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RESEARCH ARTICLE

Identification of Nonlinear Channels in Bandpass Communication Systems with OFDM Inputs

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Abstract

A general approach to the derivation of simple yet accurate formulas for identifying bandpass nonlinear channels in orthogonal frequency-division multiplexing (OFDM) systems is proposed in this paper. The OFDM is an widely adopted technique for wideband communication applications. An OFDM signal could be highly vulnerable to nonlinearities in a communication link. This makes proper estimation of the nonlinear channel essential for an OFDM system. A nonlinear bandpass communication channel is commonly described by a complex Voletrra series, in which the task of nonlinear channel identification is to determine the Volterra kernels. This task can become very difficult when the order of the nonlinear system increases. In this paper, we explore higher-order statistical properties of the OFDM signal to develop a universal scheme for solving the Volterra kernel identification problem. Based on this scheme, a simple yet accurate formula for identifying Volterra kernels of 5th-order nonlinear OFDM systems is derived. The resulting solution is shown to attain the minimum mean square error (MMSE) in both theory and computer simulation.

Keywords: OFDM, Nonlinear channel, Volterra kernel, System identification, Higher-order statistics

1. Introduction

The orthogonal frequency-division multiplexing (OFDM) [1] is a multicarrier modulation scheme that has been used in numerous communication applications [2–6]. An OFDM signal tends to yield a high peak-to-average power ratio (PAR) due to the possible constructive combination of its subcarriers. The undesirable feature often causes the OFDM signal to be corrupted by out-of-band radiations and inter-subcarrier interference generated from nonlinearities in communication systems [7–9]. To combat this problem, proper identification of the nonlinear channels in communication systems can play an essential role.

For a nonlinear bandpass communication channel, its nonlinearities are often originated from a power amplifier and are commonly modeled by a bandpass Volterra series [10–12]. By incorporating the nonlinear effect of the power amplifier into the channel response, the resulting nonlinear channel can be described by a bandpass Volterra model which relates the complex envelopes of the channel input and output [11,13]. Several methods for identifying the Volterra kernels of the bandpass nonlinear channel can be found in the literature. Namely, to determine the time-domain Volterra kernels of bandpass channels in PSK and QAM systems, algorithms based on the assumption that the input signal is independent and identically-distributed (i.i.d.) have been derived [14,15]. For the OFDM system, which is the focus of this paper, methods by assuming the OFDM signal to be asymptotically i.i.d. and complex Gaussian [16,17] have also been developed. Although the Gaussianity assumption greatly simplifies the mathematics in these methods, unsatisfactory estimate of the kernels could result when the Gaussianity of the input signal can not be guaranteed.

Recently, a computationally efficient method for determining the frequency-domain Volterra kernels of cubically nonlinear bandpass channels in OFDM

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systems was derived [18]. Instead of making the Gaussianity assumption, this method took advantage of the random multisine [19] nature of the OFDM signal to attain a simple solution. Specifically, this method utilized certain higher-order auto-moment spectral properties of the random multisine signal to come up with closed-form expressions for frequency-domain Volterra kernels. The acquired kernel estimates by this method are optimal in the minimum mean square error (MMSE) sense. A limitation of the solution in [18], however, is that it is designed for nonlinear OFDM systems up to the third order. It has been shown that some nonlinear communication channels (such as the satellite communication channel in [13]) require a Volterra series model up to the 5th order. For these nonlinear channel identification problems, the solution in [18] is inadequate.

In this paper, we extend the development we have done in [18] for cubically nonlinear OFDM systems, so that a simple Volterra Kernel estimation solution can be acquired for nonlinear OFDM systems higher than the 3rd order. Moreover, a systematic method for generating pseudo random test sequences which guarantee the attainment of the optimal MMSE solution is also developed. By applying the proposed approach, we show that how a simple formula for a 5th-order nonlinear OFDM channel can be derived. The correctness of the derived formula is justified by computer simulation.

2. Nonlinear bandpass channel and Volterra series

Nonlinear bandpass channels are often resulted from power amplifiers in communication systems operating at the radio frequency range [20]. The AM/AM and AM/PM conversions of the power amplifier cause the bandpass communication system to become nonlinear. As mentioned in Section 1, the combination of the power amplifier and the channel response results in a nonlinear channel which can be described by a bandpass Volterra model. The goal of this paper is to perform channel estimation at the receiver by estimating the kernels of the bandpass Volterra model using the channel input and output.

It has been shown that the input and output of the nonlinear bandpass channel can be related by a discrete time-domain complex-valued Volterra model as follows [13]:

$$y[n]=\sum_{k=0}^{K}\sum_{n_1=0}^{N}\sum_{n_2=0}^{N}\cdots\sum_{n_{2k+1}=0}^{N}h_{2k+1}[n_1,n_2,\ldots,n_{2k+1}]\cdot\prod_{i=1}^{k+1}x[n-n_i]\prod_{j=k+2}^{2k+1}x^*[n-n_j]+e[n] \tag{1}$$

where $x[n]$ and $y[n]$ are the input and output of the nonlinear channel, $h_{2k+1}[n_1,n_2,\ldots,n_{2k+1}]$ is the $(2k+1)$th-order baseband equivalent Volterra kernel (will simply be referred to as the Volterra kernel hereafter), $e[n]$ is the modeling error, N is the memory length, and $2K+1$ is the order of the Volterra model. By taking the discrete Fourier transform (DFT) of (1), one obtains frequency-domain Volterra model expression as follows:

$$Y(m)=\sum_{k=0}^{K}\sum_{\substack{-M\le m_1,m_2,\cdots,m_{2k+1}\le M\\(m_1+m_2+\cdots+m_{2k+1}=m)}}H_{2k+1}(m_1,m_2,\cdots,m_{2k+1})\prod_{i=1}^{k+1}X(m_i)\prod_{j=k+2}^{2k+1}X^*(-m_j)+\epsilon(m)=\widehat{Y}(m)+\epsilon(m), \tag{2}$$

where $X(m)$ and $Y(m)$ are the DFTs of $x[n]$ and $y[n]$, respectively, $H_{2k+1}(m_1,m_2,\ldots,m_{2k+1})$ is the $(2k+1)$th-order frequency-domain Volterra kernel, $\widehat{Y}(m)$ is the model output, and $\epsilon(m)$ is the modeling error. Based on (2), the problem of nonlinear channel identification using the frequency-domain Volterra model is depicted in Fig. 1, where the goal is to determine the various Volterra kernels. As one can see in (2), the number of the Volterra kernel coefficients increases rapidly with the order of the Volterra model. This suggests that the difficulty of determining the Volterra kernels may also increase dramatically with the order of the Volterra model.

2.1. Symmetry properties of the Volterra kernels

For notational simplicity, we define the multi-dimensional indices $m|_i^j$ and $\underline{m}|_i^j$ as follows:

$$m|_i^j=(m_i,m_{i+1},\ldots,m_j) \tag{3}$$

$$\underline{m}|_i^j=\underline{(m_i,m_{i+1},\ldots,m_j)} \tag{4}$$

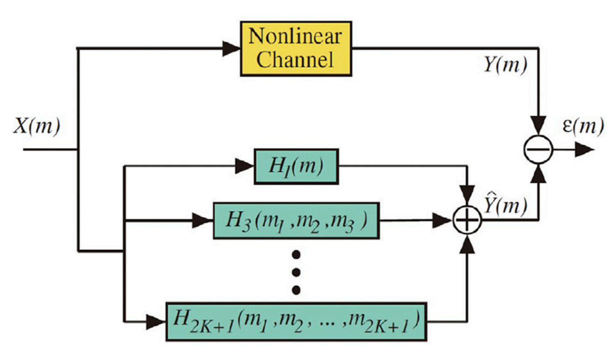

Fig. 1. The frequency-domain Volterra model for a nonlinear channel.

where the underline in (4) denotes any possible interchange of the indices. For instance,

$$m|_1^3 = (m_1, m_2, m_3), \tag{5}$$

and

$$\begin{aligned} \underline{m}|_1^3 &= (\underline{m_1, m_2, m_3}) \\ &\in \{(m_1,m_2,m_3),(m_1,m_3,m_2),(m_2,m_1,m_3), \\ &\quad (m_2,m_3,m_1),(m_3,m_1,m_2),(m_3,m_2,m_1)\} \end{aligned} \tag{6}$$

Using (3), the $(2k+1)$th-order Volterra kernel in (2) can be rewritten as

$$H_{2k+1}(m_1, m_2, \ldots, m_{2k+1}) = H_{2k+1}(m|_1^{k+1}, m_{k+2}^{2k+1}). \tag{7}$$

Since $H_{2k+1}(m|_1^{k+1}, m_{k+2}^{2k+1})$ and $H_{2k+1}(\underline{m}|_1^{k+1}, \underline{m}|_{k+2}^{2k+1})$ are both multiplied by the same input product term $\prod_{i=1}^{k+1} X(m_i) \prod_{j=k+2}^{2k+1} X^*(-m_j)$ in (2), one can assume without loss of generality that

$$H_{2k+1}(m|_1^{k+1}, m_{k+2}^{2k+1}) = H_{2k+1}(\underline{m}|_1^{k+1}, \underline{m}|_{k+2}^{2k+1}). \tag{8}$$

The total number of identical Volterra kernel coefficients in (8) varies with the number of distinct values in $m|_1^{k+1}$ and $m|_{k+2}^{2k+1}$. Assuming there are I distinct values in $m|_1^{k+1}$ with the i-th value appearing p_i times, we have

$$p_1 + p_2 + \ldots + p_I = k + 1. \tag{9}$$

The total number of permutations in $\underline{m}|_1^{k+1}$, say, $P(m|_1^{k+1})$, is equal to

$$P(m|_1^{k+1}) = \frac{I!}{p_1! p_2! \cdots p_I!} \tag{10}$$

where $I!$ is the factorial of I. Likewise, assuming there are J distinctive values in $m|_{k+2}^{2k+1}$ with the j-th value occurring q_j times, we obtain

$$q_1 + q_2 + \cdots + q_J = k \tag{11}$$

$$P(m|_{k+2}^{2k+1}) = \frac{J!}{q_1! q_2! \cdots q_J!} \tag{12}$$

Combing (10) and (12), we see that the total number of indistinguishable Volterra kernel coefficients in (8), say $P(m|_1^{k+1}, m_{k+2}^{2k+1})$, is

$$\begin{aligned} P(m|_1^{k+1}, m_{k+2}^{2k+1}) &= P(m|_1^{k+1}) P(m|_{k+2}^{2k+1}) \\ &= \frac{I! J!}{p_1! p_2! \cdots p_I! q_1! q_2! \cdots q_J!} \end{aligned} \tag{13}$$

By considering (8) and (13), the number of Volterra kernel coefficients requiring to be estimated can be significantly reduced.

3. OFDM signal and its spectral properties

In an OFDM communication system, the data are transmitted in parallel via equally-spaced orthogonal subcarriers, where the sequence of input data symbols to each subcarrier is often QAM or PSK. The summation of the parallel modulated signals constitutes the OFDM signal. Specifically, a baseband OFDM signal with $2M+1$ subcarriers at frequencies m/N, $m = -M, \ldots, M$ is usually generated by taking the inverse fast Fourier transform (IFFT) of the parallel QAM or PSK symbols as follows:

$$x[n] = \sum_{m=-M}^{M} X(m) e^{j2\pi mn/N}, \quad 0 \le t \le T \tag{14}$$

where $X(m)$ is the QAM or PSK complex data symbol for the subcarrier at the frequency m/N, and N is the number of FFT points.

The higher-order moments of the QAM and PSK signals exhibit certain properties due to their circularly symmetric characteristics [17,21]. Specifically, for a QAM signal of the form $X = a + jb$, where a and b are assume to be i.i.d. random variables with a symmetric distribution, we have [17]

$$E[X^p X^{*q}] = \begin{cases} E\left[|X|^{2\min(p,q)} \cdot X^{|p-q|}\right], \text{if } |p-q| \bmod 4 = 0 \\ 0, \text{ otherwise} \end{cases} \tag{15}$$

This can be explained as follows. If $p \ge q$, than

$$E[X^p X^{*q}] = E\left[|X|^{2q} X^{(p-q)}\right] \tag{16}$$

If on the contrary $p < q$, than

$$E[X^p X^{*q}] = E\left[|X|^{2p} X^{*(q-p)}\right] \tag{17}$$

Note that, due to the circularly symmetric characteristics of the QAM signal, its symbols must appear in complex conjugate pairs. That is, for any symbol $X_0 = r_0 e^{j\theta_0}$, there must exist another symbol $X_0' = r_0 e^{-j\theta_0}$. The contribution of these two symbols to the expectation in Eq. (17) can be expressed as

$$\begin{aligned} &|X_0|^{2p} X_0^{*(q-p)} + |X_0'|^{2p} (X_0')^{*(q-p)} \\ &\quad = r_0^{2p} r_0^{(q-p)} \left[e^{-j(q-p)\theta_0} + e^{j(q-p)\theta_0}\right] \\ &\quad = 2 r_0^{p+q} \cos[(q-p)\theta], \end{aligned} \tag{18}$$

which is a real number. This indicates that, the expectation term in Eq. (17) must also be real. Since

$x = x^*$ when x is real, we see that Eq. (17) can be written as

$$\begin{aligned} E[X^pX^{*q}] &= E[|X|^{2p}X^{*(q-p)}] \\ &= E[|X|^{2p}X^{*(q-p)}]^* \\ &= E[|X|^{2p}X^{(q-p)}] \end{aligned} \tag{19}$$

Combining Eqs. (16) and (19) one obtains

$$E[X^pX^{*q}] = E\left[|X|^{2\min(p,q)} \cdot X^{|p-q|}\right] \tag{20}$$

In addition, since $E[X^pX^{*q}]$ is real, we have $E[X^pX^{*q}] = E[X^pX^{*q}]^* = E[X^qX^{*p}]$ This indicates that interchanging p and q does not alter the result of $E[X^pX^{*q}]$ Based on the observation, we will only discuss the case of $p \geq q$.

Assuming that $(p-q) \bmod 4 = i$, one can write $p-q = 4m+i$ for some integers m and i. Note that $i \in \{0,1,2,3\}$. Now we want to show that Eq. (20) is equal to zero for $(p-q) \bmod 4 \neq 0$. First we start with the 4QAM. Its symbols are on a circle of a radius r in the complex plane and can be denoted by

$$X = re^{j\theta},\ \theta \in \left\{\theta_0 + \frac{k\pi}{2} \middle| k = 0,1,2,3\right\} \tag{21}$$

where (r, θ_0) is the Polar coordinate of the symbol in the first quadrant. By substituting (21) into (20), one obtains

$$\begin{aligned} E[X^pX^{*q}] &= E[r^{2q}r^{(p-q)}e^{j(p-q)\theta}] \\ &= r^{p+q}E[e^{j(4m+i)\theta}] \\ &= \frac{r^{p+q}e^{j(4m+i)\theta_0}}{4}\sum_{k=0}^{3} e^{j\left(\frac{ik\pi}{2}\right)} \\ &= \begin{cases} r^{p+q}e^{j4m\theta_0}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{22}$$

This justifies (15) for 4QAM. The result can be easily extended to higher-level QAMs by recognizing that a higher-level QAM can be divided into subsets of 4 QAMs. For example, the 16 QAM in Fig. 2 is divided into 4 subsets of 4QAMs with different graphic symbols. Since the contributions from the 4 data symbols in each subset would cancel out, we see that higher-level QAMs also satisfy (15).

For an $M-\text{PSK}$ signal of the form $X = r \cdot \exp^{j(2\pi m/M)}$, $m = 0,1,\ldots,M-1$, we have [17]

$$E[X^pX^{*q}] = \begin{cases} r^{p+q}, & \text{if } |p-q| \bmod M = 0 \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

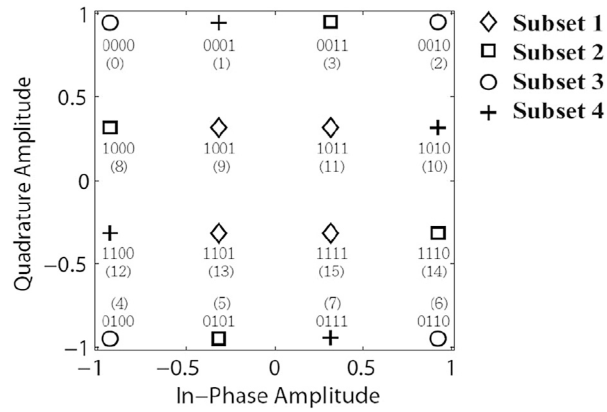

Fig. 2. The division of 16-QAM constellation into 4 subsets of 4QAMs with different graphic symbols. The 16QAM is used for the OFDM subcarriers in the simulation. Each data symbol is labeled with its 4-bit representation and its symbol number inside a parenthesis.

This can be explained as follows. Assuming that $p \geq q$. and $(p-q) \bmod M = i$, one can write $p-q = kM+i$ for some integers k and i. Note that $i \in \{0,1,2,\ldots,M-1\}$. By substituting the $M-\text{PSK}$ symbol $X = re^{j(2\pi m/M)}$, $m = 0,1,\ldots,M-1$ into $E[X^pX^{*q}]$, we obtain

$$\begin{aligned} E[X^pX^{*q}] &= \left[\sum_{m=0}^{M-1}\left(re^{j(2\pi m/M)}\right)^p\left(re^{j(2\pi m/M)}\right)^{*q}\right] \Big/ M \\ &= \frac{r^{p+q}}{M}\sum_{m=0}^{M-1} e^{j(2\pi m/M)(p-q)} \\ &= \frac{r^{p+q}}{M}\sum_{m=0}^{M-1} e^{j(2\pi m/M)(kM+i)} = \frac{r^{p+q}}{M}\sum_{m=0}^{M-1} e^{j(2\pi im/M)} \\ &= \begin{cases} r^{p+q}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{24}$$

For $p < q$ and $(q-p) \bmod M = i$, one can write $q-p = kM+i$ for some integers k and i. We have

$$\begin{aligned} E[X^pX^{*q}] &= E[X^{p*}X^q]^* = E[X^qX^{p*}]^* \\ &= \begin{cases} r^{q+p}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \tag{25}$$

where (24) has been used in deriving (25). The combination of (24) and (25) justifies (23).

From (15) and (23) we see that, for both QAM and PSK signals, most of their higher-order moments are equal to zero. These properties may come in handy when we develop the higher-order moment spectra of the OFDM signal.

The derivation of the proposed Volterra kernel identification method in this paper is based on analyzing the higher-order auto-moment spectra of the OFDM signal. As we have seen from (14), the DFT of the OFDM signal at any frequency m (i.e., $X(m)$) is equal to the complex data symbol (taken from the QAM or PSK alphabet) fed to the m-th subcarrier. Therefore, the higher-order auto-moment spectra of the OFDM signal is composed of the higher-order moments of the QAM or PSK signals. By using the higher-order moment properties of QAM and PSK given by (15) and (23), respectively, we can obtain the higher-order auto-moment spectra of the OFDM signal.

Since the QAM or PSK symbols fed to different subcarriers are independent, a higher-order auto-moment spectrum of the OFDM signal with k distinct frequencies has the following property:

$$E[f_1(X(m_1))\bullet f_2(X(m_2))\cdots f_k(X(m_k))] = E[f_1(X(m_1))]\bullet E[f_2(X(m_2))]\cdots E[f_k(X(m_k))], \tag{26}$$

where m_1, m_2, …, m_k are the k distinct frequencies, and $f_i(X)$ $(i = 1,\ldots,k)$ denotes an arbitrary function of X. Note that the separability of the expectation in (26) is due to the independency of the QAM and PSK symbols among different subcarriers, and each separated expectation term $E[f_i(X(m_i))]$ can be determined via (15) or (23).

4. Identification of the Volterra kernels for nonlinear bandpass channels

In this section we will derive the optimal MMSE estimate of the Volterra kernels in (2). That is to determine the optimal $H_{2k+1}(m_1,m_2,\ldots,m_{2k+1})$ (where $-M\le m_1,m_2,\ldots, m_{2k+1}\le M$ and $m_1+ m_2+\cdots+m_{2k+1}=m$), $k= 0, 1, \ldots, K$which minimize the cost function

$$J=E\left[|\epsilon(m)|^2\right] = E\left[|Y(m)-\widehat{Y}(m)|^2\right]=E\left[\left|Y(m)-\sum_{k=0}^{K}\sum_{\substack{-M\le m_1,m_2,\cdots,m_{2k+1}\le M\\(m_1+m_2+\cdots+m_{2k+1}=m)}}H_{2k+1}(m_1,m_2,\cdots,m_{2k+1})\prod_{i=1}^{k+1}X(m_i)\prod_{j=k+2}^{2k+1}X^*(-m_j)\right|^2\right] \tag{27}$$

Note that (2) has been used in deriving (27). By applying the orthogonality principle [22], one can see that the cost function J is minimized when the error $\epsilon(m)$ is orthogonal to all the input terms in (2). This can be stated mathematically as follows:

$$E\left[\prod_{i=1}^{k+1}X^*(m_i)\prod_{j=k+2}^{2k+1}X(-m_j)\epsilon(m)\right]=0, -M\le m_1,m_2,\ldots, m_{2k+1}\le M, m_1+m_2+\cdots+m_{2k+1}=m,\ 0\le k\le K \tag{28}$$

By using (2) one can rewrite (28) as

$$E\left[\prod_{i=1}^{k+1}X^*(m_i)\prod_{j=k+2}^{2k+1}X(-m_j)Y(m)\right]=\sum_{k'=0}^{K}\sum_{\substack{-M\le m'_1,m'_2,\cdots,m'_{2k+1}\le M\\(m'_1+m'_2+\cdots+m'_{2k+1}=m)}} H_{2k'+1}(m'_1,m'_2,\cdots,m'_{2k'+1})\bullet E\left[\prod_{i=1}^{k+1}X^*(m_i)\prod_{j=k+2}^{2k+1}X(-m_j)\prod_{i=1}^{k'+1}X(m'_i)\prod_{j=k'+2}^{2k'+1}X^*(-m'_j)\right], -M\le m_1,m_2,\ldots,m_{2k+1}\le M, m_1+m_2+\cdots+m_{2k+1}=m,\ 0\le k\le K \tag{29}$$

To obtain the MMSE estimates of the Volterra kernels, one needs to solve the system of equations in the form of (29) for $-M\le m_1,m_2,\ldots,m_{2k+1}\le M$ (subject to the constraint $m_1+m_2+\cdots+m_{2k+1}=m$) and k=0,1,…,K simultaneously. This is a very challenging task due to the extremely large number of kernel coefficients involved in (29). Note that the term expressed as the expectation of the input product on the righthand side of (29) (i.e., $E[\prod_{i=1}^{k+1}X^*(m_i)\prod_{j=k+2}^{2k+1}X(-m_j)\prod_{i=1}^{k'+1}X(m'_i)\prod_{j=k'+2}^{2k'+1}X^*(-m'_j)]$) is in fact a higher-order auto-moment spectrum [23] of the OFDM signal in the form of (26). Suppose there are L distinct frequencies involved in the expectation term, and the distinct frequencies are denoted by n_1, n_2, …, n_L. By applying (26), the expectation term can be rewritten as:

$$E\left[\prod_{i=1}^{k+1}X^*(m_i)\prod_{j=k+2}^{2k+1}X(-m_j)\prod_{i=1}^{k'+1}X(m'_i)\prod_{j=k'+2}^{2k'+1}X^*(-m'_j)\right]= E[X^{p_1}(n_1)X^{*q_1}(n_1)]\bullet E[X^{p_2}(n_2)X^{*q_2}(n_2)]\cdots E[X^{p_L}(n_L)X^{*q_L}(n_L)] \tag{30}$$

where p_l and q_l $(l = 1,2,\ldots,L)$ fulfill

$$p_1+\cdots+p_L=k+k'+1, \tag{31}$$

$$q_1+\cdots+q_L=k+k'+1. \tag{32}$$

For (30) to be nonzero, each of the individual expectations $E[X^{p_l}(n_l)X^{*q_l}(n_l)]$, $l=1,\ldots,L$ on the righthand side of (30) must be nonzero. However, one can see from (15) and (23) that $E[X^{p_l}(n_l)X^{*q_l}(n_l)]$ is equal to zero under most circumstances. This implies that, given a combination of $(m_1,\ldots,m_{k+1},-m_{k+2},\ldots,-m_{2k+1})$, Eq. (30) will be nonzero only for few combinations of $(m'_1,\ldots,m'_{k'+1},-m'_{k'+2},\ldots,-m'_{2k'+1})$, which in turn implies that at most only few Volterra kernel coefficients in the form of $H_{2k'+1}(m'_1,m'_2,\cdots,m'_{2k'+1})$ can actually survive in the expression of (29). This makes the derivation of a simple closed-from solution for the Volterra kernels from the system equations given by (29) possible.

Based on the above analysis, in the following we consider the derivation of the optimal MMSE estimate of the Volterra kernels for a 5th-order nonlinear OFDM channel. In this case, the input and output of the channel is related by (2) with $K=2$, hence we need to solve the equations described by (29) with $K=2$ and $k=0,1,2$. We assume the QAM is adopted in the subchannels of OFDM in the derivation. The PSK counterpart is simply a special case of the derivation. By defining

$$\mu_n(m)=E[|X(m)|^n],\ n=2,4,6,8,10, \tag{33}$$

We consider different combinations of (k,k') in (30) as follows.

For $(k,k')=(0,0)$ in (30), we have

$$E[X^*(m)X(m)]=\mu_2(m) \tag{34}$$

For $(k,k')=(0,1)$ in (30), we have

$$E\left[X^*(m)X(m'_1)X(m'_2)X^*(-m'_3)\right]=\begin{cases}\mu_4(m), & \langle m,m'_1,m'_2,-m'_3\rangle \quad \cdots 1\\ \mu_2(m)\mu_2(m'_2), & \langle m,m'_1\rangle\langle m'_2,-m'_3\rangle \quad \cdots 2,\\ 0, & otherwise\end{cases} \tag{35}$$

where (15) has been used in deriving (35). In addition, we have used the notation $\langle\ \rangle$ to denote that all the indices within the same $\langle\ \rangle$ are equal, but the indices in different $\langle\ \rangle$ are different. Furthermore, one should keep in mind that m'_1 and m'_2 (but not $-m'_3$) are interchangeable.

For $(k,k')=(0,2)$ in (30), we apply (15) to obtain

$$E\left[X^*(m)X(m'_1)X(m'_2)X(m'_3)X^*(-m'_4)X^*(-m'_5)\right]=\begin{cases}\mu_6(m), & \langle m,m'_1,m'_2,m'_3,-m'_4,-m'_5\rangle\\ \mu_4(m'_1)\mu_2(m), & \langle m,m'_3\rangle\langle m'_1,m'_2,-m'_4,-m'_5\rangle\\ \mu_4(m)\mu_2(m'_1), & \langle m,m'_2,m'_3,-m'_4\rangle\langle m'_1,-m'_5\rangle\\ \mu_2(m)\mu_2(m'_1)\mu_2(m'_2), & \langle m,m'_3\rangle\langle m'_1,-m'_5\rangle\langle m'_2,-m'_4\rangle\\ 0, & otherwise\end{cases} \tag{36}$$

Similarly in (36), it is understood that m'_1, m'_2, and m'_3 are interchangeable, and $-m'_4$ and $-m'_5$ are interchangeable.

For $(k,k')=(1,1)$ in (30), we again apply (15) to yield

$$E\left[X^*(m_1)X^*(m_2)X(-m_3)X(m'_1)X(m'_2)X^*(-m'_3)\right]=\begin{cases}\mu_6(m), & \langle m,m_1,m_2,-m_3,m'_1,-m'_2,-m'_3\rangle \quad \cdots 1\\ \mu_4(m_1)\mu_2(-m_3), & \langle m_1,m'_1,m_2,m'_2\rangle\langle -m_3,-m'_3\rangle \quad \cdots 2\\ \mu_4(m_2)\mu_2(m), & \langle m,m_1,m'_1\rangle\langle m_2,-m_3,m'_2,-m'_3\rangle \quad \cdots 3\\ \mu_4(m)\mu_2(m_{2'}), & \langle m,m_1,m_2,-m_3,m'_1\rangle\langle m'_2,-m'_3\rangle \quad \cdots 4\\ \mu_4(m)\mu_2(m_2), & \langle m,m'_1,m'_2,-m_3,m_1\rangle\langle m_2,-m'_3\rangle \quad \cdots 5\\ \mu_2(m_1)\mu_2(m_2)\mu_2(-m_3), & \langle m_1,m'_1\rangle\langle m_2,m'_2\rangle\langle -m_3,-m'_3\rangle \quad \cdots 6\\ \mu_2(m)\mu_2(m_2)\mu_2(-m_3), & \langle m,m_1,m'_1\rangle\langle m_2,-m_3\rangle\langle m'_2,-m'_3\rangle \quad \cdots 7\\ 0, & otherwise\end{cases} \tag{37}$$

In (37), it is understood that m_1 and m_2 are interchangeable, and m'_1 and m'_2 are interchangeable.

For $(k,k')=(1,2)$ in (30), we follow the same scenario to get

$$
\begin{aligned}
&E\left[X^*(m_1)X^*(m_2)X(-m_3)X(m_1')X(m_2')X(m_3')X^*(-m_4')X^*(-m_5')\right] = \\
&\begin{cases}
\mu_8(m_1), \\
\langle m_1, m_2, -m_3, m_1', m_2', m_3', -m_4', -m_5'\rangle & \cdots 1 \\
\mu_6(m_1)\mu_2(m_3'), \\
\langle m_1, m_2, -m_3, m_1', m_2', -m_5'\rangle\langle m_3', -m_4'\rangle & \cdots 2 \\
\mu_6(m_1)\mu_2(m_2), \\
\langle m_1, m_1', m_2', m_3', -m_4', -m_5'\rangle\langle m_2, -m_3\rangle, & \cdots 3 \\
\langle m_1, -m_3, m_1', m_2', -m_4', -m_5'\rangle\langle m_2, m_3'\rangle & \cdots 4 \\
\mu_6(m_1)\mu_2(-m_3), \\
\langle m_1, m_2, m_1', m_2', m_3', -m_5'\rangle\langle -m_3, -m_4'\rangle & \cdots 5 \\
\mu_4(m_1)\mu_4(m_2'), \\
\langle m_1, m_2, -m_3, m_1'\rangle\langle m_2', m_3', -m_4', -m_5'\rangle & \cdots 6 \\
\mu_4(m_1)\mu_4(m_2), \\
\langle m_1, m_1', m_2', -m_5'\rangle\langle m_2, -m_3, m_3', -m_4'\rangle & \cdots 7 \\
\mu_4(m_1)\mu_4(-m_3), \\
\langle m_1, m_2, m_1', m_2'\rangle\langle -m_3, m_3', -m_4', -m_5'\rangle & \cdots 8 \\
\mu_4(m_1)\mu_2(m_2')\mu_2(m_3'), \\
\langle m_1, m_2, -m_3, m_1'\rangle\langle m_2', -m_4'\rangle\langle m_3', -m_5'\rangle & \cdots 9 \\
\mu_4(m_1)\mu_2(m_2)\mu_2(m_3'), \\
\langle m_1, m_1', m_2', -m_4'\rangle\langle m_2, -m_3\rangle\langle m_3', -m_5'\rangle, & \cdots 10 \\
\langle m_1, -m_3, m_1', -m_4'\rangle\langle m_2, m_2'\rangle\langle m_3', -m_5'\rangle & \cdots 11 \\
\mu_4(m_1)\mu_2(-m_3)\mu_2(m_3'), \\
\langle m_1, m_2, m_1', m_2'\rangle\langle -m_3, -m_4'\rangle\langle m_3', -m_5'\rangle & \cdots 12 \\
\mu_4(m_1)\mu_2(m_2)\mu_2(-m_3), \\
\langle m_1, m_1', m_2', -m_4'\rangle\langle m_2, m_3'\rangle\langle -m_3, -m_5'\rangle & \cdots 13 \\
\mu_4(-m_3)\mu_2(m_1)\mu_2(m_2), \\
\langle -m_3, m_1', -m_4', -m_5'\rangle\langle m_1, m_2'\rangle\langle m_2, m_3'\rangle & \cdots 14 \\
\mu_4(m_1')\mu_2(m_1)\mu_2(m_2), \\
\langle m_1', m_2', -m_4', -m_5'\rangle\langle m_1, m_3'\rangle\langle m_2, -m_3\rangle & \cdots 15 \\
\mu_2(m_1)\mu_2(m_2)\mu_2(m_2')\mu_2(m_3'), \\
\langle m_1, m_1'\rangle\langle m_2, -m_3\rangle\langle m_2', -m_4'\rangle\langle m_3', -m_5'\rangle & \cdots 16 \\
\mu_2(m_1)\mu_2(m_2)\mu_2(-m_3)\mu_2(m_3'), \\
\langle m_1, m_1'\rangle\langle m_2, m_2'\rangle\langle -m_3, -m_4'\rangle\langle m_3', -m_5'\rangle & \cdots 17 \\
0, \ \textit{otherwise}
\end{cases}
\end{aligned}
\tag{38}
$$

In (38), it is understood that m_1 and m_2 are interchangeable, m_1', m_2', and m_3' are interchangeable, and $-m_4'$ and $-m_5'$ are interchangeable. Notice that there are two cares which lead to the same result $\mu_6(m_1)\mu_2(m_2)$, they are listed separatelyar (38-3) and (38-4). Similarly, the two cares (38-10) and (38-11) both yield the same result $\mu_4(m_1)\mu_2(m_2)\mu_2(m_3')$ and are also listed separately.

Finally, for $(k, k') = (2, 2)$ in (30), the analysis can be categorized into the following cases.

Case 1: all the frequency indices are equal, we will refer to this case as the $\langle 10\rangle$ case hereafter. In this case,

$$
\begin{aligned}
\text{Eq.}(30) = {} & E\left[X^*(m_1)X^*(m_2)X^*(m_3)X(-m_4)X(-m_5)\right. \\
& \left. X(m_1')X(m_2')X(m_3')X^*(-m_4')X^*(-m_5')\right] \\
= {} & \mu_{10}(m_1), \ \langle m_1, m_2, m_3, -m_4, -m_5, m_1', m_2', m_3', \\
& -m_4', -m_5'\rangle
\end{aligned}
\tag{39}
$$

Case 2: the $\langle 8\rangle\langle 2\rangle$ case. That is, eight of the ten frequency indices are equal but different from the other two, and the rest two frequency indices are equal. In this case,

$$
\text{Eq. }(30) =
\begin{cases}
\mu_8(m_1)\mu_2(m_3'), \\
\langle m_1, m_2, m_3, -m_4, -m_5, m_1', m_2' - m_4'\rangle\langle m_3', -m_5'\rangle \cdots 1 \\
\mu_8(m_1)\mu_2(m_3), \\
\langle m_1, m_2, -m_4, m_1', m_2', m_3', -m_4', -m_5'\rangle\langle m_3, -m_5\rangle, \cdots 2 \\
\langle m_1, m_2, -m_4, -m_5, m_1', m_2', -m_4', -m_5'\rangle\langle m_3, m_3'\rangle \cdots 3 \\
\mu_8(m_1)\mu_2(-m_5), \\
\langle m_1, m_2, m_3, -m_4, m_1', m_2', m_3', -m_4'\rangle\langle -m_5, \ -m_5'\rangle \cdots 4 \\
0, \qquad \textit{otherwise}
\end{cases}
\tag{40}
$$

Case 3: the $\langle 6\rangle\langle 4\rangle$ case.

$$\text{Eq.}(30) = \begin{cases} \mu_6(m_1)\mu_4(m_2'), & \\ \langle m_1,m_2,m_3,-m_4,-m_5,m_1'\rangle\langle m_2',m_3',-m_4',-m_5'\rangle & \cdots 1 \\ \mu_6(m_1)\mu_4(-m_5), & \\ \langle m_1,m_2,m_3,-m_4,m_1',m_2'\rangle\langle -m_5,m_3',-m_4',-m_5'\rangle, & \cdots 2 \\ \langle m_1,m_2,m_3,m_1',m_2',m_3'\rangle\langle -m_4,-m_5,-m_4',-m_5'\rangle & \cdots 3 \\ \mu_6(m_1)\mu_4(m_3), & \\ \langle m_1,m_2,m_1',m_2',m_3',-m_4'\rangle\langle m_3,-m_4,-m_5,-m_5'\rangle, & \cdots 4 \\ \langle m_1,-m_4,-m_5,m_1',-m_4',-m_5'\rangle\langle m_2,m_3,m_2',m_3'\rangle, & \cdots 5 \\ \langle m_1,m_2,-m_4,-m_5,m_1',-m_4'\rangle\langle m_3,m_2',m_3',-m_5'\rangle, & \cdots 6 \\ \langle m_1,m_1',m_2',m_3',-m_4',-m_5'\rangle\langle m_2,m_3,-m_4,-m_5\rangle, & \cdots 7 \\ \langle m_1,-m_4,m_1',m_2',-m_4',-m_5'\rangle\langle m_2,m_3,-m_5,m_3'\rangle, & \cdots 8 \\ \langle m_1,m_2,-m_4,m_1',m_2',-m_4'\rangle\langle m_3,-m_5,m_3',-m_5'\rangle & \cdots 9 \\ 0, \quad \textit{otherwise} & \end{cases} \tag{41}$$

Case 4: the $\langle 6\rangle\langle 2\rangle\langle 2\rangle$ case.

$$\text{Eq.}(30) = \begin{cases} \mu_6(m_1)\mu_2(m_2')\mu_2(m_3'), & \\ \langle m_1,m_2,m_3,-m_4,-m_5,m_1'\rangle\langle m_2',-m_4'\rangle\langle m_3',-m_5'\rangle & \cdots 1 \\ \mu_6(m_1)\mu_2(-m_5)\mu_2(m_3'), & \\ \langle m_1,m_2,m_3,-m_4,m_1',m_2'\rangle\langle -m_5,-m_4'\rangle\langle m_3',-m_5'\rangle & \cdots 2 \\ \mu_6(m_1)\mu_2(m_3)\mu_2(m_3'), & \\ \langle m_1,m_2,-m_4,m_1',m_2',-m_4'\rangle\langle m_3,-m_5\rangle\langle m_3',-m_5'\rangle, & \cdots 3 \\ \langle m_1,m_2,-m_4,-m_5,m_1',-m_4'\rangle\langle m_3,m_2'\rangle\langle m_3',-m_5'\rangle & \cdots 4 \\ \mu_6(m_1)\mu_2(-m_4)\mu_2(-m_5), & \\ \langle m_1,m_2,m_3,m_1',m_2',m_3'\rangle\langle -m_4,-m_4'\rangle\langle -m_5,-m_5'\rangle & \cdots 5 \\ \mu_6(m_1)\mu_2(m_3)\mu_2(-m_5), & \\ \langle m_1,m_2,m_1',m_2',m_3',-m_4'\rangle\langle m_3,-m_4\rangle\langle -m_5,-m_5'\rangle, & \cdots 6 \\ \langle m_1,m_2,-m_4,m_1',m_2',-m_4'\rangle\langle m_3,m_3'\rangle\langle -m_5,-m_5'\rangle & \cdots 7 \\ \mu_6(m_1)\mu_2(m_2)\mu_2(m_3), & \\ \langle m_1,-m_4,m_1',m_2',-m_4',-m_5'\rangle\langle m_2,-m_5\rangle\langle m_3,m_3'\rangle, & \cdots 8 \\ \langle m_1,m_1',m_2',m_3',-m_4',-m_5'\rangle\langle m_2,-m_4\rangle\langle m_3,-m_5\rangle, & \cdots 9 \\ \langle m_1,-m_4,-m_5,m_1',-m_4',-m_5'\rangle\langle m_2,m_2'\rangle\langle m_3,m_3'\rangle & \cdots 10 \\ 0, \quad \textit{otherwise} & \end{cases} \tag{42}$$

Case 5: the $\langle 4\rangle\langle 4\rangle\langle 2\rangle$ case.

$$\text{Eq.}(30) = \begin{cases} \mu_4(m_1)\mu_4(m_2)\mu_2(m_3), & \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,-m_5,m_2',-m_5'\rangle\langle m_3,m_3'\rangle, & \cdots 1 \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,m_2',m_3',-m_5'\rangle\langle m_3,-m_5\rangle, & \cdots 2 \\ \langle m_1,-m_4,-m_5,-m_4'\rangle\langle m_2,m_1',m_2',-m_5'\rangle\langle m_3,m_3'\rangle & \cdots 3 \\ \mu_4(m_1)\mu_4(m_2)\mu_2(-m_5), & \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,m_3,m_2',m_3'\rangle\langle -m_5,-m_5'\rangle, & \cdots 4 \\ \langle m_1,m_2,-m_4,m_1'\rangle\langle m_3,m_2',m_3',-m_4'\rangle\langle -m_5,-m_5'\rangle & \cdots 5 \\ \mu_4(m_1)\mu_4(-m_5)\mu_2(m_3), & \\ \langle m_1,m_2,-m_4,m_1'\rangle\langle -m_5,m_2',-m_4',-m_5'\rangle\langle m_3,m_3'\rangle, & \cdots 6 \\ \langle m_1,m_2,m_1',m_2'\rangle\langle -m_4,-m_5,-m_4',-m_5'\rangle\langle m_3,m_3'\rangle, & \cdots 7 \\ \langle m_1,m_2,m_1',m_2'\rangle\langle -m_4,m_3',-m_4',-m_5'\rangle\langle m_3,-m_5\rangle & \cdots 8 \\ \mu_4(m_1)\mu_4(m_2)\mu_2(m_3'), & \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,m_3,-m_5,m_2'\rangle\langle m_3',-m_5'\rangle, & \cdots 9 \\ \langle m_1,m_2,-m_4,-m_5\rangle\langle m_3,m_1',m_2',-m_4'\rangle\langle m_3',-m_5'\rangle, & \cdots 10 \\ \langle m_1,-m_4,-m_5,-m_4'\rangle\langle m_2,m_3,m_1',m_2'\rangle\langle m_3',-m_5'\rangle & \cdots 11 \\ \mu_4(m_1)\mu_4(m_2')\mu_2(m_3), & \\ \langle m_1,m_2,-m_4,m_1'\rangle\langle m_2',m_3',-m_4',-m_5'\rangle\langle m_3,-m_5\rangle, & \cdots 12 \\ \langle m_1,m_2,-m_4,-m_5\rangle\langle m_1',m_2',-m_4',-m_5'\rangle\langle m_3,m_3'\rangle & \cdots 13 \\ 0, \quad \textit{otherwise} & \end{cases} \tag{43}$$

Case 6: the $\langle 4\rangle\langle 2\rangle\langle 2\rangle\langle 2\rangle$ case.

$$\mathrm{Eq.}(30) = \begin{cases} \mu_4(m_1)\mu_2(m_2)\mu_2(m_3)\mu_2(m_3'), \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,-m_5\rangle\langle m_3,m_2'\rangle\langle m_3',-m_5'\rangle, & \cdots 1 \\ \langle m_1,m_1',m_2',-m_4'\rangle\langle m_2,-m_4\rangle\langle m_3,-m_5\rangle\langle m_3',-m_5'\rangle, & \cdots 2 \\ \langle m_1,-m_4,-m_5,-m_4'\rangle\langle m_2,m_1'\rangle\langle m_3,m_2'\rangle\langle m_3',-m_5'\rangle & \cdots 3 \\ \mu_4(m_1')\mu_2(m_1)\mu_2(m_2)\mu_2(m_3), \\ \langle m_1',m_2',-m_4',-m_5'\rangle\langle m_1,-m_4\rangle\langle m_2,-m_5\rangle\langle m_3,m_3'\rangle & \cdots 4 \\ \mu_4(m_1)\mu_2(m_3)\mu_2(m_2')\mu_2(m_3'), \\ \langle m_1,m_2,-m_4,m_1'\rangle\langle m_3,-m_5\rangle\langle m_2',-m_4'\rangle\langle m_3',-m_5'\rangle, & \cdots 5 \\ \langle m_1,m_2,-m_4,-m_5\rangle\langle m_3,m_1'\rangle\langle m_2',-m_4'\rangle\langle m_3',-m_5'\rangle & \cdots 6 \\ \mu_4(m_1)\mu_2(-m_4)\mu_2(m_3')\mu_2(m_3), \\ \langle m_1,m_2,m_1',m_2'\rangle\langle -m_4,-m_4'\rangle\langle m_3',-m_5'\rangle\langle m_3,-m_5\rangle, & \cdots 7 \\ \langle m_1,m_2,-m_4,m_1'\rangle\langle -m_5,-m_4'\rangle\langle m_2',-m_5'\rangle\langle m_3,m_3'\rangle & \cdots 8 \\ \mu_4(m_1)\mu_2(m_3)\mu_2(-m_4)\mu_2(-m_5), \\ \langle m_1,m_2,m_1',m_2'\rangle\langle m_3,m_3'\rangle\langle -m_4,-m_4'\rangle\langle -m_5,-m_5'\rangle & \cdots 9 \\ \mu_4(m_1)\mu_2(m_2)\mu_2(m_3)\mu_2(-m_5), \\ \langle m_1,-m_4,m_1',-m_4'\rangle\langle m_2,m_2'\rangle\langle m_3,m_3'\rangle\langle -m_5,-m_5'\rangle, & \cdots 10 \\ \langle m_1,m_1',m_2',-m_4'\rangle\langle m_2,-m_4\rangle\langle m_3,m_3'\rangle\langle -m_5,-m_5'\rangle & \cdots 11 \\ \mu_4(-m_4)\mu_2(m_1)\mu_2(m_2)\mu_2(m_3), \\ \langle -m_4,m_1',-m_4',-m_5'\rangle\langle m_1,-m_5\rangle\langle m_2,m_2'\rangle\langle m_3,m_3'\rangle, & \cdots 12 \\ \langle -m_4,-m_5,-m_4',-m_5'\rangle\langle m_1,m_1'\rangle\langle m_2,m_2'\rangle\langle m_3,m_3'\rangle & \cdots 13 \\ 0, & otherwise \end{cases} \tag{44}$$

Case 7: the $\langle 2\rangle\langle 2\rangle\langle 2\rangle\langle 2\rangle\langle 2\rangle$ case.

$$\mathrm{Eq.}(30) = \begin{cases} \mu_2(m_1)\mu_2(m_2)\mu_2(m_3)\mu_2(-m_5)\mu_2(m_3'), \\ \langle m_1,-m_4\rangle\langle m_2,m_1'\rangle\langle m_3,m_2'\rangle\langle -m_5,-m_4'\rangle\langle m_3',-m_5'\rangle & \cdots 1 \\ \mu_2(m_1)\mu_2(m_2)\mu_2(m_3)\mu_2(m_2')\mu_2(m_3'), \\ \langle m_1,-m_4\rangle\langle m_2,-m_5\rangle\langle m_3,m_1'\rangle\langle m_2',-m_4'\rangle\langle m_3',-m_5'\rangle & \cdots 2 \\ \mu_2(m_1)\mu_2(m_2)\mu_2(m_3)\mu_2(-m_4)\mu_2(-m_5), \\ \langle m_1,m_1'\rangle\langle m_2,m_2'\rangle\langle m_3,m_3'\rangle\langle -m_4,-m_4'\rangle\langle -m_5,-m_5'\rangle & \cdots 3 \\ 0, & otherwise \end{cases} \tag{45}$$

Case 8: other than Cases 1 to 7.

$$\mathrm{Eq.}(30) = 0 \tag{46}$$

One should keep in mind that, In (39)–(45), we implicitly imply m_1, m_2, and m_3 are interchangeable, $-m_4$ and $-m_5$ are interchangeable, m_1', m_2', and m_3' are interchangeable, and $-m_4'$ and $-m_5'$ are interchangeable.

Note that only $k \le k'$ is considered in the above derivation, one can easily see that the result for the $k > k'$ case is simply the conjugate of that for the $k < k'$ case.

5. Derivation of the formula for 5th-order nonlinear OFDM channels

The analysis in Section 4 indicates that, most of the higher-order auto-moment spectra in (29) are equal to zero, and the remaining non-zero ones can be expressed in terms of $\mu_n(m)$, $n = 2,4,6,8,10$, as shown in (34)–(45).

By substituting the obtained higher-order auto-moment spectra into (29), we are able to solve the system equations in the form of (29) and obtain a simple formula for the Volterra kernels. As explained in (8)–(13), there are some indistinguishable Volterra kernel coefficients due to their symmetry properties. To simplify the expression of equations in the derivation, we define the following modified Volterra kernels:

$$\mathscr{H}_3(\underline{m}|_1^2, m_3) = H_3(m|_1^2, m_3)P(m|_1^2, m_3) \tag{47}$$

$$\mathscr{H}_5(\underline{m}|_1^3, \underline{m}|_4^5) = H_5(m|_1^3, m|_4^5)P(m|_1^3, m|_4^5) \tag{48}$$

where $P(m|_1^2, m_3)$ and $P(m|_1^3, m|_4^5)$ are defined in (13). The modified Volterra kernels are simply the sums of their corresponding original Volterra kernels which are considered as the same under the symmetry properties. For example, the two kernel coefficients $H_3(1,2,3)$ and $H_3(2,1,3)$ are multiplied by the same input product $X(1)X(2)X^*(3)$ and hence are considered as equal. Therefore, $\mathscr{H}_3(\underline{1,2},3) = 2H_3(1,2,3)$. Similarly, there are 12 kernel coefficients in the form of $H_5(\underline{1,2,3},\underline{4,5})$ that are multiplied by the same input product $X(1)X(2)X(3)X^*(4)X^*(5)$, they are considered as equal and hence $\mathscr{H}_5(\underline{1,2,3},\underline{4,5}) = 12H_5(1,2,3,4,5)$.

In the following, we will first solve $\mathscr{H}_5(\underline{m}|_1^3, \underline{m}|_4^5)$ and $\mathscr{H}_3(\underline{m}|_1^2, m_3)$ then use (48) and (47) to obtain $H_5(m|_1^3, m|_4^5)$ and $H_3(m|_1^2, m_3)$, respectively.

We start from considering $k = 2$ and $K = 2$ in (29), where the lefthand side of the equation is $E[X^*(m_1)X^*(m_2)X^*(m_3)X(-m_4)X(-m_5)Y(m)]$. Assume there are I distinct values in the set $\{m_1, m_2, m_3, -m_4, -m_5\}$, we can denote the I distinct values as i_1, i_2, …, i_I.

We first consider $I = 5$. Under the circumstances, the only possible situation where (30) can be nonzero is given by (45-3) in Case 7 (i.e., Case ⟨2⟩⟨2⟩⟨2⟩⟨2⟩⟨2⟩). This is because that in any other situation there must be at least one $X(i)$ which can not be paired with its own kind and hence would be equal to zero due to the fact that $X(i)$ is zero mean. Taking the 3rd-order kernel coefficient $H_3(1,2,3)$ in Eq. (29) as an example, it is multiplied by the quantity $E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_5)\cdot X(i_1)X(i_2)X^*(i_3)]$. Since $X(i_1),\ldots,X(i_5)$ are zero-mean independent random variables, we have

$$E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_5)\cdot X(i_1)X(i_2)X^*(i_3)] = E[|X(i_1)|^2]E[|X(i_2)|^2]E[(X^*(i_3))^2]E[X(i_4)] E[X(i_5)] = 0 \quad (49)$$

as a consequence of $E[X(i_4)] = E[X(i_5)] = 0$. The fact that only Case ⟨2⟩⟨2⟩⟨2⟩⟨2⟩⟨2⟩ survives in the right-hand side of (29) makes (29) to become

$$E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_5)Y(m)] = \mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\mu_2(i_4)\mu_2(i_5)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_4,i_5}), \quad (50)$$

where (45-3) and (48) have been used in deriving (50).

Next we consider $I = 4$. This implies that two of the indices in the set of $\{m_1, m_2, m_3, -m_4, -m_5\}$ are equal. Recall that m_1, m_2, and m_3 are interchangeable, and $-m_4$ and $-m_5$ are interchangeable. This suggests that the two equal indices can be both in $\{m_1, m_2, m_3\}$ or both in $\{-m_4, -m_5\}$, or one in $\{m_1, m_2, m_3\}$ and the other in $\{-m_4, -m_5\}$ Therefore, there are three possible distinct equations in the form of (29) for $k = 2$ and $I = 4$. For equal indices both in $\{m_1, m_2, m_3\}$, the only possible situation where (30) can be nonzero is given by (44-9) in Case 6 (i.e., the ⟨4⟩⟨2⟩⟨2⟩⟨2⟩ case). This causes (29) to become

$$E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_3)X(i_4)Y(m)] = \mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\mu_2(i_4)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_3,i_4}). \quad (51)$$

Similarly, for equal indices both in $\{-m_4, -m_5\}$, one can easily see that (29) becomes (by using (44-13) in Case 6)

$$E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_4)Y(m)] = \mu_4(i_4)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},i_4,i_4). \quad (52)$$

Following the same Scenario, for one in $\{m_1, m_2, m_3\}$ and the other in $\{-m_4, -m_5\}$, we find that there are several nonzero terms on the right hand side of (29). These terms are resulted from the situations given by (38-17), (44-3), (44-8), (44-10), and (45-1). This simplifies (29) to become

$$\begin{aligned} &E[X^*(i_1)X^*(i_3)X^*(i_4)X(i_2)X(i_4)Y(m)] = \\ &\mu_2(i_4)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathbf{H}_{35}(i_1,i_3,i_2)+ \\ &\mu_4(i_3)\mu_2(i_1)\mu_2(i_2)\mu_2(i_4)\cdot\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})+ \\ &\mu_4(i_2)\mu_2(i_1)\mu_2(i_3)\mu_2(i_4)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2)+ \\ &\mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\mu_2(i_4)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})+ \\ &(\mu_4(i_4)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)- \\ &\mu_2(i_4)^2\mu_2(i_1)\mu_2(i_2)\mu_2(i_3))\cdot\mathscr{H}_5(\underline{i_1,i_3,i_4},\underline{i_2,i_4}), \end{aligned} \quad (53)$$

where the new symbol $\mathbf{H}_{35}(\cdot,\cdot,\cdot)$ denotes the sum of certain 3rd- and 5th-order Volterra kernel coefficients and is defined by

$$\mathbf{H}_{35}(i,j,k) = \mathscr{H}_3(\underline{i,j},k)+ \sum_{k'=-M,k'\neq i,j,k}^{M}\mu_2(k')\cdot\mathscr{H}_5(\underline{i,j,k'},\underline{k,-k'}). \quad (54)$$

For $I = 3$, the three distinct indices can be distributed as ⟨3⟩⟨1⟩⟨1⟩ (i.e., 3 out of the 5 indices are equal) or ⟨2⟩⟨2⟩⟨1⟩ (i.e., two pairs of equal indices). For the ⟨3⟩⟨1⟩⟨1⟩ case, the three equal indices can be all in $\{m_1, m_2, m_3\}$, two in $\{m_1, m_2, m_3\}$, or just one in $\{m_1, m_2, m_3\}$ In these situations, Eq. (29) yields the following three equations accordingly:

$$E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_2)X(i_3)Y(m)] = \mu_6(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(i_1,i_1,i_1,\underline{i_2,i_3}) \quad (55)$$

$$\begin{aligned} &E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_1)X(i_2)Y(m)] = \\ &\mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathbf{H}_{35}(i_1,i_3,i_2)+ \\ &\mu_4(i_1)\mu_4(i_3)\mu_2(i_2)\cdot\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})+ \\ &\mu_4(i_1)\mu_4(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2)+ \\ &\mu_6(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2}) \end{aligned} \quad (56)$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_2)X(i_2)Y(m)] = \\
&\quad \mu_4(i_2)\mu_2(i_1)\mu_2(i_3)\bullet \mathbf{H}_{35}(i_1,i_3,i_2)+ \\
&\quad \mu_4(i_2)\mu_4(i_3)\mu_2(i_1)\bullet \mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})+ \\
&\quad \mu_6(i_2)\mu_2(i_1)\mu_2(i_3)\bullet \mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2)+ \\
&\quad \mu_4(i_1)\mu_4(i_2)\mu_2(i_3)\bullet \mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})
\end{aligned}
\tag{57}
$$

where (42-5) has been used in deriving (55), Eqs. (38−13), (42−7), (43−3), (43−5), and (44-10) have been used in deriving (56), and Eqs. (38−14), (42−10), (43−6), and (44−12) have been used in deriving (57).

For the $\langle 2\rangle\langle 2\rangle\langle 1\rangle$ case, the two pairs can be one in $\{m_1, m_2, m_3\}$ the other in $\{-m_4, -m_5\}$, one in $\{m_1, m_2, m_3\}$ the other splits across $\{m_1, m_2, m_3\}$ and $\{-m_4, -m_5\}$, or both split across $\{m_1, m_2, m_3\}$ and $\{-m_4, -m_5\}$ These correspondingly simplify (29) to the following three equations:

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_2)X(i_2)Y(m)] = \\
&\quad \mu_4(i_1)\mu_4(i_2)\mu_2(i_3)\bullet \mathscr{H}_5(\underline{i_1,i_1,i_3},i_2,i_2)
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_2)X(i_3)Y(m)] = \\
&\quad \mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\bullet \mathbf{H}_{35}(i_1,i_1,i_2)+ \\
&\quad \mu_6(i_1)\mu_2(i_2)\mu_2(i_3)\bullet \mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2}) + \quad \mu_4(i_1)\mu_4(i_2)\mu_2(i_3)\bullet \mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2) + \quad (\mu_4(i_1)\mu_4(i_3)\mu_2(i_2) - \mu_4(i_1)\mu_2(i_3)^2\mu_2(i_2)
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_2)X(i_3)Y(m)] = \\
&\quad \mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\bullet \mathbf{H}_{135}(i_1) + (\mu_4(i_2)\mu_2(i_1)\mu_2(i_3) - \\
&\quad \mu_2(i_2)^2)\mu_2(i_1)\mu_2(i_3)\bullet \mathbf{H}_{35}(i_1,i_2,i_2)+ \\
&\quad \mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\bullet \mathbf{H}_{35}(i_1,i_1,i_1)+ \\
&\quad (\mu_4(i_3)\mu_2(i_1)\mu_2(i_2) - \mu_2(i_3)^2\mu_2(i_1)\mu_2(i_2))\bullet \\
&\quad \mathbf{H}_{35}(i_1,i_3,i_3) + \mu_6(i_1)\mu_2(i_2)\mu_2(i_3)\bullet \\
&\quad \mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) + (\mu_6(i_2)\mu_2(i_1)\mu_2(i_3) - \\
&\quad \mu_4(i_2)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3))\bullet \mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2)+ \\
&\quad (\mu_4(i_1)\mu_4(i_2)\mu_2(i_3) - \mu_4(i_1)\mu_2(i_2)^2\mu_2(i_3))\bullet \\
&\quad \mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2}) + (\mu_6(i_3)\mu_2(i_1)\mu_2(i_2) - \\
&\quad \mu_4(i_3)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3))\bullet \mathscr{H}_5(\underline{i_1,i_3,i_3},i_3,i_3)+ \\
&\quad (\mu_4(i_1)\mu_4(i_3)\mu_2(i_2) - \mu_4(i_1)\mu_2(i_3)^2\mu_2(i_2))\bullet \\
&\quad \mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_3}) + (\mu_4(i_2)\mu_4(i_3)\mu_2(i_1) - \\
&\quad \mu_4(i_2)\mu_2(i_3)^2\mu_2(i_1) - \mu_4(i_3)\mu_2(i_2)^2\mu_2(i_1)+ \\
&\quad \mu_2(i_2)^2\mu_2(i_3)^2\mu_2(i_1))\bullet \mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_2,i_3})
\end{aligned}
\tag{60}
$$

where the new symbol $\mathbf{H}_{135}(\bullet)$ in (60) denotes the sum of certain first-, 3rd-, and 5th-order Volterra kernel coefficients and is defined by

$$
\mathbf{H}_{135}(i) = H(i) + \sum_{j'=-M, j'\neq i}^{M} \mu_2(j')\bullet \mathbf{H}_{35}(i,j',j') \tag{61}
$$

Note that (43-7) has been used in deriving (58), Eqs. (38−10), (38−12), (38−16), (42−2), (43−11), and (44−7) have been used in deriving (59), and Eqs. (36−4), (38−9), (38−10), (38−16), (42−1), (42−4), (43−1), (43−12), (44−1), (44−5), (44−6), (45−1), and (45−2) have been used in deriving (60).

Similarly, For $I = 2$, the two distinct indices can only be distributed as $\langle 3\rangle\langle 2\rangle$ or $\langle 4\rangle\langle 1\rangle$. For the $\langle 3\rangle\langle 2\rangle$ case, the three equal indices can be all in $\{m_1, m_2, m_3\}$, one in $\{m_1, m_2, m_3\}$ and two in $\{-m_4, -m_5\}$, or two in $\{m_1, m_2, m_3\}$ and one in $\{-m_4, -m_5\}$. Under these circumstances, Eq. (29) results in the following three equations accordingly:

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_2)X(i_2)Y(m)] = \\
&\quad \mu_6(i_1)\mu_4(i_2)\bullet \mathscr{H}_5(i_1,i_1,i_1,i_2,i_2)
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_2)X(i_2)Y(m)] = \\
&\quad \mu_4(i_2)\mu_4(i_1)\bullet \mathbf{H}_{35}(i_1,i_1,i_2)+ \\
&\quad \mu_6(i_1)\mu_4(i_2)\bullet \mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2})+ \\
&\quad \mu_6(i_2)\mu_4(i_1)\bullet \mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_1)X(i_2)Y(m)] = \\
&\quad \mu_4(i_1)\mu_2(i_2)\cdot\mathbf{H}_{135}(i_1) + \\
&\quad \left(\mu_4(i_1)\mu_4(i_2) - \mu_4(i_1)\mu_2(i_2)^2\right)\cdot\mathbf{H}_{35}(i_1,i_2,i_2) + \\
&\quad \mu_6(i_1)\mu_2(i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) + \\
&\quad \mu_8(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) + \\
&\quad \left(\mu_6(i_2)\mu_4(i_1) - \mu_4(i_1)\mu_4(i_2)\mu_2(i_2)\right)\cdot \\
&\quad \mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2) + \left(\mu_6(i_1)\mu_4(i_2) - \mu_6(i_1)\mu_2(i_2)^2\right)\cdot \\
&\quad \mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})
\end{aligned}
\tag{64}
$$

where (41-3) has been used in deriving (62), Eqs. (38−8), (41−2), (41−5), and (43−4) have been used in deriving (63), and Eqs. (36−3), (38−2), (38−7), (38−10), (40−1), (41−6), (41−8), (42−3), (43−1), (43−10), and (44−2) have been used in deriving (64).

For the $\langle 4\rangle\langle 1\rangle$ case, the only different index can be in either $\{-m_4, -m_5\}$ or $\{m_1, m_2, m_3\}$ These make (29) to give the following two equations correspondingly:

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_1)X(i_2)Y(m)] = \\
&\quad \mu_6(i_1)\mu_2(i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_2) + \\
&\quad \mu_8(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2}) + \\
&\quad \mu_6(i_1)\mu_4(i_2)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_2)X^*(i_2)X(i_2)X(i_2)Y(m)] = \\
&\quad \mu_4(i_1)\mu_2(i_2)\cdot\mathbf{H}_{135}(i_1) + \left(\mu_6(i_2)\mu_2(i_1) - \right. \\
&\quad \left.\mu_4(i_2)\mu_2(i_1)\mu_2(i_2)\right)\cdot\mathbf{H}_{35}(i_1,i_2,i_2) + \\
&\quad \mu_4(i_1)\mu_4(i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) + \mu_6(i_1)\mu_4(i_2)\cdot \\
&\quad \mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) + \left(\mu_8(i_2)\mu_2(i_1) - \mu_4(i_1)^2\mu_2(i_1)\right)\cdot \\
&\quad \mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2) + \left(\mu_6(i_2)\mu_4(i_1) - \right. \\
&\quad \left.\mu_4(i_1)\mu_4(i_2)\mu_2(i_2)\right)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})
\end{aligned}
\tag{66}
$$

where Eqs. (38−5), (40−4), (41−4), and (42−6) have been used in deriving (65), and Eqs. (36−3), (38−4), (38−6), (38−10), (38−15), (40−3), (41−1), (41-8), (42−8), (43−9), (43−10), (43−12), and (44−2) have been used in deriving (66).

Finally, for $I = 1$, all the indices are equal. Eq. (29) yields (by using Eqs. (36−1), (38−1), (38−3), (39), (40−2), (41−7), and (42−9))

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_1)X(i_1)Y(m)] = \\
&\quad \mu_6(i_1)\cdot\mathbf{H}_{135}(i_1) + \mu_8(i_1)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) + \\
&\quad \mu_{10}(i_1)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)
\end{aligned}
\tag{67}
$$

We next consider $k = 1$ and $K = 2$ in (29), where the lefthand side of the equation is $E[X^*(m_1)X^*(m_2)X(-m_3)Y(m)]$. Assume there are I distinct values in the set $\{m_1, m_2, -m_3\}$, we can denote the distinct values as $i_1, i_2, \ldots i_I$. Following the same scenario, one can easily see that $I = 3$ leads (29) to (by using (37-6), (38-13), (38-14), and (38-17))

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_3)X(i_2)Y(m)] = \\
&\quad \mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathbf{H}_{35}(i_1,i_3,i_2) + \\
&\quad \mu_4(i_3)\mu_2(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3}) + \\
&\quad \mu_4(i_2)\mu_2(i_1)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2) + \\
&\quad \mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})
\end{aligned}
\tag{68}
$$

The $I = 2$ case can cause (29) to give either one of the following two equations:

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X(i_2)Y(m)] = \\
&\quad \mu_4(i_1)\mu_2(i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_2) + \\
&\quad \mu_6(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2}) + \\
&\quad \mu_4(i_1)\mu_4(i_2)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_2)X(i_2)Y(m)] = \\
&\quad \mu_2(i_1)\mu_2(i_2)\cdot\mathbf{H}_{135}(i_1) + \left(\mu_4(i_2)\mu_2(i_1) - \right. \\
&\quad \left.\mu_2(i_2)^2\mu_2(i_1)\right)\cdot\mathbf{H}_{35}(i_1,i_2,i_2) + \mu_4(i_1)\mu_2(i_2)\cdot \\
&\quad \mathbf{H}_{35}(i_1,i_1,i_1) + \mu_6(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) + \\
&\quad \left(\mu_6(i_2)\mu_2(i_1) - \mu_4(i_2)\mu_2(i_1)\mu_2(i_2)\right)\cdot \\
&\quad \mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2) + \left(\mu_4(i_1)\mu_4(i_2) - \mu_4(i_1)\mu_2(i_2)^2\right)\cdot \\
&\quad \mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})
\end{aligned}
\tag{70}
$$

where (37-2), (38-5), (38-8), and (38-12) have been used in deriving (69), and (35-2), (37-4), (37-5), (37-7), (38-3), (38-4), (38-7), (38-10), (38-11), (38-15), and (38-16) have been used in deriving (70). The $I = 1$ case can only lead (29) to (by using (35-1), (36-1), (37-4), (38-1), (38-2), (38-6), and (38-9))

$$
\begin{aligned}
&E[X^*(i_1)X^*(i_1)X(i_1)Y(m)] = \\
&\quad \mu_4(i_1)\cdot\mathbf{H}_{135}(i_1) + \mu_6(i_1)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) + \\
&\quad \mu_8(i_1)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)
\end{aligned}
\tag{71}
$$

Finally, we consider $k = 0$ and $K = 2$ in (29), where the lefthand side of the equation is $E[X^*(m_1)Y(m)]$. Since these is only one distinct index, we assume $m_1 = i_1$ and (29) becomes (by using (34), (35-1), (35-2), (36-2), (36-3), (36-4), and (37-1))

$$
\begin{aligned}
&E[X^*(i_1)Y(m)] = \mu_2(i_1)\cdot\mathbf{H}_{135}(i_1) + \\
&\quad \mu_4(i_1)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) + \mu_6(i_1)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)
\end{aligned}
\tag{72}
$$

Note that (51)–(53), (55)–(60), and (62)–(72) comprise all the different-type system equations in the form of (29) for a 5th-order OFDM nonlinear channel. By solving these system equations, we can obtain the Volterra kernels of the 5th-order OFDM nonlinear channel. We will proceed by solving the 5th-order Volterra kernel coefficients first, the 3rd-order ones next, and the first-order ones last.

By observing (50), (51), (52), (55), (58), and (62) we find that, there in only one Volterra kernel coefficient in each of the equations. Therefore, these Volterra kernel coefficients can be easily solved independently as follows, respectively:

$$\mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_4,i_5}) = \frac{E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_5)Y(m)]}{\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\mu_2(i_4)\mu_2(i_5)} \tag{73}$$

$$\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_3,i_4}) = \frac{E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_3)X(i_4)Y(m)]}{\mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\mu_2(i_4)} \tag{74}$$

$$\mathscr{H}_5(\underline{i_1,i_2,i_3},i_4,i_4) = \frac{E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_4)X(i_4)Y(m)]}{\mu_4(i_4)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)} \tag{75}$$

$$\mathscr{H}_5(i_1,i_1,i_1,\underline{i_2,i_3}) = \frac{E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_2)X(i_3)Y(m)]}{\mu_6(i_1)\mu_2(i_2)\mu_2(i_3)} \tag{76}$$

$$\mathscr{H}_5(\underline{i_1,i_1,i_3},i_2,i_2) = \frac{E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_2)X(i_2)Y(m)]}{\mu_4(i_1)\mu_4(i_2)\mu_2(i_3)} \tag{77}$$

$$\mathscr{H}_5(i_1,i_1,i_1,i_2,i_2) = \frac{E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_2)X(i_2)Y(m)]}{\mu_6(i_1)\mu_4(i_2)} \tag{78}$$

Next we observe that (56), (57), and (68) contain the same set of unknown kernel coefficients (i.e., $\mathbf{H}_{35}(i_1,i_3,i_2)$, $\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})$, $\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2)$, and $\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})$), hence they can not be solved independently and should be solved together. A closer look at the equations one can find that $\mu_4(i_1)\times(68) - \mu_2(i_1)\times(56)$ eliminates 3 out of the 4 unknowns and gives $\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})$ as follows:

$$\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2}) = \{\mu_4(i_1)\cdot E[X^*(i_1)X^*(i_3)X(i_2)Y(m)] - \mu_2(i_1)\cdot E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_1)X(i_2)Y(m)]\} / U_1(i_1,i_2,i_3), \tag{79}$$

where $U_1(i_1,i_2,i_3)$, is defined by (103) in the Appendix.

Similarly, $\mu_4(i_2)\times(68) - \mu_2(i_2)\times(57)$ also eliminates 3 out of the 4 unknowns and gives

$$\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2) = \{\mu_4(i_2)\cdot E[X^*(i_1)X^*(i_3)X(i_2)Y(m)] - \mu_2(i_2)\cdot E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_2)X(i_2)Y(m)]\}/U_1(i_1,i_2,i_3). \tag{80}$$

Note that the kernel coefficient $\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})$ becomes $\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})$ when i_1 and i_3 are interchanged. Therefore, the formula for $\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})$ can be obtained from (79) by just interchanging i_1 and i_3. By substituting $\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3})$, $\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2)$, and $\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})$ into (68), one can easily solve the leftover unknown $\mathbf{H}_{35}(i_1,i_3,i_2)$ as follows:

$$\begin{aligned}\mathbf{H}_{35}(i_1,i_3,i_2) = \{&E[X^*(i_1)X^*(i_3)X(i_2)Y(m)] - \\ &\mu_4(i_3)\mu_2(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(\underline{i_1,i_3,i_3},\underline{i_2,i_3}) - \\ &\mu_4(i_2)\mu_2(i_1)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_2,i_3},i_2,i_2) - \\ &\mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_2})\}/ \\ &\{\mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\}.\end{aligned} \tag{81}$$

We further notice that the similarity between (53) and (68). With a closer look we find that (53) $-\ \mu_2(i_4)\ \times$ (68) leads to the following formula for $\mathscr{H}_5(\underline{i_1,i_3,i_4},\underline{i_2,i_4})$:

$$\mathscr{H}_5(\underline{i_1,i_3,i_4},\underline{i_2,i_4}) = \{E[X^*(i_1)X^*(i_3)X^*(i_4)X(i_2)X(i_4)Y(m)] - \mu_2(i_4)\cdot E[X^*(i_1)X^*(i_3)X(i_2)Y(m)]\}/U_2(i_1,i_2,i_3), \tag{82}$$

where $U_1(i_1,i_2,i_3)$, is defined by (104) in the Appendix.

The three equations (63), (65) and (69) have the same three unknown kernel coefficients $\mathbf{H}_{35}(i_1,i_1,i_2)$, $\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2})$, and $\mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)$, hence they should be solved together. Specifically, by conducting $\mu_6(i_1)\times$(69) $-$ $\mu_4(i_1)\times$(65) one can obtain $\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2})$ as follows:

$$\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2}) = \{\mu_6(i_1)\cdot E[X^*(i_1)X^*(i_1)X(i_2)Y(m)] - \mu_4(i_1)\cdot E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_1)X(i_2)Y(m)]\} / U_3(i_1,i_2), \tag{83}$$

where $U_3(i_1,i_2)$ is defined by (105) in the Appendix.

Similarly, by conducting $\mu_4(i_2)\times$(69) $-$ $\mu_2(i_2)\times$(63) one gets

$$\mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2) = \{\mu_4(i_2)\cdot E[X^*(i_1)X^*(i_1)X(i_2) Y(m)] - \mu_2(i_2)\cdot E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_2) X(i_2)Y(m)]\}/U_4(i_1,i_2), \tag{84}$$

where $U_4(i_1,i_2)$ is defined by (106) in the Appendix.

Once $\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2})$ and $\mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)$ are solved via (83) and (84), they can be substituted into (69), which results in

$$\mathbf{H}_{35}(i_1,i_1,i_2) = \{E[X^*(i_1)X^*(i_1)X(i_2)Y(m)] - \mu_6(i_1)\mu_2(i_2)\cdot\mathscr{H}_5(i_1,i_1,i_1,\underline{i_1,i_2}) - \mu_4(i_1)\mu_4(i_2)\cdot \mathscr{H}_5(\underline{i_1,i_1,i_2},i_2,i_2)\}/\{\mu_4(i_1)\mu_4(i_2)\}. \tag{85}$$

In addition, by observing the similarity between (59) and (69) we find that $\mu_2(i_3)\times$ (69) $-$ (59) leads to

$$\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_2,i_3}) = \{\mu_2(i_3)\cdot E[X^*(i_1)X^*(i_1)X(i_2) Y(m)] - E[X^*(i_1)X^*(i_1)X^*(i_3)X(i_2)X(i_3) Y(m)]\}/U_5(i_1,i_2,i_3), \tag{86}$$

where $U_5(i_1,i_2,i_3)$ is defined by (107) in the Appendix.

Similarly, the three equations (67), (71) and (72) have the same three unknown kernel coefficients $\mathbf{H}_{135}(i_1)$, $\mathbf{H}_{35}(i_1,i_1,i_1)$, and $\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)$, hence they are solved together as follows. By conducting $\mu_6(i_1)\times$(72) $-$ $\mu_2(i_1)\times$(67) and $\mu_4(i_1)\times$(72) $-$ $\mu_2(i_1)\times$(71), one can get two equations, say, Eq. (A) and Eq. (B), with only the two unknowns $\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)$ and $\mathbf{H}_{35}(i_1,i_1,i_1)$. By conducting $U_8(i_1)\times$ [Eq. (A)] - $U_6(i_1)\times$ [Eq. (B)], one can get

$$\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) = \{U_8(i_1)\cdot\{\mu_6(i_1)\cdot E[X^*(i_1)Y(m)] - \mu_2(i_1)\cdot E[X^*(i_1)X^*(i_1)X^*(i_1)X(i_1)\ X(i_1)Y(m)]\} - U_6(i_1)\cdot\{\mu_4(i_1)\cdot E[X^*(i_1)Y(m)] - \mu_2(i_1)\cdot E[X^*(i_1)X^*(i_1)X(i_1)Y(m)]\}\} / [U_8(i_1)\cdot U_7(i_1) - U_6(i_1)^2], \tag{87}$$

where $U_6(i_1)$, $U_7(i_1)$, and $U_8(i_1)$ are defined by (108), (109), and (110), respectively in the Appendix.

Given (87), one then can use Eq. (B) to obtain

$$\mathbf{H}_{35}(i_1,i_1,i_1) = \{\mu_4(i_1)\cdot E[X^*(i_1)Y(m)] - \mu_2(i_1)\cdot E[X^*(i_1)X^*(i_1)X(i_1)Y(m)] - U_6(i_1)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)\}/U_8(i_1) \tag{88}$$

Once $\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)$ and $\mathbf{H}_{35}(i_1,i_1,i_1)$ are solved via (87) and (88), they can be substituted into (72) to solve $\mathbf{H}_{135}(i_1)$.

$$\mathbf{H}_{135}(i_1) = \{E[X^*(i_1)Y(m)] - \mu_4(i_1)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) - \mu_6(i_1)\cdot\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)\}/\mu_2(i_1) \tag{89}$$

Following the same scenario, we find the three equations (64), (66) and (70) have the same three unknown kernel coefficients $\mathbf{H}_{35}(i_1,i_2,i_2)$, $\mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2)$, and $\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})$. The other kernel coefficients $\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)$, $\mathbf{H}_{35}(i_1,i_1,i_1)$, and $\mathbf{H}_{135}(i_1)$ in the three equations have already been solved in (87)–(89) and are known by now. Therefore, by conducting $\mu_2(i_1)\times$ (64) $-$ $\mu_4(i_1)\times$ (70), one can obtain

$$\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2}) = \{\mu_2(i_1)\cdot E[X^*(i_1)X^*(i_1)X^*(i_2) X(i_1)X(i_2)Y(m)] - \mu_4(i_1)\cdot E[X^*(i_1)X^*(i_2) X(i_2)Y(m)] - U_9(i_1,i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) - U_{10}(i_1,i_2)\cdot \mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)\}/U_{11}(i_1,i_2), \tag{90}$$

where $U_9(i_1,i_2)$, $U_{10}(i_1,i_2)$, and $U_{11}(i_1,i_2)$, are defined by (111)–(113) respectively in the Appendix.

By conducting $U_{13}(i_1,i_2)\times$ (64) $-$ $U_{12}(i_1,i_2)\times$ (66) one obtains

$$\mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2) = \{U_{13}(i_1,i_2)\cdot\{E[X^*(i_1)X^*(i_1) X^*(i_2)X(i_1)X(i_2Y(m)] - U_{12}(i_1,i_2)\cdot E[X^*(i_1) X^*(i_2)X^*(i_2)X(i_2)X(i_2Y(m)]\} - U_{14}(i_1,i_2)\cdot \mathbf{H}_{135}(i_1) - U_{15}(i_1,i_2)\cdot\mathbf{H}_{35}(i_1,i_1,i_1) - U_{16}(i_1,i_2)\cdot \mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) - U_{18}(i_1,i_2)\cdot\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})\} /U_{17}(i_1,i_2), \tag{91}$$

where $U_{12}(i_1,i_2)$, $U_{13}(i_1,i_2)$, $U_{14}(i_1,i_2)$, $U_{15}(i_1,i_2)$, $U_{16}(i_1,i_2)$, $U_{17}(i_1,i_2)$, and $U_{18}(i_1,i_2)$ are defined by (114)–(120) respectively in the Appendix.

Once $\mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2)$ and $\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})$ are solved via (90) and (91), one can substitute them into (64) to yield

$$\begin{aligned}\mathbf{H}_{35}(i_1,i_2,i_2) = \{&E[X^*(i_1)X^*(i_1)X^*(i_2)X(i_1)X(i_2)\\ &Y(m)] - \mu_4(i_1)\mu_2(i_2)\bullet\mathbf{H}_{135}(i_1) - \mu_6(i_1)\mu_2(i_2)\bullet\\ &\mathbf{H}_{35}(i_1,i_1,i_1) - \mu_8(i_1)\mu_2(i_2)\bullet\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1)\\ &- U_{19}(i_1,i_2)\bullet\mathscr{H}_5(\underline{i_1,i_2,i_2},\underline{i_2,i_2}) - U_{20}(i_1,i_2)\bullet\\ &\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2})\}/U_{12}(i_1,i_2),\end{aligned} \tag{92}$$

where $U_{19}(i_1,i_2)$ and $U_{20}(i_1,i_2)$ are defined by (112) and (122), respectively, in the Appendix.

The last 5th-order kernel coefficient we need to solve is the $\mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_2,i_3})$ in (60). Note that all the other kernel coefficients involved in (60) have already been solved via (87)–(92) except $\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_3})$, $\mathscr{H}_5(\underline{i_1,i_3,i_3},i_3,i_3)$, and $\mathbf{H}_{35}(i_1,i_3,i_3)$. However, the three kernel coefficients can simply be solved by replacing i_2 in (90)–(92) by i_3, respectively. Therefore, the three kernel coefficients can be considered known given (90)–(92). Bearing this in mind, one can solve $\mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_2,i_3})$ by simply using (60) as follows:

$$\begin{aligned}\mathscr{H}_5(\underline{i_1,i_2,i_3},\underline{i_2,i_3}) = \{&E[X^*(i_1)X^*(i_2)X^*(i_3)X(i_2)X(i_3)\\ &Y(m)] - \mu_2(i_1)\mu_2(i_2)\mu_2(i_3)\bullet\mathbf{H}_{135}(i_1) - U_{21}(i_1,i_2,i_3)\bullet\\ &\mathbf{H}_{35}(i_1,i_2,i_2) - \mu_4(i_1)\mu_2(i_2)\mu_2(i_3)\bullet\mathbf{H}_{35}(i_1,i_1,i_1) -\\ &U_{21}(i_1,i_2,i_3)\bullet\mathbf{H}_{35}(i_1,i_3,i_3) - \mu_6(i_1)\mu_2(i_2)\mu_2(i_3)\bullet\\ &\mathscr{H}_5(i_1,i_1,i_1,i_1,i_1) - U_{22}(i_1,i_2,i_3)\bullet\mathscr{H}_5(\underline{i_1,i_2,i_2},i_2,i_2) -\\ &U_{23}(i_1,i_2,i_3)\bullet\mathscr{H}_5(\underline{i_1,i_1,i_2},\underline{i_1,i_2}) -\\ &U_{22}(i_1,i_2,i_3)\bullet\mathscr{H}_5(\underline{i_1,i_3,i_3},i_3,i_3) -\\ &U_{23}(i_1,i_2,i_3)\bullet\mathscr{H}_5(\underline{i_1,i_1,i_3},\underline{i_1,i_3})\}/U_{24}(i_1,i_2,i_3),\end{aligned} \tag{93}$$

where $U_{21}(i_1,i_2,i_3)$, $U_{22}(i_1,i_2,i_3)$, $U_{23}(i_1,i_2,i_3)$, and $U_{24}(i_1,i_2,i_3)$ are defined by (123)–(126) respectively in the Appendix.

Having solved all the 5th-order kernel coefficients, we now move on to solve the 3rd-order kernel coefficients. Given (81) and (82), one can easily use (54) to yield

$$\mathscr{H}_3(\underline{i_1,i_3},i_2) = \mathbf{H}_{35}(i_1,i_3,i_2) - \sum_{k'=-M,k'\neq i_1,i_2,i_3}^{M} \mu_2(k')\bullet\mathscr{H}_5(i_1,i_3,k',i_2,k') \tag{94}$$

Given (85) and (86), one can use (54) to get

$$\mathscr{H}_3(i_1,i_1,i_2) = \mathbf{H}_{35}(i_1,i_1,i_2) - \sum_{k'=-M,k'\neq i_1,i_2}^{M} \mu_2(k')\bullet\mathscr{H}_5(i_1,i_1,k',i_2,-k') \tag{95}$$

Similarly, given (88) and (90), one can again use (54) to obtain

$$\mathscr{H}_3(i_1,i_1,i_1) = \mathbf{H}_{35}(i_1,i_1,i_1) + \sum_{k'=-M,k'\neq i_1}^{M} \mu_2(k')\bullet\mathscr{H}_5(\underline{i_1,i_1,k'},\underline{i_1,-k'}) \tag{96}$$

Finally, given (92) and (93), one can still use (54) to obtain

$$\mathscr{H}_3(\underline{i_1,i_2},i_2) = \mathbf{H}_{35}(i_1,i_2,i_2) + \sum_{k'=-M,k'\neq i_1,i_2}^{M} \mu_2(k')\bullet\mathscr{H}_5(\underline{i_1,i_2,k'},\underline{i_2,-k'}) \tag{97}$$

This completes the solving of the 3rd-order kernel coefficients.

For the first-order kernel coefficients, they can be simply solved by

$$H(i_1) = \mathbf{H}_{135}(i_1) - \sum_{j'=-M,j'\neq i_1}^{M} \mu_2(j')\bullet\mathbf{H}_{35}(i_1,j',j') \tag{98}$$

Note that $\mathbf{H}_{135}(i_1)$ and $\mathbf{H}_{35}(i_1,j',j')$ are already solved in (89) and (92), respectively.

The equations (73)–(98) constitute the complete formulas for estimating the Volterra kernels of 5th-order nonlinear OFDM channels. However, one should keep in mind that the obtained 3rd- and 5th-order kernels are the modified Volterra kernels defined in (47) and (48). They can be easily converted back to the original Volterra kernels using (47) and (48).

6. Design of pseudo random test sequences

In Section 4, we have analyzed the theoretical values of the higher-order auto-moment spectra for OFDM signals. However, the estimated higher-order auto-moment spectra from a finite amount of random data may not be exactly equal to the theoretical ones. Generally speaking, they can only approach to their theoretical values as we increase the amount of random data. This suggests that the higher-order auto-moment spectral properties described in Section 4 can only be fulfilled approximately when the amount of random data is limited.

To deal with this issue, we proposed in [18] to use a specially designed input data sequence instead of random data for the OFDM signal, so that the higher-order auto-moment spectral properties in Section 4 can be fulfilled precisely. Specifically, note that the OFDM signal contains multiple parallel QAM symbol streams. Each OFDM symbol interval contains a particular permutation of parallel QAM symbols. In [18] we chose the input parallel QAM symbols to the subcarriers in a way that every possible permutation of parallel QAM symbols appears the same number of times in the input data. This guarantees that the estimated higher-order auto-moment spectra coincide with the theoretical ones precisely. The reader is referred to [18] for further details.

Despite the idea of using a designed test sequence in [18] does secure the usage of the higher-order auto-moment spectral properties, it does not shed light on how a test sequence can be constructed efficiently. In the following, we propose a systematic way to design the test sequence. Suppose that K-QAM is used at the N subcarriers. In this case, the input sequence $X(m)$ is chosen from a set of K QAM data symbols with an equal probability. Let F_i denotes the ith-order auto-moment spectrum of the input. It has the following form:

$$F_i = f_i\left(X(0),\ X(1),\ \ldots,\ X(N-1)\right) \tag{99}$$

where $f_i(\cdot)$ is the function which generates the ith-order auto-moment spectrum of the input. Since each $X(m)$ (m = 0, ..., $N-1$) can be one of the K data symbols, we see from (99) that for a particular F_i there are K^N different input permutations. Therefore, if the test sequence includes all the K^N permutations with each permutation showing up the same number of times, the estimated higher-order auto-moment spectrum will match its analytical result exactly. This secures the usage of the higher-order auto-moment spectral properties derived in Section 4 and hence guarantees the Volterra kernel estimate acquired by the proposed method is optimum in the MMSE sense.

An efficient way to generate such a sequence is to use a feedback shift register specified by a degree N primitive polynomial over a Galois field with K elements (defined as $\mathrm{GF}(K)$) [24]. The K-ary pseudo random sequence generated by the degree N primitive polynomial (or its corresponding feedback shift register) with a non-zero initial condition has the so-called window property that if a window of width N is slid along the sequence, each of the K^N-1 nonzero K-ary N-tuples is seen exactly once before the initial condition reappears. The primitive polynomial and the Galois field have been widely used in error-correcting codes and the reader can refer to [25] for their definitions and further properties. The pseudo random sequence, when augmented with an all-zero N-tuples, forms a length K^N test sequence. The general form of a primitive polynomial $p(x)$ of degree N can be written as

$$p(x) = \sum_{k=0}^{N} h_k x^k \tag{100}$$

where $h_k \in \mathrm{GF}(K)$. The feedback shift register corresponding to the primitive polynomial $p(x)$ is shown in Fig. 3, where $a_i \in \mathrm{GF}(K)$ and the addition and multiplication are defined in the $\mathrm{GF}(K)$ sense. More details on generating a pseudo random sequence can be found in [24]. Given the output sequence $\{a_i\}$, one only needs to map the K-QAM symbols to the K elements in the $\mathrm{GF}(K)$ to obtain the required test sequence for the Volterra kernel estimation task.

7. Computer simulation

In this paper, we use a 5th-order nonlinear channel model that was used to characterize the traveling wave tube (TWT) power amplifier for satellite communications [13] to verify the correctness of the derived formula. The time-domain input and output of the channel are related by (1) with K = 2. The time-domain Volterra kernels of the satellite channel were derived in [13] and are shown in Table 1.

Since the purpose of the simulation is to justify the effectiveness of the proposed method, for simplicity, we set M = 2 in (14) to generate the OFDM signal

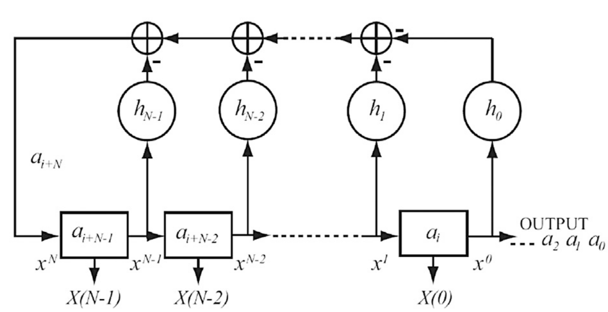

Fig. 3. The feedback shift register corresponding to the primitive polynomial $p(x)$ in (100) for generating the test sequence.

Table 1. The time-domain volterra kernels of a satellite communication system.

Order	Kernel Coefficients
1st	$h_1[0] = 1.22 + j0.646$
	$h_1[1] = 0.063 - j0.001$
	$h_1[2] = -0.024 - j0.014$
	$h_1[3] = 0.036 + j0.031$
3rd	$h_3[0,0,2] = 0.039 - j0.022$
	$h_3[3,3,0] = 0.018 - j0.018$
	$h_3[0,0,1] = 0.035 - j0.035$
	$h_3[0,0,3] = -0.040 - j0.009$
	$h_3[1,1,0] = -0.010 - j0.017$
5th	$h_5[0,0,0,1,1] = 0.039 - j0.022$

Table 2. The states of the shift register in Fig. 4.

State No.	State				
0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0
1	1 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0
2	0 1 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
3	0 0 1 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1
4	1 0 0 1	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0
⋮	⋮	⋮	⋮	⋮	⋮
1,048,573	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0	0 0 1 0
1,048,574	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1	0 0 0 1
1,048,575 = 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0
1,048,576 = 1	1 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0	0 1 0 0
⋮	(REPEATS)				

$x[n]$ as the input to the nonlinear satellite channel. For each OFDM symbol interval, the parallel symbols (i.e., $X(m)$ in (14)) applied to the subcarriers were chosen from the 16-QAM constellation shown in Fig. 2.

The pseudo-random parallel symbol sequences were generated by the feedback shift register method we described in Section 6. Specifically, we adopted the following degree 20 primitive polynomial over GF(2) [24]:

$$p(x) = x^{20} + x^3 + 1. \tag{101}$$

The shift register corresponding to the primitive polynomial $p(x)$ in (101) is shown in Fig. 4, where the 20-bit initial condition of the shift register is [1000 1000 1000 1000 1000]. According to the theory we mentioned in Section 6, as we slide the 20-bit widow through the output sequence, each of the $2^{20}-1$=1,048,575 non-zero 20-bit permutations would be seen exactly once before the initial condition reappears. The simulation result indeed verified this argument, as shown in Table 2, where state 1,048,575 is equal to state 0 and the states repeat afterward. The 1,048,575 non-zero states, when augmented with the all-zero state, constitute all the possible 2^{20} 20-bit permutations with each permutation appearing exactly once. Next we divided the 20 bits of each state in Table 2 into five 4-bit groups and mapped each 4-bit group to a 16-QAM symbol according to the 16-QAM constellation shown in Fig. 2, we obtained Table 3, where the numbers under the "16-QAM Symbols" columns indicate the symbol numbers labeled inside the parentheses in Fig. 2. Note that the bottom row of Table 3 is the augmented all-zero state.

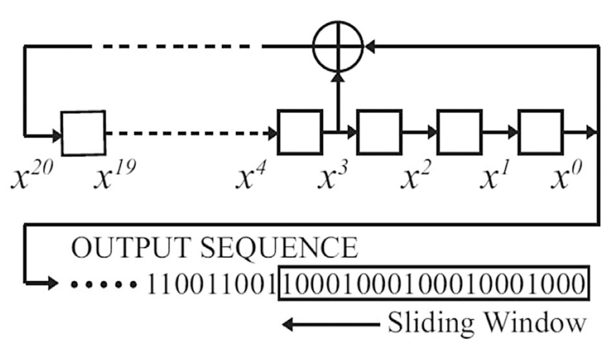

Fig. 4. The shift register corresponds to the primitive polynomial $p(x) = x^{20} + x^3 + 1$ in (101). The 20-bit sequence inside the sliding window indicates the initial condition of the shift register.

The data in Table 3 were used as the parallel symbol sequences for the 5 subcarriers (i.e., $m = -2, -1, \ldots, 2$) to generate the OFDM input signal $x[n]$. Note that, for the 5 subcarriers we used in the simulation, there are 16^5=1,048,576 possible permutations of parallel symbols, which are already completely accounted for in Table 3. Therefore, the input signal $x[n]$ contained 1,048,576 OFDM symbol intervals. Given $x[n]$, the output $y[n]$ of the nonlinear channel was generated by using Table 1 and (1) with $K = 2$. The power spectra of the input and output are shown in Fig. 5. Notice that, although the discrete frequency range for the input is from −2 to 2, the output's discrete frequency range is from −10 to 10. This is due to the fact that a nonlinear system can yield out-of-band frequency components [12,20].

These data were used by the proposed method to estimate the frequency-domain Volterra kernels of the nonlinear channel. The normalized mean square errors (NMSEs) of the estimated linear, cubic, 5th-

Table 3. The corresponding 16-QAM symbols for the output sequence in Table 2.

State No.	16-QAM Symbols				
0	8	8	8	8	8
1	12	4	4	4	4
2	6	2	2	2	2
3	3	1	1	1	1
4	9	8	8	8	8
⋮	⋮	⋮	⋮	⋮	⋮
1,048,573	2	2	2	2	2
1,048,574	1	1	1	1	1
Augmented	0	0	0	0	0

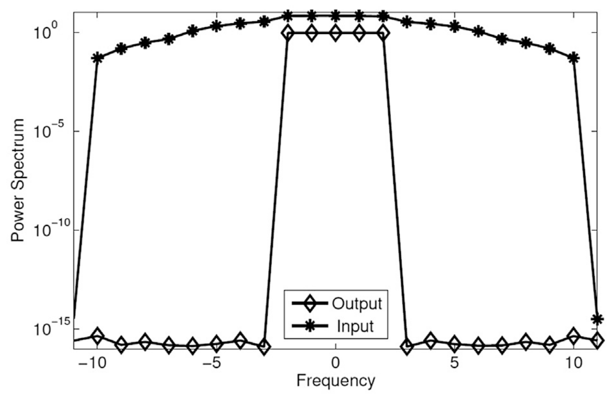

Fig. 5. The power spectra of the input and output signals of the simulated nonlinear satellite channel.

order, and total frequency-domain Volterra kernels under various signal-to-noise ratios (SNRs) are shown in Table 4 and are also plotted in Fig. 6. Here the NMSE is defined as

Table 4. The NMSES for the linear, cubic, 5th-order, and total frequency-domain Volterra Kernels achieved by the proposed method under various SNRS.

SNR(dB)	NMSE			
	Linear	Cubic	5th-order	Total
0	1.716×10^{-3}	2.921×10^{-2}	1.559×10^{-3}	3.319×10^{-3}
10	5.669×10^{-5}	2.905×10^{-3}	1.367×10^{-4}	2.302×10^{-4}
20	4.566×10^{-6}	2.932×10^{-4}	1.966×10^{-5}	2.285×10^{-5}
30	1.716×10^{-6}	2.921×10^{-5}	1.559×10^{-6}	3.319×10^{-6}
40	8.397×10^{-8}	2.277×10^{-6}	1.524×10^{-7}	2.191×10^{-7}
50	1.716×10^{-8}	2.921×10^{-7}	1.559×10^{-8}	3.319×10^{-8}
60	1.117×10^{-9}	3.086×10^{-8}	2.041×10^{-9}	2.945×10^{-9}
70	4.337×10^{-11}	3.907×10^{-9}	2.631×10^{-10}	2.889×10^{-10}

$$\mathrm{NMSE} = ||\widehat{\mathbf{h}} - \mathbf{h}||^2 / ||\mathbf{h}||^2, \tag{102}$$

where $\mathbf{h}$ and $\widehat{\mathbf{h}}$ are vectors containing the actual and estimated frequency-domain Volterra kernel coefficients, respectively. Note that the actual frequency-domain Volterra kernel coefficients were calculated from the time-domain Volterra kernel coefficients shown in Table 1 by using multidimensional Fourier transforms. Generally speaking, one can see from Table 4 that the estimation result is quite accurate under various SNRs. For example, even under SNR $= 0$ dB, the estimated linear, cubic, 5th-order, and total frequency-domain Volterra kernels still achieved relatively good NMSEs of 1.716×10^{-3}, 2.921×10^{-2}, and 1.559×10^{-3}, 3.319×10^{-3}, respectively. In addition, the consistent trend of decreasing NMSE with increasing SNR shown in Fig. 6 suggests the reliability of the proposed method. These results justify the correctness of the derived formula.

One may have noticed that the linear kernel has a better NMSE performance than the cubic and 5th-order kernels in most cases in the simulation. A possible reason for this phenomenon is that the linear kernel is dominant in the simulated satellite channel. Specifically, the linear kernel coefficient $h_1[0] = 1.22 + j0.646$ (see Table 1), which is almost two order of magnitude larger than those cubic and 5th-order kernel coefficients. Given the definition of the NMSE in (102), the kernel having a larger $||\mathbf{h}||$ tends to have a smaller NMSE.

In [18], the proposed approach was compared to the method in [17] using the same simulated

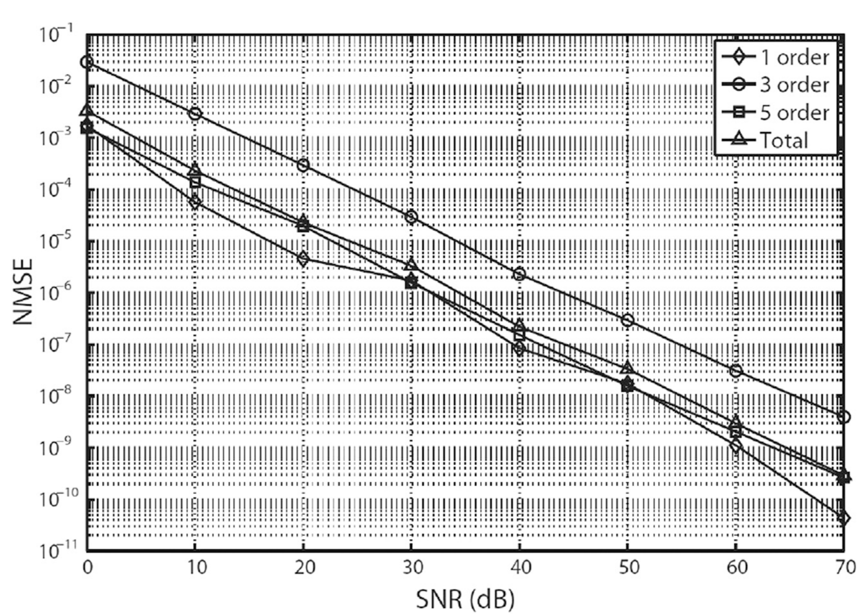

Fig. 6. The NMSEs of the estimated linear, cubic, 5th-order, and total frequency-domain Volterra kernels versus SNRs.

satellite channel in this paper but up to the 3rd order. The result showed that the proposed approach achieved a significantly better NMSE. This is due to the fact that the derivation of the method in [17] relies on the complex Gaussianity assumption on the OFDM signal. In practice, whether the OFDM signal is sufficiently Gaussian could be in doubt. The deviation from Gaussianity of the OFDM signal can lead to an unsatisfactory estimation result. The proposed approach, on the other hand, makes the assumption that the complex data sequences for all the subcarriers of the OFDM signal are zero mean and mutually independent. The satisfication of this assumption, however, can be guaranteed in practice by using the pseudo random test sequences designed in Section 6. This in turn guarantees the proposed method to obtain the optimal minimum mean square error (MMSE) estimate of the Volterra kernels.

8. Conclusion

In this paper, we have proposed a general approach to derive simple formulas for estimating the frequency-domain Volterra kernels of bandpass nonlinear OFDM systems. By exploring higher-order auto-moment spectral properties of the OFDM signal, we have shown that the proposed approach can greatly simplify the complexity of the kernel estimation process. Based on this approach, a simple and computationally efficient formula for identifying 5th-order nonlinear OFDM channels is derived. In addition, the estimated Volterra kernels by the proposed method are optimal in the MMSE sense. Furthermore, a shift-register based method for systematically generating test sequences that guarantee the attainment of the optimal MMSE solution is developed. This suggests that the proposed method can indeed offer a simple yet accurate way to identify the frequency-domain Volterra kernels of nonlinear OFDM systems. The result of this paper can not only consolidate the Volterra modeling theory for nonlinear OFDM systems, but also come in handy for Volterra model based nonlinear channel compensators.

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Appendix

The following are the definitions of the various notations used in Section 5. Their values can be easily determined given $\mu_n(m)$, $-M \le m \le M$, $n = 2,4,6,8,10$.

$$U_1(i_1,i_2,i_3) = \mu_4(i_1)^2\mu_2(i_2)\mu_2(i_3) - \mu_6(i_1)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3) \tag{103}$$

$$U_2(i_1,i_2,i_3) = \mu_4(i_4)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3) - \mu_2(i_4)^2\mu_2(i_1)\mu_2(i_2)\mu_2(i_3) \tag{104}$$

$$U_3(i_1,i_2) = \mu_6(i_1)^2\mu_2(i_2) - \mu_8(i_1)\mu_4(i_1)\mu_2(i_2) \tag{105}$$

$$U_4(i_1,i_2) = \mu_4(i_2)^2\mu_4(i_1) - \mu_6(i_2)\mu_4(i_1)\mu_2(i_2) \tag{106}$$

$$U_5(i_1,i_2,i_3) = \mu_4(i_1)\mu_2(i_3)^2\mu_2(i_2) - \mu_4(i_1)\mu_4(i_3)\mu_2(i_2) \tag{107}$$

$$U_6(i_1) = \mu_6(i_1)\mu_4(i_1) - \mu_8(i_1)\mu_2(i_1) \tag{108}$$

$$U_7(i_1) = \mu_6(i_1)^2 - \mu_{10}(i_1)\mu_2(i_1) \tag{109}$$

$$U_8(i_1) = \mu_4(i_1)^2 - \mu_6(i_1)\mu_2(i_1) \tag{110}$$

$$U_9(i_1,i_2) = \mu_6(i_1)\mu_2(i_1)\mu_2(i_2) - \mu_4(i_1)^2\mu_2(i_2) \tag{111}$$

$$U_{10}(i_1,i_2) = \mu_8(i_1)\mu_2(i_1)\mu_2(i_2) - \mu_6(i_1)\mu_4(i_1)\mu_2(i_2) \tag{112}$$

$$U_{11}(i_1,i_2) = \mu_6(i_1)\mu_4(i_2)\mu_2(i_1) - \mu_6(i_1)\mu_2(i_2)^2\mu_2(i_1) - \mu_4(i_1)^2\mu_4(i_2) + \mu_4(i_1)^2\mu_2(i_2)^2 \tag{113}$$

$$U_{12}(i_1,i_2) = \mu_4(i_1)\mu_4(i_2) - \mu_4(i_1)\mu_2(i_2)^2 \tag{114}$$

$$U_{13}(i_1,i_2) = \mu_6(i_2)\mu_2(i_1) - \mu_4(i_2)\mu_2(i_1)\mu_2(i_2) \tag{115}$$

$$U_{14}(i_1,i_2) = U_{13}(i_1,i_2) \times \mu_4(i_1)\mu_2(i_2) - U_{12}(i_1,i_2) \times \mu_4(i_2)\mu_2(i_1) \tag{116}$$

$$U_{15}(i_1,i_2) = U_{13}(i_1,i_2) \times \mu_6(i_1)\mu_2(i_2) - U_{12}(i_1,i_2) \times \mu_4(i_1)\mu_4(i_2) \tag{117}$$

$$U_{16}(i_1,i_2) = U_{13}(i_1,i_2) \times \mu_8(i_1)\mu_2(i_2) - U_{12}(i_1,i_2) \times \mu_6(i_1)\mu_4(i_2) \tag{118}$$

$$U_{17}(i_1,i_2) = U_{13}(i_1,i_2)\times U_{19}(i_1,i_2) - U_{12}(i_1,i_2)\times[\mu_8(i_2)\mu_2(i_1) - \mu_4(i_2)^2\mu_2(i_1)] \tag{119}$$

$$U_{18}(i_1,i_2) = U_{13}(i_1,i_2)\times U_{20}(i_1,i_2) - U_{12}(i_1,i_2)\times[\mu_6(i_2)\mu_4(i_1) - \mu_4(i_1)\mu_4(i_2)\mu_2(i_2)] \tag{120}$$

$$U_{19}(i_1,i_2) = \mu_6(i_1)\mu_4(i_2) - \mu_4(i_1)\mu_4(i_2)\mu_2(i_2) \tag{121}$$

$$U_{20}(i_1,i_2) = \mu_6(i_1)\mu_4(i_2) - \mu_6(i_1)\mu_2(i_2)^2 \tag{122}$$

$$U_{21}(i_1,i_2,i_3) = \mu_4(i_2)\mu_2(i_1)\mu_2(i_3) - \mu_2(i_2)^2\mu_2(i_1)\mu_2(i_3) \tag{123}$$

$$U_{22}(i_1,i_2,i_3) = \mu_6(i_3)\mu_2(i_1)\mu_2(i_2) - \mu_4(i_3)\mu_2(i_1)\mu_2(i_2)\mu_2(i_3) \tag{124}$$

$$U_{23}(i_1,i_2,i_3) = \mu_4(i_1)\mu_4(i_3)\mu_2(i_2) - \mu_4(i_1)\mu_2(i_3)^2\mu_2(i_2) \tag{125}$$

$$U_{24}(i_1,i_2,i_3) = \mu_4(i_2)\mu_4(i_3)\mu_2(i_1) - \mu_4(i_2)\mu_2(i_3)^2\mu_2(i_1) - \mu_4(i_3)\mu_2(i_2)^2\mu_2(i_1) + \mu_2(i_2)^2\mu_2(i_3)^2\mu_2(i_1) \tag{126}$$

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