \\ \cos D_{FX} \end{bmatrix} = \begin{bmatrix} \cos L_F & -\sin L_F \\ \sin L_F & \cos L_F \end{bmatrix} \begin{bmatrix} \sin L_X \\ \cos L_X \cos DLo_{FX} \end{bmatrix}.$$

By expanding the above equation, we have

$$\sin D_{FX} \cos C = \cos L_F \sin L_X - \sin L_F \cos L_X \cos DLo_{FX}\,, \tag{6}$$

$$\cos D_{FX} = \sin L_F \sin L_X + \cos L_F \cos L_X \cos DLo_{FX}\,. \tag{7}$$

In addition, the square relation of trigonometric functions is

$$\sin^2 L_F + \cos^2 L_F = 1\,. \tag{8}$$

We find that substituting Eq. (7) into Eq. (5) and introducing Eq. (8) can yield Eq. (6). Similarly, substituting Eq. (4) into Eq. (6) and introducing Eq. (8) can yield Eq. (5). Therefore, Eqs. (5) and (6) are equivalent. Actually, both are five-parts formulae of spherical trigonometry.

3. Derivation of the Governing Equations

1) The Waypoints

Condition 1: Giving the GC distance to obtain the latitude and the longitude

Rewriting Eq. (4) in the following form as

$$\sin L_X = \sin L_F \cos D_{FX} + \cos L_F \cos C \sin D_{FX} . \quad (9)$$

Rearranging Eq. (7) yields

$$\cos DLo_{FX} = \frac{\cos D_{FX} - \sin L_F \sin L_X}{\cos L_F \cos L_X} . \quad (10)$$

The above two equations are used to obtain the waypoints by giving the GC distance.

Condition 2: Giving the longitude to obtain the latitude

Substituting Eq. (3) into Eq. (6) or substituting Eqs. (3) and (7) into Eq. (5) and eliminating the variable D_{FX}, we have

$$\tan L_X = \frac{\cos C \sin DLo_{FX} + \sin L_F \sin C \cos DLo_{FX}}{\cos L_F \sin C} . \quad (11)$$

2) The GC Distance and Initial Course Angle

The preliminaries of the GCT are departure point and destination point. Taking the latter to replace the waypoints and rearranging Eqs. (7) and (4), we have

$$\cos D = \sin L_F \sin L_T + \cos L_F \cos L_T \cos DLo , \quad (12)$$

$$\cos C = \frac{\sin L_T - \sin L_F \cos D}{\cos L_F \sin D} . \quad (13)$$

3) The Equator Crossing Point

The latitude of the equator crossing point must be zero, that is, $L_E = 0$. Substituting it into Eq. (11) rearranging, we have

$$\tan DLo_{FE} = -\sin L_F \tan C . \quad (14)$$

4) The Vertex

The vertices of a GC are the points nearest the poles. Accordingly, let the first derivative of Eq. (11) be zero and rearrange it, one has

$$\tan DLo_{FV} = \frac{1}{\sin L_F \tan C} . \quad (15)$$

Substituting the result of the above equation into Eq. (11) can yield

$$\tan L_V = \frac{\cos C \sin DLo_{FV} + \sin L_F \sin C \cos DLo_{FV}}{\cos L_F \sin C} . \quad (16)$$

Noted that the multiplication of Eqs. (14) and (15) is equal to -1, the difference of longitudes between the equator crossing point and the vertex is 090°.

III. COMPUTATION PROCEDURES AND NUMERICAL PROGRAM

As mentioned in previous section, the great circle sailing is to obtain waypoints along the GCT. Then, the course and distance of each RL between adjacent waypoints can be computed by using the spheroidal RL sailing (Bennett, 1996; Royal Navy, 2008). First, the meridional parts (M) formula is adopted to determine the course and expressed as:

$$M = a_e \left[\ln \tan\left(45° + \frac{L}{2}\right) + \frac{e}{2} \ln\left(\frac{1 - e\sin L}{1 + e\sin L}\right) \right], \quad (17)$$

in which M is the number of meridional parts between the equator and the given latitude (L, in degrees), $a_e = 10800/\pi = 3437.74677078$ nautical miles (nm) is the equatorial radius of the Earth (Bowditch, 2002), and e = 0.081819190842622, eccentricity of the Earth, is derived from WGS-84 (Bennett, 1996; NIMA, 2000; Royal Navy, 2008). Then, the meridional arc length (m) formula is adopted to compute the distance and expressed as (Bennett, 1996; Royal Navy, 2008):

$$m = a\left[\left(1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256}\right)L - \frac{3}{8}\left(e^2 + \frac{e^4}{4} + \frac{15e^6}{128}\right)\sin 2L + \frac{15}{256}\left(e^4 + \frac{3e^6}{4}\right)\sin 4L - \frac{35e^6}{3072}\sin 6L \right], \quad (18)$$

in which m is international nautical miles of 1852 meters and a = 3443.918467 nm is the semi-major axis of the WGS-84 (NIMA, 2000; Royal Navy, 2008). Note that L, the latitude, is in radians. Finally, the spheroidal RL course and distance can be obtained by the following formulae (Bennett, 1996; Royal Navy, 2008).

$$dlo = 60'\left(\lambda_{X_{i+1}} - \lambda_{X_i}\right), \quad (19a)$$

$$\Delta M = M_{X_{i+1}} - M_{X_i}, \quad (19b)$$

$$\Delta m = m_{X_{i+1}} - m_{X_i}, \quad (19c)$$

$$\tan c_{RL} = \frac{dlo}{\Delta M}, \quad (20)$$

$$d_{RL} = \begin{cases} \Delta m \sec c_{RL} & , c_{RL} \neq 90° \\ \dfrac{a\pi dlo \cos L_X}{10800\left(1 - e^2 \sin^2 L_X\right)^{1/2}} & , c_{RL} = 90°, \end{cases} \quad (21)$$

in which dlo is in minutes of arc, L_X is in degrees and d_{RL} is in nm.

1. Constructing Computation Procedure

Step 1. Calculate needed and extra information of the GC.

(1) Needed information: Calculating the GC distance (D) and initial course angle (C) using Eqs. (12) and (13), respectively.

(2) Extra information: The equator crossing point can be determined by using Eq. (14). The longitude, λ_E, can be obtained when the DLo_{FE} and the known λ_{F} are available. As for the vertex, it can be determined by using Eqs. (15) and (16), in which the longitude, λ_V, can be obtained when the $DLo_{\mathrm{F}V}$ and the known λ_{F} are available.

Step 2. Calculate the waypoints along the GCT.

Condition 1: Giving the GC distance to obtain the latitude and the longitude. Calculating the latitudes of the waypoints by using Eq. (9) and calculating the longitudes of the waypoints by using Eq. (10), that is, λ_X can be yielded when the known λ_{F} and calculated $DLo_{\mathrm{F}X}$ are available.

Condition 2: Giving the longitude to obtain the latitude. With given λ_X and known λ_{F}, the $DLo_{\mathrm{F}X}$ can be obtained. Then L_X can be yielded by using Eq. (11).

Step 3. According to the spheroidal RL sailing, the course and distance of each RL between adjacent waypoints can be determined by using Eqs. (17), (18), (19a), (19b), (19c), (20) and (21).

2. Numerical Program

A GCT program with the graphical user interface (GUI), namely GCTPro_RT, has been developed by using JavaScript (JS) with advantages of free running in the browsers of any digital devices such as laptops, tablets and mobile phones. In addition, for the reasonable number of waypoints, a diagram of total rhumb lines distance versus waypoints number (called tRLd-n diagram) is provided for the navigator. Since the GCTPro_RT is free of charge and open to the navigator; therefore, it also plays a role of sharing scientific knowledge for the researcher.

IV. DEMONSTRATED EXAMPLES

Example 1: *A vessel is proceeding from* ***San Francisco*** *(USA) to* ***Sydney*** *(AUSTRALIA). The navigator wants to use great circle sailing from* L37°47.5′N, λ122°27.8′W *to* L33°51.7′S, λ151°12.7′E. (Bowditch, 1981, pp. 616-618)

Required: Using the GCTPro_RT to calculate the latitudes and longitudes of the waypoints on the GCT 360 nm (6°) apart, and the GC information, such as the GC distance, initial course, the equator crossing point and the vertices can be acquired.

Solution: The GCTPro_RT is run to solve the waypoints on the GCT under condition 1. Results including the GC information and the waypoints on the GCT are all shown in Fig. 3.

GCTPro_RT

GC Information

The spherical GC initial course is 240.3° (N119.7°W).
The spherical GC distance is 6445.22′.
The spheroidal RL distance is 6460.63′.
The difference of distance between GC and RL is 15.41′.

The equator crossing points:
(0°, 169°30.0′W) (0°, 10°30.0′E)
One of the equator crossing points is on the GC track.

The vertexes:
(46°39.5′N, 79°30.0′W) (46°39.5′S, 100°30.0′E)
The vertexes are out of the GC track.

Waypoints

wp.	latitude	longitude	RL course	RL distance
F	37°47.5′N	122°27.8′W	238.5°	360.70′
1	34°38.7′N	128°47.9′W	235.0°	360.49′
2	31°11.7′N	134°39.0′W	232.1°	360.27′
3	27°30.0′N	140°4.5′W	229.7°	360.07′
4	23°36.6′N	145°8.4′W	227.8°	359.88′
5	19°33.8′N	149°54.4′W	226.2°	359.71′
6	15°23.9′N	154°26.1′W	225.0°	359.57′
7	11°8.7′N	158°47.1′W	224.2°	359.46′
8	6°49.9′N	163°0.5′W	223.7°	359.40′
9	2°28.9′N	167°9.3′W	223.5°	359.37′
10	1°52.9′S	171°16.6′W	223.7°	359.39′
11	6°14.0′S	175°25.0′W	224.1°	359.45′
12	10°33.2′S	179°37.6′W	224.9°	359.55′
13	14°49.0′S	176°2.7′E	226.0°	359.69′
14	18°59.7′S	171°32.6′E	227.5°	359.85′
15	23°3.6′S	166°48.8′E	229.4°	360.04′
16	26°58.5′S	161°47.6′E	231.8°	360.24′
17	30°42.0′S	156°25.3′E	234.5°	325.62′
T	33°51.7′S	151°12.7′E		

Fig. 3. Results of condition 1 by running GCTPro_RT in example 1.

Remark: It is found that the computerized solution is always more accurate than tabular methods (the Ageton method) because it is free of rounding errors shown in Table 2, which is also reported in (Bowditch, 1981).

Example 2: *A vessel is proceeding from* ***Sydney*** *(AUSTRALIA) to* ***Balboa*** *(PANAMA). The master would like to use the great circle sailing from* L33°515.5′S, λ151°13.0′E *to* L08°53.0′N, λ079°31.0′W. (Chen, 2003, pp. 69-71)

Required: Using the GCTPro_RT to calculate the latitudes and longitudes of the waypoints along the GCT at longitude 170°E and at each 20 degrees of longitude thereafter to longitude 090°W, and the GC information, such as the GC distance, initial course, the equator crossing point and the vertices are acquired.

Solution: The GCTPro_RT is run to solve the waypoints on

Table 2. A comparison of the GC information obtained by the tabular method and the current approach in example 1.

D_{FX}	Tabular method*	GCTPro_RT
6°	34°39.0′N, 128°48.3′W	34°38.7′N, 128°47.9′W
12°	31°12.0′N, 134°39.3′W	31°11.7′N, 134°39.0′W
18°	27°30.0′N, 140°04.3′W	27°30.0′N, 140°04.5′W
36°	15°24.0′N, 154°26.3′W	15°23.9′N, 154°26.1′W
54°	02°29.0′N, 167°09.3′W	02°28.9′N, 167°09.3′W
60°	01°52.5′S, 171°17.3′W	01°52.9′S, 171°16.6′W
D	6445.5′	6445.22′
C	240°17.5′	240.3°
E	-	00°00.0′N, 169°30.0′W 00°00.0′N, 010°30.0′E
V	46°39.5′S, 100°29.7′E	46°39.5′N, 079°30.0′W 46°39.5′S, 100°30.0′E

* Resource: Bowditch, 1981, pp. 616-618.

GCTPro_RT

GC Information

The spherical GC initial course is 106.1° (S73.9°E).
The spherical GC distance is 7635.14′.
The spheroidal RL distance is 7820.64′.
The difference of distance between GC and RL is 185.50′.

The equator crossing points:
(0°, 91°27.7′W) (0°, 88°32.3′E)
One of the equator crossing points is on the GC track.

The vertexes:
(37°3.5′S, 178°32.3′E) (37°3.5′N, 1°27.7′W)
One of the vertexes is on the GC track.

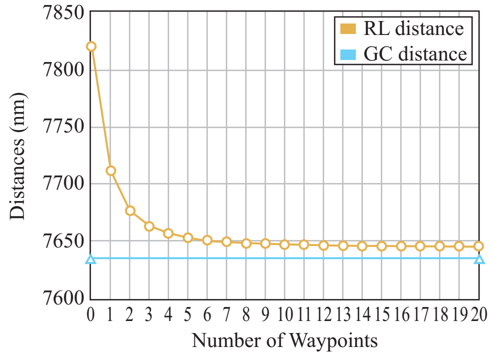
Waypoints

wp.	latitude	longitude	RL course	RL distance
F	33°51.5′S	151°13.0′E	100.6°	938.37′
1	36°45.1′S	170°0.0′E	89.1°	966.02′
2	36°30.3′S	170°0.0′W	77.3°	1014.46′
3	32°47.2′S	150°0.0′W	66.6°	1145.06′
4	25°11.8′S	130°0.0′W	58.3°	1330.66′
5	13°30.1′S	110°0.0′W	53.8°	1476.87′
6	1°6.2′N	90°0.0′W	53.5°	780.55′
T	8°53.0′N	97°31.0′W		

Fig. 4. Results of condition 2 by running GCTPro_RT in example 2.

Waypoints Number Analysis

✦ total rhumb lines distance versus waypoints number (tRLd-n)



no.	DLo of each RL	tRLd	Difference between tRLd and GC distance
0	129°16.0′	7820.64′	185.50′
1	64°38.0′	7712.10′	76.96′
2	43°5.3′	7677.16′	42.01′
3	32°19.0′	7663.54′	28.40′
4	25°51.2′	7656.97′	21.82′
5	21°32.7′	7653.33′	18.18′
6	18°28.0′	7651.11′	15.97′
7	16°9.5′	7649.66′	14.52′
8	14°21.8′	7648.67′	13.52′
9	12°55.6	7647.95′	12.81′
10	11°45.1′	7647.42′	12.27′
11	10°46.3′	7647.02′	11.87′
12	9°56.6′	7646.70′	11.56′
13	9°14.0′	7646.45′	11.31′
14	8°37.1′	7646.25′	11.10′
15	8°4.7′	7646.08′	10.94′
16	7°36.2′	7645.95′	10.80′
17	7°10.9′	7645.83′	10.69′
18	6°48.2′	7645.73′	10.59′
19	6°27.8′	7645.65′	10.51′
20	6°9.3′	7645.58′	10.43′

Fig. 5. The tRLd-n diagram by running GCTPro_RT in example 2.

the GCT under condition 2. Results including of the GC information and the waypoints on the GCT are all shown in Fig. 4. In addition, a tRLd-n diagram and its detailed data are shown in Fig. 5.

Remark:

(1) It is found that the Eq. (11) used for condition 2 is equivalent to those reported in References (Chen, 2003; Chen et al., 2004; Chen et al., 2014).

(2) A tRLd-n diagram and the relationship of the total spheroidal RL distance and waypoints number are presented in Fig. 5. It shows that the total RL distance of 20 waypoints is nearly equal to that of 10 waypoints and their distance difference is less than 2 nm. A reasonable number of waypoints can be considered for navigator in practice. This is a new discovery and an interesting issue for further discussion.

Example 3: *A ship leaves* ***Cape Town*** *(SOUTH AFRICA) bound for* ***New York*** *City (USA). The captain decides to use great circle sailing from* L33°53.3′S, λ018°23.1′E *(near Green Point Light) to* L40°27.1′N, λ073°49.4′W *(near Ambrose Light).* (Bowditch, 1981, pp. 619-620)

Required: Using the GCTPro_RT to calculate each of the

GCTPro_RT

GC Information

The spherical GC initial course is 304.5° (S124.5°W).
The spherical GC distance is 6762.72′.
The spheroidal RL distance is 6786.84′.
The difference of distance between GC and RL is 24.11′.

The equator crossing points:
(0°, 20°41.2′W) (0°, 159°18.8′E)
One of the equator crossing points is on the GC track.

The vertexes:
(46°49.3′S, 69°18.8′E) (46°49.3′N, 110°41.2′W)
The vertexes are out of the GC track.

Waypoints

wp.	latitude	longitude	RL course	RL distance
F	33°53.3′S	18°23.1′E	305.7°	300.36′
1	30°57.8′S	13°34.7′E	308.1°	300.22′
2	27°52.3′S	9°4.0′E	310.1°	300.07′
3	24°38.5′S	4°48.6′E	311.8°	299.94′
4	21°17.9′S	0°46.3′E	313.2°	299.82′
5	17°51.7′S	3°5.1′W	314.4°	299.71′
6	14°21.2′S	6°47.6′W	315.3°	299.62′
7	10°47.3′S	10°23.2′W	316.0°	299.55′
8	7°11.0′S	13°53.7′W	316.4°	299.50′
9	3°33.1′S	17°20.8′W	316.6°	299.47′
10	0°5.5′N	20°46.4′W	316.6°	299.47′
11	3°44.2′N	24°11.9′W	316.4°	299.50′
12	7°22.0′N	27°39.2′W	315.9°	299.55′
13	10°58.2′N	31°10.0′W	315.3°	299.62′
14	14°31.9′N	34°45.8′W	314.3°	299.71′
15	18°2.3′N	38°28.7′W	313.2°	299.82′
16	21°28.2′N	42°20.6′W	311.7°	299.95′
17	24°48.5′N	46°23.5′W	310.0°	300.08′
18	28°1.9′N	50°39.6′W	307.9°	300.22′
19	31°7.0′N	55°11.2′W	305.5°	300.37′
20	34°1.9′N	60°0.5′W	302.8°	300.52′
21	36°44.8′N	65°9.7′W	299.6°	300.66′
22	39°13.3′N	70°40.5′W	296.9°	163.11′
T	40°27.1′N	73°49.4′W		

Fig. 6. Results of condition 1 by running GCTPro_RT in example 3.

following under different given initial condition.

(1) Calculate the latitudes and longitudes of the waypoints along the GCT at equal interval of GC distance, 300 nm (5°), from the departure point. (condition 1)

(2) Calculate the latitudes and longitudes of the waypoints on the GCT at longitude 015°E and at each 5 degrees of longitude thereafter to longitude 070°W. (condition 2)

Solution:

(1) The GCTPro_RT is run to solve the waypoints along the GCT under a given GC distance. Results including of the GC information and the waypoints on the GCT are shown in Fig. 6.

GCTPro_RT

GC Information

The spherical GC initial course is 304.5° (S124.5°W).
The spherical GC distance is 6762.72′.
The spheroidal RL distance is 6786.84′.
The difference of distance between GC and RL is 24.11′.

The equator crossing points:
(0°, 20°41.2′W) (0°, 159°18.8′E)
One of the equator crossing points is on the GC track.

The vertexes:
(46°49.3′S, 69°18.8′E) (46°49.3′N, 110°41.2′W)
The vertexes are out of the GC track.

Waypoints

wp.	latitude	longitude	RL course	RL distance
F	33°53.3′S	18°23.1′E	305.3°	209.52′
1	31°52.0′S	15°0.0′E	307.5°	327.45′
2	28°32.4′S	10°0.0′E	309.8°	349.92′
3	24°47.5′S	5°0.0′E	311.9°	372.67′
4	20°37.7′S	0°0.0′E	313.7°	394.35′
5	16°4.4′S	5°0.0′W	315.0°	413.27′
6	11°10.7′S	10°0.0′W	316.0°	427.63′
7	6°1.6′S	15°0.0′W	316.5°	435.83′
8	0°43.9′S	20°0.0′W	316.6°	436.84′
9	4°35.0′N	25°0.0′W	316.2°	430.55′
10	9°47.2′N	30°0.0′W	315.3°	417.74′
11	14°45.6′N	35°0.0′W	314.1°	399.89′
12	19°25.0′N	40°0.0′W	312.4°	378.79′
13	23°41.5′N	45°0.0′W	310.4°	356.18′
14	27°33.2′N	50°0.0′W	308.1°	333.53′
15	30°59.7′N	55°0.0′W	305.6°	311.88′
16	34°1.6′N	60°0.0′W	302.8°	291.93′
17	36°40.0′N	65°0.0′W	299.8°	274.04′
18	38°56.5′N	70°0.0′W	297.1°	198.86′
T	40°27.1′N	73°49.4′W		

Fig. 7. Results of condition 2 by running GCTPro_RT in example 3.

(2) The GCTPro_RT is run to solve the waypoints on the GCT under a given longitude. Results including of the GC information and the waypoints on the GCT are shown in Fig. 7.

Remark: Examples 1 and 2 have validated the accuracy of the proposed approach. In this example, both of conditions 1 and 2 are included to yield the accurate solution effectively for solving the GCT problems. It is found that the proposed approach shows the advantages of completeness and practical applications.

V. CONCLUSIONS

In this paper, an idea of the plane of the celestial meridian in celestial navigation is proposed to determine the waypoints along the GCT in the equator system and the horizontal one, respectively. Then, the governing equations can be derived by using the rotation transformation. With respect to different given initial conditions, the waypoints along the GCT can thus be obtained. A program for calculating GCT problems has been developed and validated by several practical examples for its completeness.

ACKNOWLEDGMENTS

The first author would like to express the gratitude to the National Science Council of Taiwan for financial support under contract number: NSC 95-2221-E-019-092- and NSC 96-2628-E-019-022-MY3.

APPENDIX

$\vec{\mathrm{N}}, \vec{\mathrm{F}}$	a set of orthonormal vectors in the horizontal coordinate
$\vec{\mathrm{P}}, \vec{\mathrm{Q}}$	a set of orthonormal vectors in the equator coordinate
P	the elevated pole, north pole or south pole which nearer the departure
F	the departure
T	the destination
X	the waypoints
X_{H}	the waypoints in the horizontal coordinate
X_{E}	the waypoints in the equator coordinate
E	the equator crossing point
V	the vertex
L	latitude
L_{F}	departure latitude
L_{T}	destination latitude
L_X	latitudes of the waypoints on the GCT; a GC arc in Fig. 2
L_E	latitudes of the equator crossing points of a GC
L_V	latitudes of the vertices of a GC
λ	longitude
λ_{F}	departure longitude
λ_{T}	destination longitude
λ_X	longitudes of the waypoints on the GCT
λ_E	longitudes of the equator crossing points of a GC
λ_V	longitudes of the vertices of a GC
DLo	difference of longitude from departure to destination
$DLo_{\mathrm{F}X}$	difference of longitude from departure to the waypoints on the GCT; a dihedral angle in Fig. 2
$DLo_{\mathrm{F}E}$	difference of longitude from departure to the equator crossing point
$DLo_{\mathrm{F}V}$	difference of longitude from departure to the vertex
D	GC distance from departure to destination
$D_{\mathrm{F}X}$	GC distance from departure to waypoints; a GC arc in Fig. 1
C	GC initial course angle from departure to destination; a dihedral angle in Fig. 1
d_{RL}	spheroidal RL distance
c_{RL}	spheroidal RL course angle
M	meridional parts
a_e	equatorial radius of the Earth
e	eccentricity of the WGS-84
m	meridional arc length
a	semi-major axis of the WGS-84
dlo	difference of longitude between two adjacent waypoints on the GCT
ΔM	difference of meridional parts
Δm	difference of meridional arc length

REFERENCES

Bennett, G. G. (1996). Practical rhumb line calculations on the spheroid. The Journal of Navigation 49(1), 112-119.

Bowditch, N. (1981). American Practical Navigator, Volume 2. Defense Mapping Agency Hydrographic/Topographic Center, Washington, D.C.

Bowditch, N. (2002). The American Practical Navigator, 2002 Bicentennial Edition. National Imagery and Mapping Agency, Bethesda, Maryland.

Chen, C. L. (2003). New computational approaches for solving the great circle sailing and astronomical vessel position. Ph.D. Dissertation, Department of Civil Engineering, National Taiwan University, Taiwan.

Chen, C. L., T. P. Hsu and J. R. Chang (2004). A novel approach to great circle sailings: The great circle equation. The Journal of Navigation 57(2), 311-320.

Chen, C. L., P. F. Liu and W. T. Gong (2014). A simple approach to great circle sailing: The COFI method. The Journal of Navigation 67(3), 403-418.

Cutler, T. J. (2004). Dutton's Nautical Navigation, 15th Edition. Naval Institute Press, Annapolis, Maryland.

Holm, R. J. (1972). Great circle waypoints for inertial equipped aircraft. NAVIGATION, Journal of the Institute of Navigation 19(2), 191-194.

Jofeh, M. L. (1981). The analysis of great-circle tracks. The Journal of Navigation 34(1), 148-149.

Miller, A. R., I. S. Moskowitz and J. Simmen (1991). Traveling on the curved earth. NAVIGATION, Journal of the Institute of Navigation 38(1), 71-78.

Nastro, V. and U. Tancredi (2010). Great circle navigation with vectorial methods. The Journal of Navigation 63(3), 557-563.

National Imagery and Mapping Agency (NIMA) (2000). Department of Defense World Geodetic System 1984: Its definition and relationships with local geodetic systems, Technical Report NIMA TR8350.2., 3rd Edition. NIMA, Missouri.

Royal Navy (2008). Admiralty Manual of Navigation: The Principles of Navigation, Volume 1, 10th Edition. The Nautical Institute, London.

Spiegel, M. R., S. Lipschutz and D. Spellman (2009). Vector Analysis, 2nd Edition. McGraw-Hill, New York.