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# THE COMPOSITE DESIGN OF H?^?-ERL SLIDING-MODE CONTROLLER

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# THE COMPOSITE DESIGN OF $H_\infty$-ERL SLIDING-MODE CONTROLLER

Shun-Min Wang, Zhi-Hao Chen, and Cheng-Neng Hwang

Key words: sliding-mode control, $H_\infty$ control theory, lag-lead compensator, popov criterion.

## ABSTRACT

In a multi-input multi-output nonlinear system, because the system is subjected to the impacts of external disturbances and parametric uncertainties, its output response may not be able to satisfy the desired specification or even may make the system unstable. The $H_\infty$-ERL sliding mode controller proposed in this paper is motivated to solve these problems.

This controller utilizes the concept of sliding mode controller with ERL (Exponential Reaching Law) as its major framework, and then uses Lyapunov stability theorem to ensure the closed-loop stability when the system encounters prescribed external disturbances and parametric uncertainties. For the optimal selection of the adjustable parameters in the proposed sliding mode controller with ERL, the $H_\infty$ control methodology and the Lag-Lead compensator are formulated together in the proposed control scheme to find optimal control gains, which are used to minimize the ill-effect of external disturbances and plant parametric uncertainties on the controlled output. The closed-loop poles of the augmented system are then placed on the specified region to match the desired performance. The Popov criterion is then applied to handle the uncanceled dynamics caused by the unmodeled uncertainties so that the system robustness can be guaranteed.

Finally, an ROV (Remotely Operated underwater Vehicle) is controlled and simulated by the proposed controller. The simulation results reveal that the proposed control law is robust to plant uncertainties and disturbances while the desired specifications assigned by the users are matched.

## I. INTRODUCTION

To design a controller for a multi-input multi-output nonlinear system, we need to consider its parametric uncertainties and external disturbances. The parametric uncertainties have direct impact on the performance of system, for example, the load mass of an elevator will impact its stability. Furthermore, the performance of a system will be influenced by external disturbances, such as unstable power supply or wind on a boat. These disturbances and uncertainties will make the system response unstable or unable to achieve the desired specification.

Sliding mode control is an extraordinary type of variable structure control, it was first proposed in 1950's in Soviet Union. The famous sliding mode control was proposed by Slotine and Sastry (1983), the design concept is to choose a sliding surface and design a controller. This controller forces the system states to arrive at sliding surface. When system states arrive at sliding surface, it will slide into equilibrium points even if the system is influenced by parametric uncertainties and external disturbances. Finally, we use Lyapunov stability theorem to prove the stability of the closed-loop system. The disadvantage of sliding mode control is chattering phenomenon, in order to remove this disadvantage, Fallaha et al. (2011) proposed a novel sliding mode control. Because of the advantages of sliding mode controller, it is widely used in industry. Some important studies of sliding mode control are published in literatures (Utkin, 1977; Slotine, 1984; Hwang, 1986; Slotine and Li, 1991; Gao, 1993; Hung et al., 1993; Park and Tsuji, 1999; Utkin et al., 1999; Young, 1999; Yu and Kaynak, 2009).

In $H_\infty$ control theory, there are two methods to solve $H_\infty$ control problem, one is polynomial approach, and the other is state space method. Polynomial approach was proposed by Slotine and Sastry (1983), Francis (1987) and Kimura (1989), it transforms the $H_\infty$ control problem into the model matching problem. The difficulty of polynomial approach is that it requires complex calculation. The state space method was proposed by Doyle et al. (1989), it only needs to solve the Riccati equation. However, this method has the limit of orthogonality assumption, so it is not easy to apply to real system. Hwang (1993) proposed the variational approach to get the same conclusion as Doyle et al., but Hwang removed the orthogonality assumption, so the state space method can easily apply to the real system, especially in the reduced order system models. The proposed theorem by Hwang (1993) is given as follows:

Consider the $H_\infty$ standard problem form as Eq. (1).

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$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $y(t) \in R^p$, $u(t) \in R^m$, $w(t) \in R^r$ and $z(t) \in R^l$ denote the system states, measured outputs, control input, exogenous input and controlled output. Suppose that $(A, B_2)$ is controllable, $(C_1, A)$ is observable, $D_{12}^T D_{12} = I$. Then, the $H_\infty$ optimal state feedback control law $u(t)$ minimizes $\|z(t)\|_2$ under the worst exogenous input in a prespecified set in $L_2[0, \infty)$ is:

$$u(t) = -(B_2^T k_1 x + D_{12}^T C_1) x(t) \tag{2}$$

where $k_1$ is the positive definite solution of the Algebraic Riccati Equation (ARE):

$$\begin{aligned} (A - B_2 D_{12}^T C_1)^T k_1 + k_1 (A - B_2 D_{12}^T C_1) + k_1 (B_1 B_1^T - B_2 B_2^T) k_1 \\ + C_1^T (I - D_{12} D_{12}^T)(I - D_{12} D_{12}^T) C_1 = 0 \end{aligned} \tag{3}$$

In order to unite the advantages of above controllers, we combine sliding mode control with $H_\infty$ control methodology into a novel controller: $H_\infty$-ERL sliding mode controller.

Finally a ROV is used to be an example to demonstrate the robustness and tracking performance of the proposed controller.

## II. SYSTEM DESCRIPTION

Consider a multi-input multi-output nonlinear system, the dynamic model can be written as follows:

$$\begin{aligned} \ddot{q}_i(t) = F_i\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta}\right) + B_i\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta}\right)\tau_i(t) \\ + \tau_{mwi}, i = 1, 2, \cdots, n \end{aligned} \tag{4}$$

where $q_i(t)$ is system state, $F_i\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta}\right)$ and $\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta}\right)$ are nonlinear functions, $\underline{\delta}$ is uncertain parameter, $\tau_i(t)$ is control input, and $\tau_{mwi}$ is external disturbance. $F_{oi}\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta_o}\right)$ and $B_{oi}\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta_o}\right)$ are the nominal values of the nonlinear system, $B_{oi}\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta_o}\right)$ is invertible ($B_{oi}^{-1}$ exists), and they can be written as:

$$\begin{cases} F_{oi}\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta_o}\right) \equiv F_{oi} \\ B_{oi}\left(\underline{q}(t), \underline{\dot{q}}(t), \underline{\delta_o}\right) \equiv B_{oi} \end{cases} \tag{5}$$

Define error $e_i = q_i - q_{di}$, $\dot{e}_i = \dot{q}_i - \dot{q}_{di}$ and $\ddot{e}_i = \ddot{q}_i - \ddot{q}_{di}$, where $q_{di}$ is the reference input.

## III. THE DESIGN OF SLIDING-MODE CONTROLLER WITH ERL

When we design sliding mode controller, we need to define the sliding surface $s$ and the desired form of $\dot{s}$ at first.

Define the sliding surface $s$ as follows:

$$s_i = \dot{e}_i + \lambda_i e_i, \lambda_i > 0, i = 1, 2, \cdots, n \tag{6}$$

Define the desired form of $\dot{s}$ as follows (Fallaha et al., 2011):

$$\dot{s}_i = -k_i \operatorname{sgn}(s_i) - Q_i s_i, i = 1, 2, \cdots, n \tag{7}$$

where $k_i = \dfrac{g_i}{N(s_i)} > 0, g_i > 0, Q_i \geq 0,$ exponential variation $N(x) = \delta_0 + (1 - \delta_0) e^{-\alpha_{ERL}|x|}, \alpha_{ERL} > 0, 0 < \delta_0 \leq 1$. It is called the exponential reaching law (ERL).

The purpose of controller is to make the $\dot{s}_i$ change into the desired form as Eq. (7). Aim at the system of Eq. (4); we can design a controller as follows:

$$\tau_i(t) = \frac{1}{B_{oi}}\left(-F_{oi} + \ddot{q}_{di} - \lambda_i \dot{e}_i - k_i \operatorname{sgn}(s_i) - Q_i s_i\right) \tag{8}$$

In real applications, the sign function sgn($s_i$) has the feature of fast switching velocity with ultrahigh frequency, and the feature makes the control force have the phenomenon of chattering, so it can not apply to real industrial system. In order to solve the above-mentioned problems, many literatures replace sign function *sgn*($s$) with saturation function sat($s$) (Slotine, 1984; Slotine and Li, 1991; Utkin et al., 1999). Therefore, the controller can redesign as follows:

$$\tau_i(t) = \frac{1}{B_{oi}}\left(-F_{oi} + \ddot{q}_{di} - \lambda_i \dot{e}_i - k_i sat(s_i) - Q_i s_i\right) \tag{9}$$

where

$$sat(s_i) = \begin{cases} \operatorname{sgn}(s_i), |s_i| > \varepsilon, \varepsilon \to 0 \\ \dfrac{s_i}{\varepsilon}, |s_i| \leq \varepsilon, \varepsilon \to 0 \end{cases}$$

## IV. THE COMPOSITE DESIGN OF $H_\infty$-ERL SLIDING-MODE CONTROLLER

In order to optimize the adjustable parameters of sliding mode control with ERL, we rewrite the controller of Eq. (9) into the following form:

$$\begin{aligned}\tau_i(t) &= \frac{1}{B_{oi}}\left[-F_{oi}+\ddot{q}_{di}-k_i sat(s_i)\right]+\frac{1}{B_{oi}}\left[-\lambda_i\dot{e}_i-Q_i s_i\right]\\ &= \frac{1}{B_{oi}}\left[-F_{oi}+\ddot{q}_{di}-k_i sat(s_i)\right]+\frac{-1}{B_{oi}}\left[\lambda_i\dot{e}_i+Q_i\dot{e}_i+Q_i\lambda_i e_i\right]\\ &= \frac{1}{B_{oi}}\left[-F_{oi}+\ddot{q}_{di}-k_i sat(s_i)\right]+\frac{-1}{B_{oi}}\left[K_{Di}\dot{e}_i+K_{Pi}e_i\right]\end{aligned} \tag{10}$$

where $K_{Di}\equiv\lambda_i+Q_i$, $K_{Pi}\equiv Q_i\lambda_i$.

We introduce $H_\infty$ control methodology to get the optimal parameters $K_{Pi}$ and $K_{Di}$, where $K_{Pi}$ and $K_{Di}$ minimize the ill-effect caused by external disturbances and plant parametric uncertainties on controlled output. Aim at the control gain $K_{Pi}$ and $K_{Di}$ of Eq. (10), we define an equivalent control input $U_{H\infty i}$ as: $U_{H\infty i}=-K_{Di}\dot{e}_i-K_{Pi}e_i$. Therefore, we can rewrite the controller as follows:

$$\tau_i(t)=\frac{1}{B_{oi}}\left[-F_{oi}+\ddot{q}_{di}-k_i sat(s_i)\right]+\frac{1}{B_{oi}}U_{H\infty i} \tag{11}$$

In Eq. (11), there are all known parameters except for $U_{H\infty i}$. For this reason, we only need to design $U_{H\infty i}$, and then we can obtain actually control input $\tau_i(t)$. Substitute the controller of Eq. (11) into the system of Eq. (4), we can obtain:

$$\begin{aligned}\ddot{q}_i-\ddot{q}_{di} &= U_{H\infty i}+\left(F_i-\frac{B_i}{B_{oi}}F_{oi}\right)\\ &\quad+\left(\frac{B_i}{B_{oi}}-1\right)\left(\ddot{q}_{di}+U_{H\infty i}\right)\\ &\quad-\frac{B_i}{B_{oi}}k_i sat(s_i)+\tau_{mwi}\end{aligned} \tag{12}$$

We define $\bar{x}_{1i}(t)=e_i$, $\bar{x}_{2i}(t)=\dot{e}_i$ and $d_i(t)=(F_i-\frac{B_i}{B_{oi}}F_{oi})+(\frac{B_i}{B_{oi}}-1)(\ddot{q}_{di}+U_{H\infty i})-\frac{B_i}{B_{oi}}k_i sat(s_i)+\tau_{mwi}$. $d_i(t)$ is external disturbance. Let $\bar{X}_i(t)=\begin{bmatrix}\bar{x}_{1i}(t)\\ \bar{x}_{2i}(t)\end{bmatrix}$, we can obtain the equivalent state space equation as follows:

$$\begin{cases}\dot{\bar{X}}_i=\bar{A}_i\bar{X}_i+\bar{B}_iU_{H\infty i}+\bar{G}_id_i\\ \bar{Y}_i=\bar{C}_i\bar{X}_i\end{cases} \tag{13}$$

where $\bar{X}_i(t)\in R^2$ is system state, $\bar{Y}_i(t)\in R^1$ is system output, $U_{H\infty i}\in R^1$ is equivalent control input, and $d_i(t)\in R^1$ is disturbance. $\bar{A}_i$, $\bar{B}_i$, $\bar{G}_i$ and $\bar{C}_i$ are constant matrix.

Following, we design the equivalent control input $U_{H\infty i}$ for the system of Eq. (13). We add servo compensator $S_{ci}$ and stabilizing compensator $S_{si}$ to compensate the system. Because the system states of Eq. (13) are the errors of original system of Eq. (4), we define the reference input $r_{nom}\equiv 0$, and we let integrator as servo compensator $S_{ci}$. The state space equation of servo compensator $S_{ci}$ is as follows:

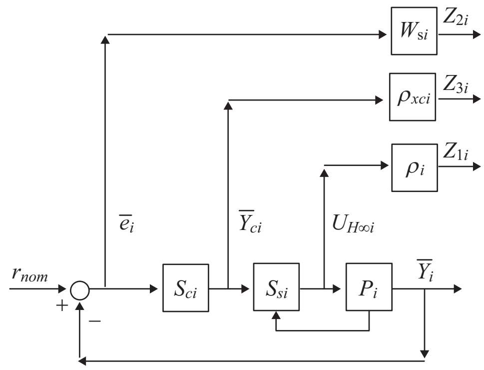

**Fig. 1. Augmented system diagram.**

$$\begin{cases}\dot{\bar{X}}_{ci}=\bar{A}_{ci}\bar{X}_{ci}+\bar{B}_{ci}(r_{nom}-\bar{Y}_i)\\ \quad\;\;=\bar{A}_{ci}\bar{X}_{ci}+\bar{B}_{ci}\bar{e}_i\\ \bar{Y}_{ci}=\bar{C}_{ci}\bar{X}_{ci}\end{cases} \tag{14}$$

where

$$\bar{A}_{ci}=0,\ \bar{B}_{ci}=1,\ \bar{C}_{ci}=1,\ \bar{X}_{ci}\in R^1 \text{ and } \bar{e}_i=(r_{nom}-\bar{Y}_i).$$

This is a regular problem that the system outputs are equal to the errors of the original system. As a result, we define the weighting function of error as $W_{si}$ and the weighting function of $Y_{ci}$ as $\rho_{xci}$. We also define the weighting function of $U_{H\infty i}$ as $\rho_i$, and let $u_i(t)=\rho_iU_{H\infty i}$, so the controlled output $z_i(t)$ is as follows:

$$z_i(t)\equiv\begin{bmatrix}z_{1i}(t)\\ z_{2i}(t)\\ z_{3i}(t)\end{bmatrix}=\begin{bmatrix}\rho_iU_{H\infty i}\\ W_{si}\bar{e}_i\\ \rho_{xci}\bar{Y}_{ci}\end{bmatrix} \tag{15}$$

The augmented system diagram is shown as Fig. 1. In Fig. 1, $P_i$ represents the system of Eq. (13).

The state space equation of weighting function $W_{si}$ is as follows:

$$\begin{cases}\dot{\bar{X}}_{si}=\bar{A}_{si}\bar{X}_{si}+\bar{B}_{si}e_i\\ z_{2i}=\bar{C}_{si}\bar{X}_{si}\end{cases} \tag{16}$$

where $\bar{A}_{si}\in R^{1\times 1}$, $\bar{B}_{si}\in R^{1\times 1}$, $\bar{C}_{si}\in R^{1\times 1}$.

Combine Eq. (13), Eq. (14) and Eq. (16), we can obtain the standard H∞ state space equation as follows:

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_{1i} w_i(t) + B_{2i} u_i(t) \\ z_i(t) = C_{1i} x_i(t) + D_{12i} u_i(t) \\ y_i(t) = C_{2i} x_i(t) + D_{21i} w_i(t) \end{cases} \tag{17}$$

where $x_i(t) = [\overline{X}_i \ \overline{X}_{ci} \ \overline{X}_{si}]^T \in R^4$ is system state. $y_i(t) = r_{nom} - \overline{Y}_i + z_{2i}(t) + z_{3i}(t) \in R^1$ is measured output. $z_i(t) \in R^3$ is controlled output. $W_i(t) = [r_{nom} \ d_i(t)]^T \in R^2$ is external input. $u_i(t) = \rho_i U_{H\infty i} \in R^{1\times 1}$ is control input. $A_i$, $B_{1i}$, $B_{2i}$, $C_{1i}$, $D_{12i}$, $C_{2i}$ and $D_{21i}$ are constant matrix.

In Eq. (17), if we use H∞ control methodology to get optimal control input $u_i(t)$, the poles of closed loop system maybe locate at the neighborhood of imaginary axis. For this reason, we replace $A_i$ with $A_i + \beta_i I$, and it ensures that the poles of system locate on left half plane of $-\beta_i$ $(\beta_i > 0)$. Therefore, we can obtain the standard H∞ state space equation as follows:

$$\begin{cases} \dot{x}_i(t) = A_{\beta i} x_i(t) + B_{1i} w_i(t) + B_{2i} u_i(t) \\ z_i(t) = C_{1i} x_i(t) + D_{12i} u_i(t) \\ y_i(t) = C_{2i} x_i(t) + D_{21i} w_i(t) \end{cases} \tag{18}$$

where $A_{\beta i} = A_i + \beta_i I$.

Design stabilizing compensator $S_{si}$ for the system of Eq. (18). According to Hwang (1993), we know that if $(A_{\beta i}, B_{2i})$ is controllable, $(C_{1i}, A_{\beta i})$ is observable and $D_{12i}^T D_{12i} = I$, the H∞ optimal state feedback control law $u_i(t)$ minimizing $\|z_i(t)\|_2$ under the worst exogenous input is:

$$u_i(t) = K_{\infty i} x_i(t) \tag{19}$$

where $K_{\infty i}$ is control gain matrix:

$$K_{\infty i} = -(B_{2i}^T k_{1i} + D_{12i}^T C_{i1}) \tag{20}$$

where $k_{1i}$ is the positive definite symmetrical solution ($k_{1i} = k_{1i}^T > 0$) of the following Algebraic Riccati Equation (ARE):

$$\begin{cases} A_{\tau i}^T k_{1i} + k_{1i} A_{\tau i} + k_{1i}(B_{1i} B_{1i}^T - B_{2i} B_{2i}^T) k_{1i} + C_{\tau i}^T C_{\tau i} = 0 \\ A_{\tau i} = A_{\beta i} - B_{2i} D_{12i}^T C_{1i} \\ C_{\tau i} = (I - D_{12i} D_{12i}^T) C_{1i} \end{cases} \tag{21}$$

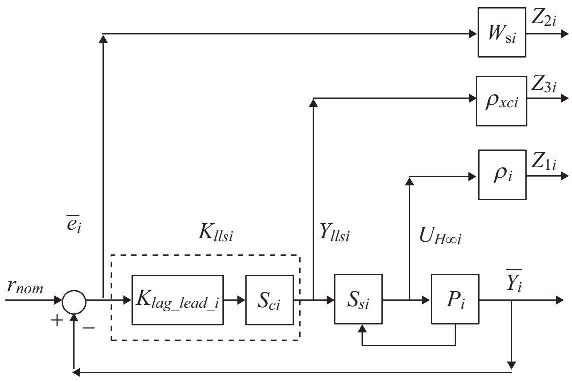


**Fig. 2. System structure with Lag-Lead compensator.**

Because $\frac{1}{\rho_i} K_{\infty i} \equiv [K_{\infty ei} \mid K_{\infty xci} \mid K_{\infty xsi}]$ and $K_{\infty ei} \equiv [-K_{Pi} \mid -K_{Di}]$, we can obtain the optimal parameters $K_{Pi}$ and $K_{Di}$.

Substitute the controller $u_i(t)$ into the system of Eq. (17), we will find the transfer function $G_{xcd\_ybar\_i}$ between $\dot{\overline{X}}_{ci}$ and $\overline{Y}_i$. If the performance (Phase margin, error constant, etc.) of $G_{xcd\_ybar\_i}$ does not satisfy our design specification, we design a Lag-Lead compensator for $G_{xcd\_ybar\_i}$. The performance of the system will satisfy our specification after we add Lag-Lead compensator. We combine servo compensator $S_{ci}(s)$ with Lag-Lead compensator $K_{lag\_lead\_i}(s)$ into complex compensator $K_{llsi}(s)$. The state space equation of complex compensator $K_{llsi}(s)$ is as follows:

$$\begin{cases} \dot{X}_{llsi}(s) = A_{llsi} X_{llsi} + B_{llsi} \overline{e}_i \\ Y_{llsi}(s) = C_{llsi} X_{llsi}) \end{cases} \tag{22}$$

where $X_{llsi} \in R^{3\times 1}$ is compensator state. $A_{llsi}$, $B_{llsi}$ and $C_{llsi}$ are constant matrix. The diagram of system structure with Lag-Lead compensator is as Fig. 2 (where the thick frame represents $K_{llsi}(s)$).

In order to get the optimization of whole performance, we need to replace servo compensator $S_{ci}(s)$ with complex compensator $K_{llsi}(s)$, and augment the system again to get optimal stabilizing compensator $S_{si}(s)$. We combine Eq. (13), Eq. (16) and Eq. (22), and then we can obtain the following standard H∞ state space Eq. (23).

$$\begin{cases} \dot{x}_{ki}(t) = A_{ki} x_{ki}(t) + B_{k1i} w_i(t) + B_{k2i} u_i(t) \\ z_{ki}(t) = C_{k1i} x_{ki}(t) + D_{k12i} u_i(t) \\ y_{ki}(t) = C_{k2i} x_i(t) + D_{k21i} w_i(t) \end{cases} \tag{23}$$

where $x_{ki}(t)=\left[\bar{X}_i\ \bar{X}_{ci}\ \bar{X}_{si}\right]^T \in R^6$ is system state. $y_{ki}(t)= r_{nom}-\bar{Y}_i+z_{2i}(t)+z_{3i}(t)\in R^1$ is measured output. $z_{ki}(t)\in R^3$ is controlled output. $W_i(t)=[r_{nom}\ d_i(t)]^T \in R^2$ is external input. $u_i(t)=\rho_i U_{H\infty i}\in R^{1\times 1}$ is control input. $A_{ki}, B_{k1i}, B_{k2i}, C_{k1i}, D_{k12i}, C_{k2i}$ and $D_{k21i}$ are constant matrix.

In order to ensure that the poles of system locate on left half plane of $-\beta_i$ ($\beta_i>0$), we replace $A_{ki}$ with $A_{ki}+\beta_i I$. Therefore, we can obtain the standard H∞ state space equation as follows:

$$\begin{cases}\dot{x}_{ki}(t)=A_{\beta ki}x_{ki}(t)+B_{k1i}w_i(t)+B_{k2i}u_i(t)\\ z_{ki}(t)=C_{k1i}x_{ki}(t)+D_{k12i}u_i(t)\\ y_{ki}(t)=C_{k2i}x_i(t)+D_{k21i}w_i(t)\end{cases} \tag{24}$$

where $A_{\beta ki}=A_{ki}+\beta_i I$.

Design stabilizing compensator $S_{si}$ for the system of Eq. (24). Suppose that $(A_{\beta ki}, B_{k2i})$ is controllable, $(C_{k1i}, A_{\beta ki})$ is observable and $D_{k12i}^T D_{k12i}=I$, the H∞ optimal state feedback control law $u_i(t)$ minimizing $\|z_{ki}(t)\|_2$ under the worst exogenous input is:

$$u_i(t)=K_{k\infty i}x_{ki}(t) \tag{25}$$

where $K_{k\infty i}$ is control gain matrix:

$$K_{k\infty i}=-(B_{k2i}^T k_{k1i}+D_{k12i}^T C_{ki1}) \tag{26}$$

where $k_{k1i}$ is the positive definite symmetrical solution ($k_{k1i}=k_{k1i}^T>0$) of the following Algebraic Riccati Equation (ARE):

$$\begin{cases}A_{k\tau i}^T k_{k1i}+k_{k1i}A_{k\tau i}+k_{k1i}(B_{k1i}B_{k1i}^T-B_{k2i}B_{k2i}^T)k_{k1i}\\ \qquad +C_{k\tau i}^T C_{k\tau i}=0\\ A_{k\tau i}=A_{\beta ki}-B_{k2i}D_{k12i}^T C_{k1i}\\ C_{k\tau i}=(I-D_{k12i}D_{k12i}^T)C_{k1i}\end{cases} \tag{27}$$

Because $\frac{1}{\rho_i}K_{k\infty i}\equiv\left[K_{k\infty ei}\,|\,K_{k\infty xci}\,|\,K_{k\infty xsi}\right]$ and $K_{k\infty ei}\equiv\left[-K_{Pi}\,|\,-K_{Di}\right]$, we can obtain the optimal parameters $K_{Pi}$ and $K_{Di}$. From the aforementioned discussions, we have the controller:

$$\tau_i(t)=\frac{1}{B_{oi}}\left[-F_{oi}+\ddot{q}_{di}-k_i sat(s_i)-K_{Di}\dot{e}_i-K_{Pi}e_i\right] \tag{28}$$

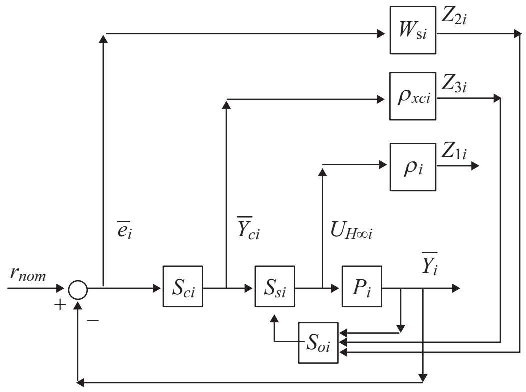


**Fig. 3. System structure with observer (Without Lag-Lead compensator).**

## State Observer

If the system state $x_i(t)$ of Eq. (17) or the system state $x_{ki}(t)$ of Eq. (23) is not measurable, we need to use the state observer $S_{oi}$ to estimate states. First, we design state observer $S_{oi}$ for the system of Eq. (17). The diagram of system structure with observer is as Fig. 3.

According to Hwang (1993), we have the following observer: if $(A_i, B_{1i})$ and $(A_i, B_{2i})$ are controllable, $(C_{1i}, A_i)$ and $(C_{2i}, A_i)$ are observable, $D_{12i}^T D_{12i}=I$ and $D_{21i}^T D_{21i}=I$, the state observer is as follows:

$$\begin{cases}\dot{\hat{x}}_i(t)=A_i\hat{x}_i(t)+B_{2i}u_i(t)+H_i\left(C_{2i}\hat{x}_i(t)-y_i(t)\right)\\ \qquad +B_{1i}w_{i\_worst}(t)\\ w_{i\_worst}(t)=B_{1i}^T k_{1i}\hat{x}_i(t)\end{cases} \tag{29}$$

where the optimal observer gain $H_i$ is:

$$H_i=-(I-h_{\infty i}k_{1i})^{-1}(h_{\infty i}C_{2i}^T+B_{1i}D_{21i}^T) \tag{30}$$

where $h_{\infty i}$ is the positive definite solution of the following Algebraic Riccati Equation (ARE):

$$\begin{cases}A_{\tau i}h_{\infty i}+h_{\infty i}A_{\tau i}^T+h_{\infty i}\left(C_{1i}^T C_{1i}-C_{2i}^T C_{2i}\right)h_{\infty i}\\ \qquad +B_{1\tau i}B_{1\tau i}^T=0\\ A_{\tau i}=A_i-B_{1i}D_{21i}^T C_{2i}\\ B_{1\tau i}=B_{1i}\left(I-D_{21i}^T D_{21i}\right)\end{cases} \tag{31}$$

Substitute the controller $u_i(t)$ into observer, and then we can obtain the H∞ optimal observer as follows:

$$\dot{\hat{x}}_i(t)=(A_i+B_{2i}K_{\infty i}+H_iC_{2i}+B_{1i}B_{1i}^T k_{1i})\hat{x}_i(t)-H_i y_i(t) \tag{32}$$

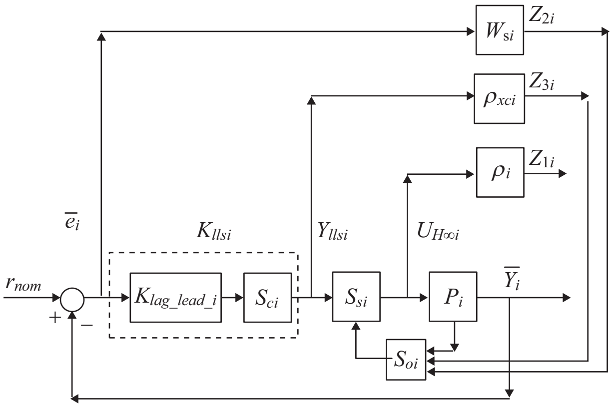


**Fig. 4. System structure with observer and Lag-Lead compensator.**

Similarly, we design state observer $S_{oi}$ for the system of Eq. (23). The diagram of system structure with observer and Lag-Lead compensator is as Fig. 4.

According to Hwang (1993), we have the following observer: if $\left(A_{ki}, B_{k1i}\right)$ and $\left(A_{ki}, B_{k2i}\right)$ are controllable, $\left(C_{k1i}, A_{ki}\right)$ and $\left(C_{k2i}, A_{ki}\right)$ are observable, $D_{12i}^T D_{12i} = I$ and $D_{21i}^T D_{21i} = I$, we have the following state observer:

$$\begin{cases} \dot{\hat{x}}_{ki}(t) = A_{ki}\hat{x}_{ki}(t) + B_{k2i}u_i(t) + H_{ki}\left(C_{k2i}\hat{x}_{ki}(t) - y_{ki}(t)\right) \\ \qquad\qquad + B_{k1i}w_{ki\_worst}(t) \\ w_{ki\_worst}(t) = B_{k1i}^T k_{k1i}\hat{x}_{ki}(t) \end{cases} \tag{33}$$

where the optimal observer gain $H_{ki}$ is as follows:

$$H_{ki} = -(I - h_{k\infty i}k_{k1i})^{-1}(h_{k\infty i}C_{k2i}^T + B_{k1i}D_{k21i}^T) \tag{34}$$

where $h_{k\infty i}$ is the positive definite solution of the following Algebraic Riccati Equation (ARE):

$$\begin{cases} A_{k\tau i}h_{k\infty i} + h_{k\infty i}A_{k\tau i}^T + h_{k\infty i}\left(C_{k1i}^T C_{k1i} - C_{k2i}^T C_{k2i}\right)h_{k\infty i} \\ \qquad + B_{k1\tau i}B_{k1\tau i}^T = 0 \\ A_{k\tau i} = A_{ki} - B_{k1i}D_{k21i}^T C_{k2i} \\ B_{k1\tau i} = B_{k1i}\left(I - D_{k21i}^T D_{k21i}\right) \end{cases} \tag{35}$$

Substitute the controller $u_i(t)$ into observer, and then we can obtain the H∞ optimal observer as follows:

$$\begin{aligned} \dot{\hat{x}}_{ki}\left(t\right) = {} & (A_{ki} + B_{k2i}K_{k\infty i} + H_{ki}C_{k2i} + B_{k1i}B_{k1i}^T k_{k1i})\hat{x}_{ki}(t) \\ & - H_{ki}y_{ki}\left(t\right) \end{aligned} \tag{36}$$

From aforementioned derivation, the proposed H∞-ERL Sliding-Mode Controller is designed in Theorem A shown as follows:

**Theorem A**

Consider a multi-input multi-output nonlinear system as Eq. (4) and the state space equation as Eq. (24). The H∞-ERL sliding mode controller is

$$\tau_i(t) = \frac{1}{B_{oi}}\left[-F_{oi} + \ddot{q}_{di} - k_i sat(s_i) - K_{Di}\dot{e}_i - K_{Pi}e_i\right] \tag{37}$$

where

1. $(A_{\beta ki}, B_{k2i})$ is controllable, $(C_{k1i}, A_{\beta ki})$ is observable and $D_{k12i}^T D_{k12i} = I$.
2. Saturation function

$$sat(s_i) = \begin{cases} \text{sgn}(s_i), \left|s_i\right| > \varepsilon, \varepsilon \to 0 \\ \dfrac{s_i}{\varepsilon}, \left|s_i\right| \le \varepsilon, \varepsilon \to 0 \end{cases}$$

3. $k_i = \dfrac{g_i}{N(s_i)} > 0,\ g_i = \Delta_{i\max} + \eta_i, \eta_i > 0$
4. $\max\left|\dfrac{B_{oi}}{B_i}F_i - F_{oi} + \ddot{q}_{di} - \dfrac{B_{oi}}{B_i}\ddot{q}_{di} + \dfrac{B_{oi}}{B_i}\dfrac{K_{Pi}}{K_{Di}}\dot{e}_i + \dfrac{B_{oi}}{B_i}\tau_{mwi}\right| \le \Delta_{i\max}$
5. $N(x) = \delta_0 + (1-\delta_0)e^{-\alpha_{ERL}|x|}, \alpha_{ERL} > 0, 0 < \delta_0 \le 1$
6. Sliding surface $s_i = \dot{e}_i + \dfrac{K_{Pi}}{K_{Di}}e_i, \dfrac{K_{Pi}}{K_{Di}} > 0, i = 1, 2, \cdots, n$
7. $K_{Pi}$ and $K_{Di}$ can be obtained by the process according to the performance of $G_{xcd\_ybar\_i}$ to determine whether we need to design Lag-Lead compensator for $G_{xcd\_ybar\_i}$. Therefore, there are two different situations to obtain $K_{Pi}$ and $K_{Di}$.

**Situation 1:**

If the performance of transfer function $G_{xcd\_ybar\_i}$ is satisfied the desired specification, we do not need to add Lag-Lead compensator. Thus, the optimal control gains $K_{Pi}$ and $K_{Di}$ are obtained by:

$$K_{\infty ei} \equiv \left[-K_{Pi} \mid -K_{Di}\right] \tag{38}$$

where $K_{\infty ei}$ is obtained by $\dfrac{1}{\rho_i}K_{\infty i} \equiv \left[K_{\infty ei} \mid K_{\infty xci} \mid K_{\infty xsi}\right]$, and $K_{\infty i}$ is solved from the following equation:

$$K_{\infty i} = -(B_{2i}^T k_{1i} + D_{12i}^T C_{i1}) \tag{39}$$

where $\rho_i$, $B_{2i}$, $D_{12i}$ and $C_{1i}$ are shown in Eq. (18), and $k_{1i}$ is the positive definite symmetrical solution ( $k_{1i} = k_{1i}^T > 0$ ) of the Algebraic Riccati Equation (ARE) in Eq. (21).

**Situation 2:**

If the performance of the transfer function $G_{xcd\_ybar\_i}$ does not satisfy the desired specification, we need to add Lag-Lead compensator. Thus, the optimal control gains $K_{Pi}$ and $K_{Di}$ are obtained by:

$$K_{k\infty ei} \equiv \left[ -K_{Pi} \mid -K_{Di} \right] \tag{40}$$

where $K_{k\infty ei}$ is obtained by $\frac{1}{\rho_i} K_{k\infty i} \equiv \left[ K_{k\infty ei} \mid K_{k\infty xci} \mid K_{k\infty xsi} \right]$, and $K_{k\infty i}$ is given by following equation:

$$K_{k\infty i} = -(B_{k2i}^T k_{k1i} + D_{k12i}^T C_{ki1}) \tag{41}$$

where $\rho_i$, $B_{k2i}$, $D_{k12i}$ and $C_{k1i}$ are shown in Eq. (24), and $k_{k1i}$ is the positive definite symmetrical solution ( $k_{k1i} = k_{k1i}^T > 0$ ) of the Algebraic Riccati Equation (ARE) in Eq. (27). Then, if the following assumption is satisfied:

$$\begin{cases} \max \left| \frac{B_{oi}}{B_i} F_i - F_{oi} + \ddot{q}_{di} - \frac{B_{oi}}{B_i} \ddot{q}_{di} + \frac{B_{oi}}{B_i} \frac{K_{Pi}}{K_{Di}} \dot{e}_i + \frac{B_{oi}}{B_i} \tau_{mwi} \right| \le \Delta_{i\max} \\ 0 < B_{i\min} \le B_i \le B_{i\max}, \quad \Delta_{i\max} \quad \text{is bounded} \end{cases} \tag{42}$$

The proposed H∞-ERL sliding mode controller shown in Eq. (37) will make $|s_i| > \varepsilon$ in finite time and ensure that the closed-loop system is asymptotically stable, while it minimizes the H∞-norm of the transfer function between the external inputs ( $W_i(t) = \left[ r_{nom} \ d_i(t) \right]^T$ ) and the controlled outputs ( $z_i(t) = \left[ \rho_i U_{H\infty i} \ W_{si} \overline{e}_i \ \rho_{xci} \overline{Y}_{ci} \right]^T$ ). And it guarantees that the desired specifications can be matched.

**Proof of Theorem A:**

Choose a Lyapunov function candidate:

$$V_i(s_i) = \frac{1}{2} s_i^2, \ i = 1, 2, \cdots, n \tag{43}$$

where $V_i$ satisfies: $V_i(s_i) > 0, \forall s_i \neq 0, V_i(0) = 0.$

Differentiate Lyapunov function candidate with respect to time, so we can obtain $\frac{dV_i}{dt} = s_i \dot{s}_i$.

From Theorem A, we know:

$$s_i = \dot{e}_i + \frac{K_{Pi}}{K_{Di}} e_i, \ \dot{s}_i = \ddot{e}_i + \frac{K_{Pi}}{K_{Di}} \dot{e}_i = \ddot{q}_i - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \tag{44}$$

Substitute the system of Eq. (4) into above equation, we can obtain:

$$\dot{s}_i = F_i + B_i \tau_i(t) + \tau_{mwi} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \tag{45}$$

Substitute the controller of Eq. (37) into $\dot{s}_i$, we can obtain:

$$\begin{aligned} \dot{s}_i &= F_i + \frac{B_i}{B_{oi}} (-F_{oi} + \ddot{q}_{di} - k_i sat(s_i) - K_{Di} \dot{e}_i - K_{Pi} e_i) \\ &\quad + \tau_{mwi} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \\ &= F_i + \frac{B_i}{B_{oi}} (-F_{oi} + \ddot{q}_{di} - k_i sat(s_i) - K_{Di} (\dot{e}_i + \frac{K_{Pi}}{K_{Di}} e_i) \\ &\quad + \tau_{mwi} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \\ &= F_i + \frac{B_i}{B_{oi}} (-F_{oi} + \ddot{q}_{di} - k_i sat(s_i) - K_{Di} s_i) + \tau_{mwi} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \\ &= F_i - \frac{B_i}{B_{oi}} F_{oi} + \frac{B_i}{B_{oi}} \ddot{q}_{di} - \frac{B_i}{B_{oi}} k_i sat(s_i) - \frac{B_i}{B_{oi}} K_{Di} s_i \\ &\quad + \tau_{mwi} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i \\ &= F_i - \frac{B_i}{B_{oi}} F_{oi} + \frac{B_i}{B_{oi}} \ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i + \tau_{mwi} - \frac{B_i}{B_{oi}} K_{Di} s_i \\ &\quad - \frac{B_i}{B_{oi}} k_i sat(s_i) \\ &= F_i - \frac{B_i}{B_{oi}} F_{oi} + \frac{B_i}{B_{oi}} \ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i + \tau_{mwi} - \frac{B_i}{B_{oi}} K_{Di} s_i \\ &\quad - \frac{B_i}{B_{oi}} \frac{g_i}{N(s_i)} sat(s_i) \end{aligned} \tag{46}$$

From $\frac{dV_i}{dt} = s_i \dot{s}_i$, we can obtain (when $|s_i| \ge \varepsilon$ ):

$$\begin{aligned} \frac{dV_i}{dt} &= s_i \dot{s}_i \\ &= s_i [F_i - \frac{B_i}{B_{oi}} F_{oi} + \frac{B_i}{B_{oi}} \ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}} \dot{e}_i + \tau_{mwi} \\ &\quad - \frac{B_i}{B_{oi}} K_{Di} s_i - \frac{B_i}{B_{oi}} \frac{g_i}{N(s_i)} sat(s_i)] \end{aligned}$$

$$
\begin{aligned}
&= s_i[F_i - \frac{B_i}{B_{oi}}F_{oi} + \frac{B_i}{B_{oi}}\ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}}\dot{e}_i + \tau_{mwi}] \\
&\quad - \frac{B_i}{B_{oi}}\frac{g_i}{N(s_i)}sat(s_i)s_i - \frac{B_i}{B_{oi}}K_{Di}s_is_i \\
&\le s_i[F_i - \frac{B_i}{B_{oi}}F_{oi} + \frac{B_i}{B_{oi}}\ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}}\dot{e}_i + \tau_{mwi}] \\
&\quad - \frac{B_i}{B_{oi}}\frac{g_i}{N(s_i)}sat(s_i)s_i \\
&= s_i[F_i - \frac{B_i}{B_{oi}}F_{oi} + \frac{B_i}{B_{oi}}\ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}}\dot{e}_i + \tau_{mwi}] \\
&\quad - \frac{B_i}{B_{oi}}\frac{g_i}{N(s_i)}|s_i| \\
&\le s_i[F_i - \frac{B_i}{B_{oi}}F_{oi} + \frac{B_i}{B_{oi}}\ddot{q}_{di} - \ddot{q}_{di} + \frac{K_{Pi}}{K_{Di}}\dot{e}_i + \tau_{mwi}] \\
&\quad - \frac{B_i}{B_{oi}}g_i|s_i| \\
&= s_i\frac{B_i}{B_{oi}}[\frac{B_{oi}}{B_i}F_i - F_{oi} + \ddot{q}_{di} - \frac{B_{oi}}{B_i}\ddot{q}_{di} + \frac{B_{oi}}{B_i}\frac{K_{Pi}}{K_{Di}}\dot{e}_i \\
&\quad + \frac{B_{oi}}{B_i}\tau_{mwi}] - \frac{B_i}{B_{oi}}g_i|s_i| \\
&\le |s_i|\frac{B_i}{B_{oi}}[\frac{B_{oi}}{B_i}F_i - F_{oi} + \ddot{q}_{di} - \frac{B_{oi}}{B_i}\ddot{q}_{di} + \frac{B_{oi}}{B_i}\frac{K_{Pi}}{K_{Di}}\dot{e}_i \\
&\quad + \frac{B_{oi}}{B_i}\tau_{mwi}] - \frac{B_i}{B_{oi}}g_i|s_i| \\
&\le |s_i|\frac{B_i}{B_{oi}}\Delta_{i\max} - \frac{B_i}{B_{oi}}g_i|s_i| \\
&= |s_i|\frac{B_i}{B_{oi}}\Delta_{i\max} - \frac{B_i}{B_{oi}}(\Delta_{i\max} + \eta_i)|s_i| \\
&= |s_i|\frac{B_i}{B_{oi}}\Delta_{i\max} - \frac{B_i}{B_{oi}}\Delta_{i\max}|s_i| - \frac{B_i}{B_{oi}}\eta_i|s_i| \\
&= -\frac{B_i}{B_{oi}}\eta_i|s_i|
\end{aligned} \tag{47}
$$

where $\frac{B_i}{B_{oi}} > 0, \ \eta_i > 0$

From above derivation, we can obtain:

$$\frac{dV_i(s_i)}{dt} < 0, \forall s_i \ne 0 \text{ and } \frac{dV_i(s_i)}{dt} = 0, s_i = 0 \tag{48}$$

The result shows that the controller makes $|s_i| \le \varepsilon$ in finite time. From Lyapunov stability theorem, we know the controller which makes the closed loop system be asymptotically stable.

Q.E.D.

As for the plant with uncanceled uncertainties, the Popov criterion will be applied to take care of it and will be discussed in Theorem B.

**Theorem B**

Substitute the controller of Theorem A into the multi-input multi-output nonlinear system of Eq. (4), then the closed loop system can be written as:

$$\begin{cases} \dot{z}_i(t) = A_i z_i(t) + B_i v_i(t) \\ y_i(t) = C_i z_i(t) \\ v_i(t) = -\phi_i(t, y_i) \end{cases} \tag{49}$$

where

$$z_i(t) = \begin{bmatrix} e_i & \dot{e}_i \end{bmatrix}^T \in R^{2\times1},\ A_i \in R^{2\times2},\ B_i \in R^{2\times1},$$
$$C_i \in R^{1\times2},\ v_i \in R^{1\times1} \text{ and } G_i(s) \equiv C_i(sI - A_i)^{-1}B_i.$$

If the following conditions are satisfied, the point $z_i = 0$ is global asymptotically stable.

**Condition 1:** The whole poles of $G_i(s)$ locate on the left half plane.

**Condition 2:** $(A_i, B_i)$ is controllable and $(C_i, A_i)$ is observable.

**Condition 3:** $\phi_i(t, y_i)$ belongs to the sector $[-\alpha_i, \beta_i]$ for $\alpha_i \ge 0$ and $\beta_i > 0$.

**Condition 4:** There exist a constant $x_i \ge 0$, such that

$$\frac{1}{\beta_i + \alpha_i} + R_e\left[(1 + jwx_i)\frac{G_i(jw)}{1 - \alpha_i G_i(jw)}\right] > 0, \forall w \ge 0.$$

**Condition 5:** The poles of $\frac{G_i(jw)}{1 - \alpha_i G_i(jw)}$ are all on left half plane.

**Proof of Theorem B:**

Substitute the controller of Theorem A into nonlinear system, and then we can obtain the following equation:

$$\ddot{e}_i + K_{Di}\dot{e}_i + K_{Pi}e_i = d_i,\ i = 1, 2, \cdots, n \tag{50}$$

where $e_i$ is error, $d_i(t) \in R^1$ is uncertainty, $K_{Pi}$ and $K_{Di}$ are optimal parameters of controller.

Let $z_i(t) = \begin{bmatrix} e_i & \dot{e}_i \end{bmatrix}^T \in R^{2\times1}$, then we can obtain the following absolute stability problem:

$$\begin{cases} \dot{z}_i(t) = A_i z_i(t) + B_i v_i(t) \\ y_i(t) = C_i z_i(t) \\ v_i(t) = -\phi_i(t, y_i) \end{cases} \tag{51}$$

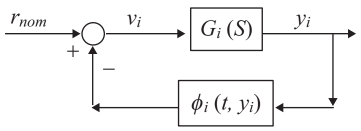


**Fig. 5. System structure in absolute stability problems.**

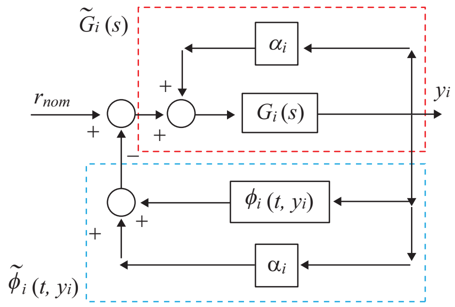


**Fig. 6. Loop transformation diagram.**

where $z_i(t) \in R^{2\times1}, A_i \in R^{2\times2}, B_i \in R^{2\times1}, C_i \in R^{1\times2}$, $v_i \in R^{1\times1}$ and $G_i(s) \equiv C_i(sI - A_i)^{-1}B_i$ with $s = jw$. $\phi_i\left(t, y_i\right)$ belongs to the sector $[-\alpha_i, \beta_i]$ for $\alpha_i \geq 0$ and $\beta_i > 0$ (i.e. $-\alpha_i y_i \leq \phi_i\left(t, y_i\right) \leq \beta_i y_i, \alpha_i \geq 0, \beta_i > 0$). The whole poles of $G_i(s)$ locate on the left half plane. $\left(A_i, B_i\right)$ is controllable and $\left(C_i, A_i\right)$ is observable. The block diagram of the absolute stability problem is as Fig. 5 $\left(r_{nom} \equiv 0\right)$. In order to use Popov criterion to show the stability of absolute stability problem, we need to transform the loop of Figure Fig. 5. Thus, the system via loop transformation can satisfy the conditions in Popov criterion. The diagram of loop transformation is as Fig. 6.

From Fig. 6, $\tilde{\phi}_i\left(t, y_i\right) = \phi_i\left(t, y_i\right) + \alpha_i$ and $\tilde{G}_i(s) = \dfrac{G_i(s)}{1-\alpha_i G_i(s)}$, where the sector of $\tilde{\phi}_i\left(t, y_i\right)$ is $\left[0, \beta_i + \alpha_i\right] = \left[0, Popov\_k_i\right]$ with $\alpha_i \geq 0$ and $\beta_i > 0$. Use Popov criterion, there exist a constant $x_i \geq 0$, such that

$$\frac{1}{\beta_i + \alpha_i} + R_e\left[(1 + jwx_i)\frac{G_i(jw)}{1-\alpha_i G_i(jw)}\right] > 0, \forall w \geq 0.$$

And the poles of $\dfrac{G_i(jw)}{1-\alpha_i G_i(jw)}$ are all on left half plane.

So, these conditions are all satisfied, the point $z_i = 0$ is global asymptotically stable.

Q.E.D.

## V. THE DESIGN PROCEDURES OF H$_\infty$-ERL SLIDING-MODE CONTROLLER

**Step 1:** Form the nonlinear system to the general equation form shown in Eq. (4).

**Step 2:** Define sliding surfaces $s_i$ and obtain the prototype of controller.

**Step 3:** Define $-K_{Di}\dot{e}_i - K_{Pi}e_i$ as an equivalent control input $U_{H\infty i}$.

**Step 4:** Substitute the controller into the system and obtain an equivalent state space equation.

**Step 5:** Choose the proper weighting function, $\beta_i$, $\gamma_i$, and the upper bound $\gamma_{upi}$ of $\gamma_i$, where $\gamma_{upi}$ is desired specification that $\left\|T_{wz}\right\|_{H\infty} < \gamma_{upi}$ ($T_{wz}$ is the transfer function between exogenous input and controlled output). Add servo compensator and stabilizing compensator to augment the system into standard H$_\infty$ state space Eq. (18).

**Step 6:** Scale and normalize the system to adjust the H$_\infty$-norm between $w_i(t)$ and $z_i(t)$ so that $\left\|T_{wz}\right\|_{H\infty}$ is squeezed to be less or equal to 1. To do that, we need to adjust $B_{1i}$ into $\gamma_i^{-0.5}B_{1i}$, adjust $B_{2i}$ into $\gamma_i^{0.5}B_{2i}$, adjust $C_{1i}$ into $\gamma_i^{-0.5}C_{1i}$ and adjust $C_{2i}$ into $\gamma_i^{0.5}C_{2i}$.

**Step 7:** Compute Eq. (21) and Eq. (31) to obtain $k_{1i}'$ and $h_{\infty i}'$. Then we get original system gain $k_{1i} = \gamma_i k_{1i}'$ and $h_{\infty i} = \gamma_i h_{\infty i}'$.

**Step 8:** If there are solutions in step 7, $k_{1i} \geq 0$, $h_{\infty i} \geq 0$, the maximum eigenvalue of $h_{\infty i}k_{1i}$ is smaller than one, then go to next step. Else, if $\gamma_i \geq \gamma_{upi}$, reduce $\rho_i$ and go back to step 5. Else, increase $\gamma_i$ and go back to step 5.

**Step 9:** If $\left\|T_{wz}\right\|_{H\infty}$ already satisfies the desired specifications, then go to step 10. Else, if $\gamma_i$ is the minimum which bases on given $\rho_i$, then go to step 10. Else, reduce $\gamma_i$ and go back to step 5.

**Step 10:** Plot the Bode plot of $G_{xcd\_ybar\_i}$.

**Step 11:** If the performance of $G_{xcd\_ybar\_i}$ do not satisfy the desired specifications, we have to design a Lag-Lead compensator $K_{lag\_lead\_i}$ for $G_{xcd\_ybar\_i}$, then augment standard H$_\infty$ state space equation again, and go to step 12, else, go to step 13.

**Step 12:** Scale the system and compute Eq. (27) and Eq. (35) to obtain $k_{k1i}$ and $h_{k\infty i}$. If there are solutions in Eq. (27) and Eq. (35), $k_{k1i} \geq 0, h_{k\infty i} \geq 0$, the maximum eigenvalue of $h_{k\infty i}k_{k1i}$ is smaller than one, then go to next step, else, go back to step 5.

**Step 13:** Find $\Delta_{i\max}$ such that the assumption of Theorem A is satisfied.

**Step 14:** Get the H$_\infty$-ERL sliding mode controller, which is in the form of Eq. (37).

**Step 15:** Do computer simulation. If the results do not satisfy the desired performance, go back to step 5.

## VI. COMPUTER SIMULATION

Consider the depth control system of an ROV. The dynamic equation of ROV can be expressed as Bessa et al. (2008):

$$M_{ROV}\ddot{z} + C_{ROV}\dot{z}|\dot{z}| + d_{sea} = u \tag{52}$$

where $M_{ROV}$ and $C_{ROV}$ are coefficient of ROV, z is the distance between ROV and sea level, $d_{sea}$ is disturbance, and $u$ is control input (Thrust force).

In this simulation, we define: $d_{sea}$ in the range of $\pm 5$ N, the upper bound of $M_{ROV}$ is $\bar{M}_{ROV} = 55$ Kg, the lower bound of $M_{ROV}$ is $\underline{M}_{ROV} = 45$ Kg, the upper bound of $C_{ROV}$ is $\bar{C}_{ROV} = 275$ Kg/m and the lower bound of $C_{ROV}$ is $\underline{C}_{ROV} = 225 Kg/m$. The nominal value of $M_{ROV}$ is chosen as $\tilde{M}_{ROV} = 1/2(\bar{M}_{ROV} + \underline{M}_{ROV}) = 50$ Kg and the nominal value of $C_{ROV}$ is chosen as $\tilde{C}_{ROV} = 1/2(\bar{C}_{ROV} + \underline{C}_{ROV}) = 250$ Kg/m. Our purpose is that the depth $z$ of ROV will track $z_d = 10 \times \dfrac{1-\cos(0.1\pi\tau)}{2}$.

The desired specification for this example is described as:

1. The phase margin of $G_{xcd\_ybar\_i}$ is greater than $55^0$.
2. The velocity error constant of $G_{xcd\_ybar\_i}$ is $K_v =$ 5.5 m/sec.

According to the design procedures, we have the following controller:

$$u(t) = \frac{1}{B_{oi}}\left[-F_{oi} + \ddot{z}_d - ksat(s) - K_D\dot{e} - K_P e\right] \tag{53}$$

where sliding surface: $s = \dot{e} + \dfrac{K_P}{K_D}e$,

$$e = z - z_d,\ K_P = 21.86,\ K_D = 7.25,$$

$$F_o = -5\dot{z}|\dot{z}|,\ B_o = \frac{1}{50},\ k = \frac{g}{N(s)},\ g = 15.1337,$$

$$N(s) = \delta_0 + (1-\delta_0)e^{-\alpha_{ERL}|s|},\ \alpha_{ERL} = 2,\quad \delta_0 = 0.8$$

Transfer function $G_{xcd\_ybar\_i}$ is as follows:

$$G_{xcd\_ybar\_i} = \frac{31.7s^3 + 1356s^2 + 14600s + 3480}{s^6 + 49.9s^5 + 786.1s^4 + 4258s^3 + 10140s^2 + 588.4s} \tag{54}$$

The velocity error constant $K_v = 5.9143 m/\sec$ and $P.M. = 55.1^0$ satisfy our specification.

The computer simulation results are as following cases: the case 1 is $M_{ROV} = 45$ and $C_{ROV} = 225$, the case 2 is $M_{ROV} = 50$ and $C_{ROV} = 250$, and the case 3 is $M_{ROV} = 55$ and $C_{ROV} = 275$. The initial state is $z = 1$ for all cases. We also show the simulation results of sliding mode controller with ERL. Therefore, we can compare the performance of H∞-ERL sliding mode controller and sliding mode controller with ERL.

**Case 1 ($M_{ROV} = 45$ and $C_{ROV} = 225$)**

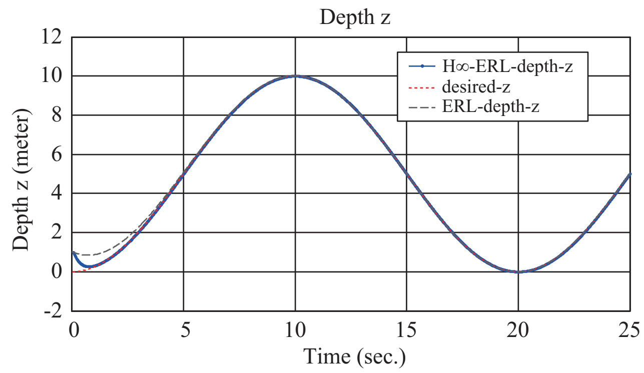


**Fig. 7. Depth z response.**

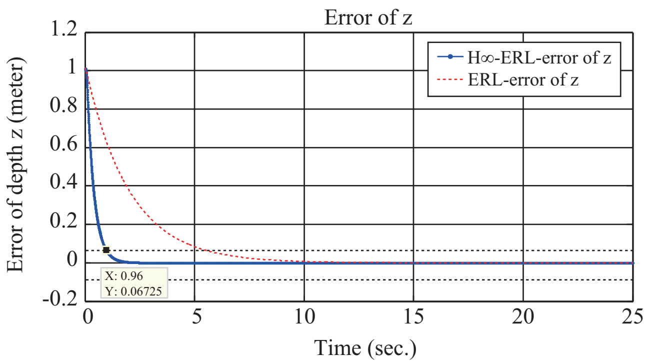


**Fig. 8. Error of depth z.**

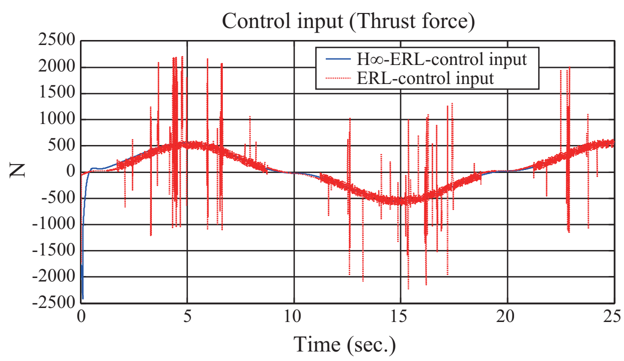


**Fig. 9. Control input *u*.**

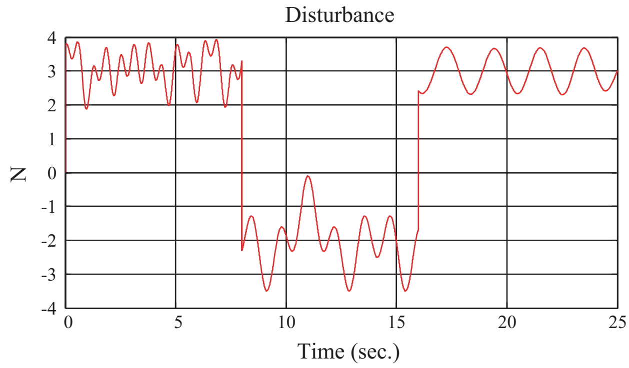


**Fig. 10. Disturbance ($d_{sea}$).**

**Case 2 ( $M_{ROV} = 50$ and $C_{ROV} = 250$ )**

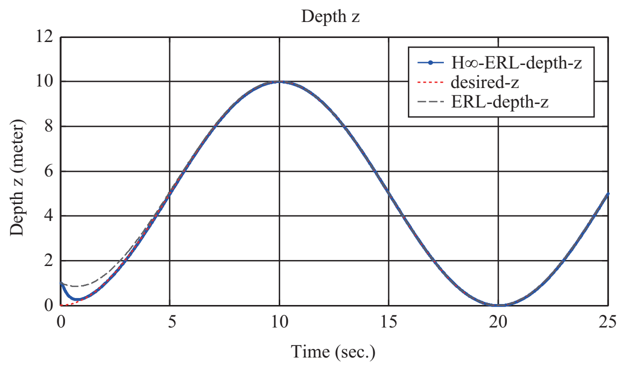


**Fig. 11. Depth z response.**

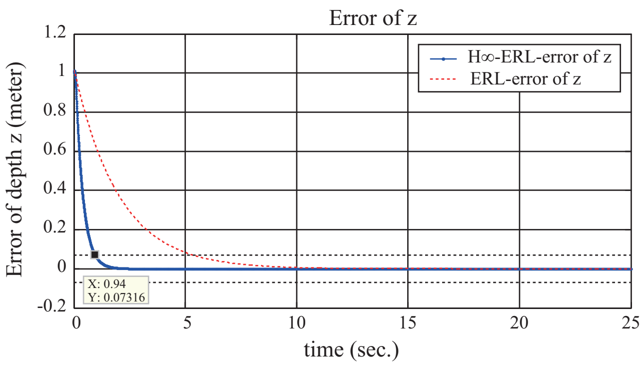


**Fig. 12. Error of depth z.**

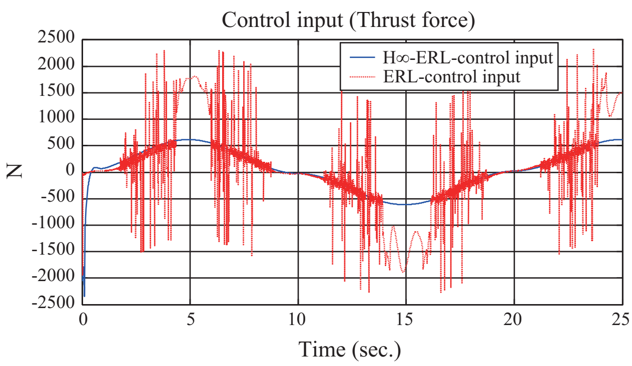


**Fig. 13. Control input *u*.**

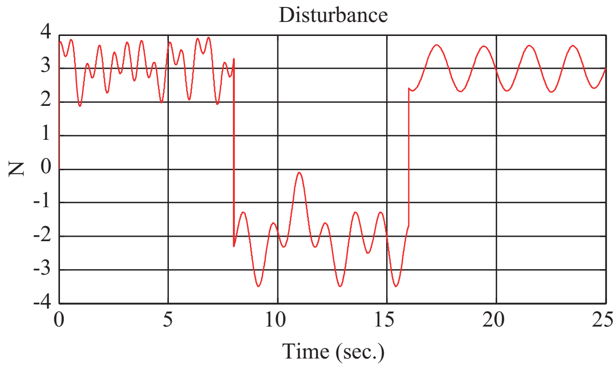


**Fig. 14. Disturbance ($d_{sea}$).**

**Case 3 ( $M_{ROV} = 55$ and $C_{ROV} = 275$ )**

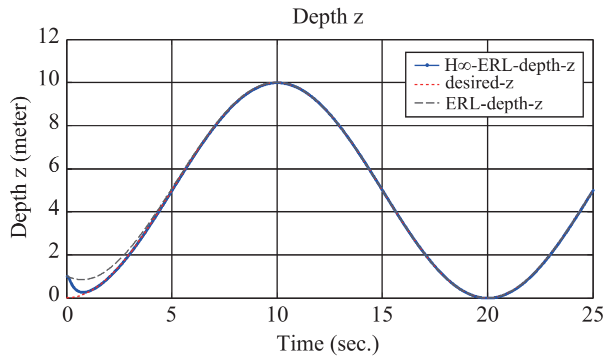


**Fig. 15. Depth z response.**

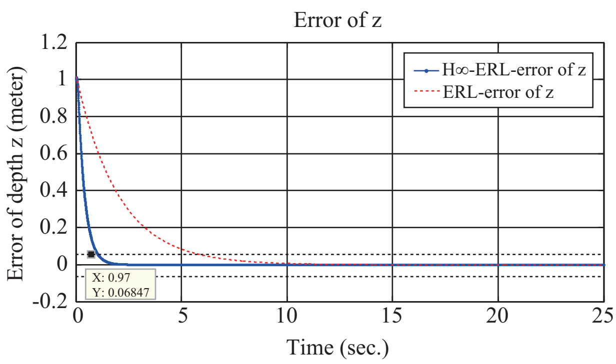


**Fig. 16. Error of depth z.**

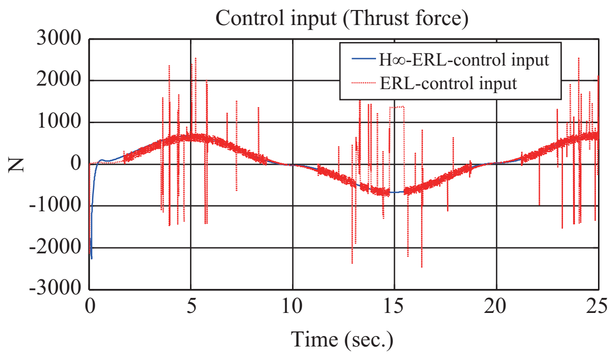


**Fig. 17. Control input *u*.**

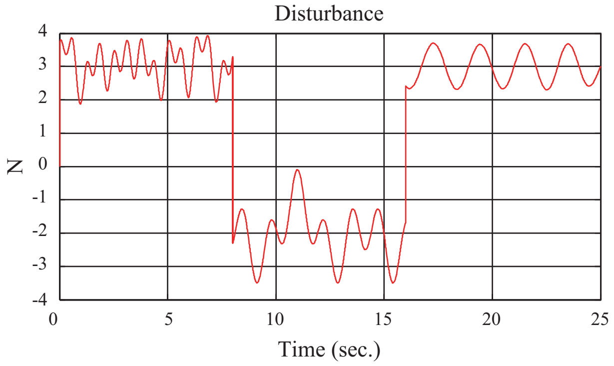


**Fig. 18. Disturbance ($d_{sea}$).**

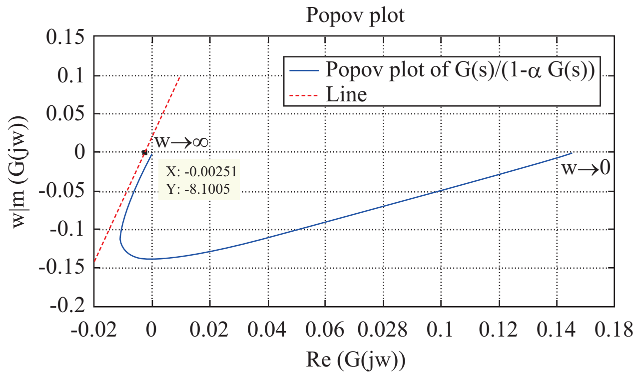


**Fig. 19. Popov plot of G(jw)/(1-αG(jw)).**

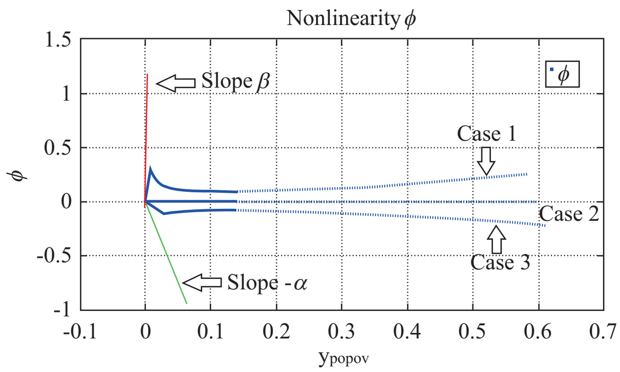


**Fig. 20. Nonlinearity $\phi$(t, $y_{popov}$) belongs to the sector [-$\alpha$, $\beta$].**

Fig. 7, Fig. 11 and Fig. 15 are response plot of depth z. Fig. 8, Fig. 12 and Fig. 16 are error response plot. From error response plot, we know the tracking performance of proposed controller is better than sliding mode controller with ERL. Fig. 9, Fig. 13 and Fig. 17 are control input. From the control input plot, we know the chattering phenomenon of proposed controller is less than sliding mode controller with ERL. Fig. 10, Fig. 14 and Fig. 18 are hypothetical disturbance.

**Apply Theorem B to show the system robustness:**

Substitute the controller of Eq. (53) into system; we can obtain the following error equation:

$$\ddot{e} + K_D \dot{e} + K_P e = d(t) \tag{55}$$

where $e$ is error, $d(t) \in R^1$ is disturbance, $K_P = 21.86$ and $K_D = 7.25$. Let $z_{popov}(t) = [e_i \quad \dot{e}_i]^T \in R^{2\times1}$, $y_{popov}(t) = e$, then we can obtain the following absolute stability problem:

$$\begin{cases} \dot{z}_{popov}(t) = Az_{popov}(t) + Bv(t) \\ y_{popov}(t) = Cz_{popov}(t) \\ v(t) = -\phi(t, y_{popov}) \end{cases} \tag{56}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -K_p & -K_D \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0], \phi(t, y_{popov}) = -d$$

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^2 + 7.25s + 21.86}$$

The poles of $G(s)$ locate at $-3.625 \pm 2.973i$.

The whole poles of $G(s)$ locate on the left half plane. $(A, B)$ is controllable and $(C, A)$ is observable.

In order to ensure that the poles of $\dfrac{G(jw)}{1-\alpha G(jw)}$ are all on the left half plane, we choose $\alpha = 15$, $\beta = 385$, $x = 0.1235$. The conditions of the system are satisfied to Theorem B. In Fig. 19 shows that the Popov plot of $\dfrac{G(jw)}{1-\alpha G(jw)}$ lies to the right of the line. From Fig. 20, we know that the point $z_{popov} = 0$ is global asymptotically stable for any nonlinearity in the sector [-$\alpha$, $\beta$], uch that the system is called asymptotically stable in the given sector.

## VII. CONCLUSION

An H∞-ERL sliding mode controller is proposed in this paper for a multi-input multi-output nonlinear system with parametric uncertainty and external disturbances. The sliding mode controller with ERL is utilized to form the main structure of the proposed controller, which ensures that the system states will arrive at the sliding surface region in a finite time and the plant output matches the desired specifications while it ensures that the closed-loop system is asymptotically stable. The H∞ control methodology and the Lag-Lead compensator are used to optimize the adjustable parameters in sliding mode controller with ERL. The optimal parameters can then minimize the ill-effect of external disturbances and plant parametric uncertainty on the controlled outputs. The closed-loop poles of the augmented system are then located on the specified region to match the desired performance. Finally, Popov criterion is applied to ensure the system stability of unmodeled dynamics. The computer simulation results reveal that the proposed $H_\infty$-ERL sliding mode controller can make the system have excellent tracking performance and robustness. These results also show that the $H_\infty$-ERL sliding mode controller may have better performance than a pure sliding mode controller with ERL because the extra $H_\infty$ formulation and Lag-Lead compensator are formulated in the proposed control structure.

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