A MODIFIED SUMNER METHOD FOR OBTAINING THE ASTRONOMICAL VESSEL POSITION

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Recommended Citation
Hsu, Tien-Pen; Weng, Guo-Yu; and Chen, Chih-Li (2017) "A MODIFIED SUMNER METHOD FOR OBTAINING THE ASTRONOMICAL VESSEL POSITION," Journal of Marine Science and Technology; Vol. 25 : Iss. 3 , Article 8. DOI: 10.6119/JMST-017-0213-1
Available at: https://jmstt.ntou.edu.tw/journal/vol25/iss3/8

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Acknowledgements
The corresponding author would like to express the gratitude to the National Taiwan Ocean University for financial support under contract number: NTOU-105-001.
A MODIFIED SUMNER METHOD FOR OBTAINING THE ASTRONOMICAL VESSEL POSITION

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Key words: celestial navigation, Sumner line, running fix, iteration method.

ABSTRACT

An adaptive boundaries technique (ABT) yielded by the geometrical properties of celestial triangles is proposed to deal with problems resulting from ex-meridian or meridian sights when the Sumner method is used. Due to the trial-and-error characteristic of the Sumner method, an iteration method is introduced to improve numerical accuracy. Combining the ABT and the iteration method into the Sumner method, this modified Sumner method (MSM) is developed so that it successfully determines the astronomical vessel position (AVP). Especially when the non-simultaneous sights condition is encountered, based on the running fix concept, middle-latitude sailing is adopted to translate Sumner points to the fix time for determining the AVP. A program developed using the proposed approach is implemented to solve the AVP problem. Three benchmark examples are conducted to validate the accuracy and versatility of the proposed approach.

I. INTRODUCTION

In the open sea, there are two major ways to obtain the vessel position. One is to obtain it using a global navigation satellite system (GNSS); while the other is to determine it using celestial navigation. Although a GNSS provides navigators with continuous vessel position conveniently, it may suffer deliberate jamming, hostile spoofing or accidental interference, leading to inaccurate positioning (NTSB, 1997; John A. Volpe National Transportation Systems Center, 2001; Carroll, 2003; Williams et al., 2008; Grant et al., 2009). Accordingly, using independent position-fixing sources to cross check the vessel position for navigational safety is necessary (Bowditch, 1984, 2002; ICS, 1998; OCIMF, 2008; Royal Navy, 2008). As modern society grows ever more dependent on computing technology, computers are an essential tool of our working environments and academic disciplines. Not surprisingly, therefore, section B-II/1 of the 2010 Manila Amendments of International Convention on Standards of Training, Certification and Watchkeeping for Seafarers (STCW) recommends developing celestial navigation calculation software for determining astronomical vessel position (AVP) (IMO, 2010), as we set out to do here.

In the history of celestial navigation development, the American sea captain Thomas H. Sumner was the first to replace circle of position (COP) by line of position (LOP) (also called the Sumner line) to simplify calculation of the AVP. Based on this concept, the Sumner method was developed. Because the geometry of the Sumner method is so simple and obvious, it gradually gained popularity and eventually came to be used on every ship in the United States Navy (Richardson, 1946). Knowledge about the Sumner method soon spreads to the European maritime countries, opening a new era in practical navigation (Oestmann, 2011). However, the Sumner method cannot be used when the sights are taken near or at the time of the meridian passage (Gradsztajn, 1979). In addition, sizable errors in the calculation resulting from replacing COP by LOP, especially in the case of high altitude observations, commonly arise (Bowditch, 1984, 2002; Culter, 2003; Chen et al., 2003; Hsu et al., 2005; Chen et al., 2014). Finally, under conditions involving non-simultaneous sights (Gibson, 1994), determining how to use the Sumner method to determine the AVP still needs to be resolved. Elimination of these obstacles to using the Sumner method thus becomes the main challenge to our research goal of developing software that uses celestial navigation to accurately calculate AVP, which we solve via the development and implementation of a modified Sumner method (MSM) that incorporates an adaptive boundaries technique (ABT) proposed to deal with problems resulting from ex-meridian or meridian sights and an iteration method that we introduce to remedy the problem of inaccuracy arising from the trial-and-error characteristic of the Sumner method.

This paper is organized as follows: Section 2 describes theoretical background of the proposed MSM and the process of running fix. Computational procedures and the developed program are presented in Section 3. In Section 4, three benchmark examples illustrate the accuracy and versatility of the proposed method. Finally, conclusions are drawn in Section 5.
II. THEORETICAL BACKGROUND

The Sumner method consists of establishing an LOP (also called a Sumner line) from the observed altitude of a celestial body by assuming two latitudes and then calculating the longitudes through which the Sumner line passes (Bowditch, 2002). The intersections of the two assumed latitudes and the COP, in which the Sumner line is connected as shown in Fig. 1, are hereafter called Sumner points. When another Sumner line is established, the AVP is determined by the intersection of the two Sumner lines. This method has been widely used due to its simplicity, which derives from the concept of LOP (Oestmann, 2011); however, as shown in Fig. 2, there are situations in which the Sumner points do not exist (i.e., the COP only intersects one of the assumed latitudes), because the Sumner method only considers the position of the observer, without regard to the geographical position and the observed altitude of the celestial body when locating the assumed latitudes. This makes the Sumner method unusable for ex-meridian or meridian sights. Besides, when the Sumner line is used to approximate the COP, the AVP is usually inaccurate (Bowditch, 1984, 2002; Culter, 2003; Chen et al., 2003; Hsu et al., 2005; Chen et al., 2014). Aimed at overcoming these shortcomings of the Sumner method, our modified Sumner method (MSM) first proposes an adaptive boundaries technique (ABT) to ensure the existence of two Sumner points and then, makes use of an iteration method to eliminate the approximation error induced by the Sumner line. Note that it is impossible to yield two Sumner lines at the same time in practice. Consequently, middle-latitude sailing is adopted to translate two pairs of Sumner points taken at two different observed times to a single fix time. Finally, the AVP is yielded by the intersection of the two resulting Sumner lines at the single fix time.

1. Adaptive Boundaries Technique

For the Sumner method, the normal practice is to take the assumed latitudes at 10° north and south of the latitude of the dead reckoning (DR) position (Cotter, 1969; Chen et al., 2014), hereafter called the initial bounds. When ex-meridian or meridian sights are encountered, the Sumner points might not be determinable as explained above. To tackle this problem, we need to go back to consider the three sides of a celestial (spherical) triangle, which describes the relation of the latitude of the DR position ($L_{DR}$), declination ($d$) and observed altitude ($H$) as shown in Fig. 3. Based on the geometrical property, that any two sides
of the spherical triangle are together greater than the third (Clough-Smith, 1966), we have

\[ d - (90^\circ - H) \leq L_{DB} \leq d + (90^\circ - H), \quad (1) \]

in which the left-hand side of \( L_{DB} \) is the lower bound (LB) and the right hand side of \( L_{DB} \) is the upper bound (UB). Thus, adaptive boundaries can be located to limit the interval of assumed latitudes, where the LB and UB are decided by the declination and the observed altitude of a celestial body.

As shown in Fig. 4, if any one of the initial bounds is lower than the LB, we set the lower assumed latitude as the LB; while if any one of the initial bounds is higher than the UB, we set the higher assumed latitude as the UB. Consequently, the lower bound (\( L_{BA} \)), upper bound (\( U_{BA} \)), and two assumed latitudes for the celestial body A (\( CBA \)) at observed time \( T_A \), \( L_A(T_A) \) and \( U_A(T_A) \), are respectively expressed as

\[ L_{BA} = d - (90^\circ - H_A), \quad (2a) \]
\[ U_{BA} = d + (90^\circ - H_A), \quad (2b) \]

and

\[
\begin{bmatrix}
L_A(T_A) - I, & L_{DA}(T_A) + I, \\
L_A(T_A), & L_{DA}(T_A)
\end{bmatrix}, 
\]

\[ \text{if } L_B \leq L_{DA}(T_A) \not\equiv I \leq U_B \]

\[
\begin{bmatrix}
L_B, & L_B + 2I, \\
U_B - 2I, & U_B
\end{bmatrix}
\]

\[ \text{if } L_{DA}(T_A) - I < L_B \]

\[
\begin{bmatrix}
U_B - 2I, & U_B \\
I, & I
\end{bmatrix}
\]

\[ \text{if } L_{DA}(T_A) + I > U_B \]

where \( d_B \) is the declination; \( H_A \) is the observed altitude; \( L_{DA}(T_A) \) is the latitude of the DR position at \( T_A \); \( I \) is the increment of the assumed latitude and \( I = 10' \) for the initial step.

As shown in Fig. 5, once the assumed latitudes (\( L_A(T_A) \) and \( U_A(T_A) \)) are available, the meridian angles (\( t_A \) and \( t_B \)) and longitudes (\( \lambda_A(T_A) \) and \( \lambda_B(T_B) \)) of Sumner points \( A_1 \) and \( A_2 \) at observed time \( T_A \) can be obtained by

\[ \cos(t_A) = \frac{\sin H_A - \sin[L_A(T_A)] \sin d_A}{\cos[L_A(T_A)] \cos d_A}, \quad (3a) \]

\[ \cos(t_B) = \frac{\sin H_A - \sin[L_B(T_B)] \sin d_B}{\cos[L_B(T_B)] \cos d_B}, \quad (3b) \]

\[ \lambda_A(T_A) = G_A + t_A, \quad (3c) \]

\[ \lambda_B(T_B) = G_B + t_B, \quad (3d) \]

where \( G_A \) is the Greenwich hour angle (GHA) of \( CBA \).

Similarly, the lower bound \( L_{BB} \), upper bound \( U_{BB} \), and two assumed latitudes for the celestial body B (\( CBB \)) at observed time \( T_B \), \( L_B(T_B) \) and \( U_B(T_B) \), can respectively be written as

\[ L_{BB} = d_B - (90^\circ - H_B), \quad (4a) \]
\[ U_{BB} = d_B + (90^\circ - H_B), \quad (4b) \]
and

\[
L_B(T_B), L_B(T_B) = \begin{cases}
L_{0B}(T_B) - I, & L_{0B}(T_B) + I, \\
L_B - L_{0B} + 2I, & L_{0B} + 2I, \\
UB_B - 2I, & UB_B
\end{cases},
\]

(4c)

where \(d_B\) is the declination; \(H_B\) is the observed altitude and \(L_{DR}(T_B)\) is the latitude of the DR position at \(T_B\).

Then, once the assumed latitudes \((\lambda_B(T_B)\) and \(\lambda_B(T_B)\)) are available, the meridian angles \((\lambda_B(T_B)\) and \(\lambda_B(T_B)\)) of Sumner points B1 and B2 at observed time \(T_B\) can be calculated by

\[
\cos(t_B) = \frac{\sin H_B \sin L_{0B}(T_B) \sin d_B}{\cos L_{0B}(T_B) \cos d_B},
\]

(5a)

\[
\cos(t_B) = \frac{\sin H_B \sin L_{0B}(T_B) \sin d_B}{\cos L_{0B}(T_B) \cos d_B},
\]

(5b)

\[
\lambda_B(T_B) = G_B + (t_B),
\]

(5c)

and

\[
\lambda_B(T_B) = G_B + (t_B),
\]

(5d)

where \(G_B\) is the Greenwich hour angle (GHA) of CB_B.

Note that north latitudes and east longitudes are positive, while south latitudes and west longitudes are negative. As for the minus-or-plus option in Eqs. (3c), (3d), (5c) and (5d), – is used for bodies east of the meridian and + is used for bodies west of the meridian.

2. Astronomical Vessel Position

By using the equation of line joining two points, simultaneous Sumner line equations at fix time \(T_F\) for CB_A and CB_B can be established as

\[
(d\lambda_A)L_F - (d\lambda_A)L_F = (L_A)(d\lambda_A) - (dL_A)(\lambda_A),
\]

(6a)

and

\[
(d\lambda_B)L_F - (d\lambda_B)L_F = (L_B)(d\lambda_B) - (dL_B)(\lambda_B),
\]

(6b)

where \(L_F\) and \(\lambda_F\) are the latitude and longitude of AVP at the fix time; \(dL_A\) and \(d\lambda_A\) are the difference of latitude and longitude between Sumner points A1 and A2, respectively; \(L_B\) and \(\lambda_B\) are the latitude and longitude of Sumner point B1, respectively; \(dL_B\) and \(d\lambda_B\) are the difference of latitude and longitude between Sumner points B1 and B2, respectively; and \(L_{DR}\) and \(\lambda_{DR}\) are the latitude and longitude of Sumner point B1, respectively.

The AVP \((L_F, \lambda_F)\) can be obtained by Cramer’s Rule as

\[
L_F = \frac{(dL_A)(d\lambda_B) - (dL_B)(\lambda_B))}{(dL_A)(d\lambda_A) - (dL_B)(d\lambda_A)},
\]

(7a)

and

\[
\lambda_F = \frac{(d\lambda_A)(L_B)(d\lambda_B) - (d\lambda_B)(d\lambda_A))}{(dL_A)(d\lambda_B) - (dL_B)(d\lambda_A)},
\]

(7b)

3. Iteration Method

Since the Sumner method is a trial-and-error method, an iteration method is introduced to eliminate the approximation error for obtaining the real AVP. Consequently, by decreasing the increment of assumed latitudes \((I)\), we have

\[
I = 10' \times 2^{-n},
\]

(8)

where \(n\) is the number of iteration.

4. Process of Running Fix

When the position \((L_A, \lambda_A)\) at departure time, the speed \((S)\) and the course angle \((C)\) of the vessel are known, the position \((L_m, \lambda_m)\) at arrival time can be obtained by using middle-latitude sailing (Bowditch, 1984, 2002; Culter, 2003). Therefore, we introduce the equations for middle-latitude sailing:

\[
dL = (S \times dT) \cos C,
\]

(9a)

\[
L_m = L_f + \frac{1}{2} dL,
\]

(9b)

and

\[
d\lambda = (S \times dT) \sin C \sec L_m,
\]

(9c)

where \(dL\) is the difference of latitude, \(L_m\) is the middle-latitude (also called the mean latitude) and \(d\lambda\) is the difference of longitude between positions at departure time and arrival time, respectively; \(dT\) is the time interval (in hours) between the departure time and arrival time. Thus, the position \((L_m, \lambda_m)\) at
Fig. 6. Illustrations of translating the Sumner points.

III. COMPUTATION PROCEDURES AND NUMERICAL PROGRAM

1. Construction of the Computational Procedures for MSM

Step 1. Calculate DR Positions at Observed Times $T_A$ and $T_B$

The DR positions at $T_A$ and $T_B$ with known DR position at time $T_{DR}$ can be calculated by using Eqs. (9a), (9b), (9c), (10a) and (10b), as shown in Fig. 6.

Step 2. Determine the Two Sumner Points at $T_A$ and $T_B$

1. At $T_A$: The lower bound and upper bound for $CB_A$ are located by using Eqs. (2a) and (2b). Then, the assumed latitudes of the Sumner points can be determined by Eqs. (2c) and the longitudes of the Sumner points $A_1$ and $A_2$ at $T_A$ can be obtained by Eqs. (3a), (3b), (3c) and (3d).

2. At $T_B$: The lower bound and upper bound for $CB_B$ are located by using Eqs. (4a) and (4b). Then, the assumed latitudes of the Sumner points can be determined by Eqs. (4c) and the longitudes of the Sumner points $B_1$ and $B_2$ at $T_B$ can be obtained by Eqs. (5a), (5b), (5c) and (5d).

Step 3. Determine the Two Sumner Points at Fix Time $T_F$

1. At $T_F$ for $CB_A$: Once the Sumner points at $T_A$ are yielded, the two Sumner points at $T_F$ can be determined by using Eqs. (9a), (9b), (9c), (10a) and (10b), as shown in Fig. 6. (2) At $T_F$ for $CB_B$: Similarly, once the Sumner points at $T_B$ are yielded, the two Sumner points at $T_F$ can be determined by using Eqs. (9a), (9b), (9c), (10a) and (10b), as shown in Fig. 6.

Step 4. Determine the AVP at Fix Time $T_F$

The AVP ($L_F$, $\lambda_F$) at $T_F$ can be obtained by using Eqs. (7a) and (7b).

Step 5. Determine the Real AVP

The real AVP is yielded by using the iteration method. Consequently, decreasing the increment of assumed latitudes ($I$) of Eq. (8) and repeating the iteration steps 1 to 5 can reach the real AVP while two successive positions obtained by Step 4 does not change to the precision required.

2. Numerical Program

Based on the proposed MSM, we developed a numerical program with graphical user interface (GUI) using Visual Basic.Net 2012, which we named the AVP-MSM Prog. For the convenience of navigators, the AVP-MSM Prog calculates the real AVP and draws the Sumner lines on the built-in small area plotting sheet.

IV. VALIDATION AND DISCUSSION

Three examples are provided to validate the proposed MSM. In Example 1, we illustrate the application of ABT. Example 2 illustrates application of the running fix concept for dealing with the non-simultaneous sights at high altitude. Example 3 describes how the MSM and the running fix concept are used in combination to solve the problem of the overdetermined AVP.

Example 1

On May 16, 1995 the ZT 20-11-26 the DR position of a vessel is 25°10.0’N, 157°10.0’W. At 20-07-43, the Kochab is spotted with a sextant. Later, at 20-11-26 the star Spica is observed. The navigator records the needed information and further reduces it from the Nautical Almanac for sight reduction as shown in Table 1. (Reediting from pp. 301-303 of Bowditch (2002)).
Table 2. Solving procedures in details by the MSM for Example 1.

<table>
<thead>
<tr>
<th>Step</th>
<th>CB&lt;sub&gt;A&lt;/sub&gt;</th>
<th>Equations</th>
<th>Kochab</th>
<th>Spica</th>
<th>Equations</th>
<th>CB&lt;sub&gt;B&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DR&lt;sub&gt;(Ta)&lt;/sub&gt;</td>
<td>(9a)-(9c)</td>
<td>25°10.0' N</td>
<td>25°10.0' N</td>
<td>(9a)-(9c)</td>
<td>DR&lt;sub&gt;(Ta)&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td>(10a), (10b)</td>
<td></td>
<td>157°10.0' W</td>
<td>157°10.0' W</td>
<td>(10a), (10b)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>LB&lt;sub&gt;B&lt;/sub&gt;</td>
<td>(2a)</td>
<td>31°24.2' N</td>
<td>68°39.7' S</td>
<td>(4a)</td>
<td>LB&lt;sub&gt;B&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td>(2b)</td>
<td>63°03.0' N</td>
<td>46°22.9' N</td>
<td>(4b)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>lower than LB&lt;sub&gt;B&lt;/sub&gt;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>L&lt;sub&gt;DR&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;) - I</td>
<td>(2c)</td>
<td>25°00.0' N</td>
<td>25°00.0' N</td>
<td>(4c)</td>
<td>L&lt;sub&gt;DR&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;) - I</td>
</tr>
<tr>
<td></td>
<td>L&lt;sub&gt;DR&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;) + I</td>
<td>(2c)</td>
<td>25°20.0' N</td>
<td>25°20.0' N</td>
<td>(4c)</td>
<td>L&lt;sub&gt;DR&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;) + I</td>
</tr>
<tr>
<td></td>
<td>L&lt;sub&gt;K&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
<td>(2c)</td>
<td>31°24.2' N</td>
<td>25°00.0' N</td>
<td>(4c)</td>
<td>L&lt;sub&gt;K&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
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<tr>
<td></td>
<td>(2b)</td>
<td>31°44.2' N</td>
<td>25°20.0' N</td>
<td>(4c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3a)</td>
<td>t&lt;sub&gt;a&lt;/sub&gt;</td>
<td>00°00.0'</td>
<td>045°55.0' E</td>
<td>(5a)</td>
<td>t&lt;sub&gt;a&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td>(3b)</td>
<td>t&lt;sub&gt;a&lt;/sub&gt;</td>
<td>010°34.5' E</td>
<td>045°40.4' E</td>
<td>(5b)</td>
<td>t&lt;sub&gt;a&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td>(3c)</td>
<td>A&lt;sub&gt;1&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
<td>31°24.2' N</td>
<td>25°00.0' N</td>
<td>(5c)</td>
<td>B&lt;sub&gt;1&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>103°43.0' W</td>
<td>172°00.7' W</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3d)</td>
<td>A&lt;sub&gt;2&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
<td>31°44.2' N</td>
<td>25°20.0' N</td>
<td>(5d)</td>
<td>B&lt;sub&gt;2&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;a&lt;/sub&gt;&lt;/sub&gt;)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>114°17.5' W</td>
<td>171°46.1' W</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>A&lt;sub&gt;1&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;B&lt;/sub&gt;&lt;/sub&gt;)</td>
<td>(9a)-(9c)</td>
<td>31°24.2' N</td>
<td>25°00.0' N</td>
<td>(9a)-(9c)</td>
<td>B&lt;sub&gt;1&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;B&lt;/sub&gt;&lt;/sub&gt;)</td>
</tr>
<tr>
<td></td>
<td>(10a), (10b)</td>
<td></td>
<td>103°43.0' W</td>
<td>172°00.7' W</td>
<td>(10a), (10b)</td>
<td></td>
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<tr>
<td></td>
<td>A&lt;sub&gt;2&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;B&lt;/sub&gt;&lt;/sub&gt;)</td>
<td>(9a)-(9c)</td>
<td>31°44.2' N</td>
<td>25°20.0' N</td>
<td>(9a)-(9c)</td>
<td>B&lt;sub&gt;2&lt;/sub&gt;(&lt;sub&gt;T&lt;sub&gt;B&lt;/sub&gt;&lt;/sub&gt;)</td>
</tr>
<tr>
<td></td>
<td>(10a), (10b)</td>
<td></td>
<td>114°17.5' W</td>
<td>171°46.1' W</td>
<td>(10a), (10b)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>F(0)</td>
<td>(7a), (7b)</td>
<td>(33°21.8' N, 165°53.6' W)</td>
<td>(7a), (7b)</td>
<td>F(0)</td>
<td></td>
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<tr>
<td></td>
<td>F(1)</td>
<td>(8), (7a), (7b)</td>
<td>(37°46.7' N, 159°17.6' W)</td>
<td>(8), (7a), (7b)</td>
<td>F(1)</td>
<td></td>
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<tr>
<td></td>
<td>F(2)</td>
<td>(8), (7a), (7b)</td>
<td>(38°59.5' N, 156°29.0' W)</td>
<td>(8), (7a), (7b)</td>
<td>F(2)</td>
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<td></td>
<td>F(3)</td>
<td>(8), (7a), (7b)</td>
<td>(39°00.0' N, 156°21.7' W)</td>
<td>(8), (7a), (7b)</td>
<td>F(3)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>F(4)</td>
<td>(8), (7a), (7b)</td>
<td>(39°00.0' N, 156°21.7' W)</td>
<td>(8), (7a), (7b)</td>
<td>F(4)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fix</td>
<td>(39°00.0' N, 156°21.7' W)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 7. Results of running the AVP-MSM Prog in Example 1.

Required
Determine the AVP at ZT 20-11-26.

Solution
The AVP (39°00.0' N, 156°21.7' W) is determined by using the MSM and the AVP-MSM Prog, respectively. Results and detailed information are shown in Table 2 and Fig. 7.

Remark
(1) We purposely moved the DR position from (L<sub>39°00.0' N, 157°10.0' W</sub>) to (L<sub>25°10.0' N, 157°10.0' W</sub>) for validation of the ABT. As shown in Table 2, the initial bounds (L<sub>DR</sub>(<sub>T<sub>a</sub></sub>) + I) of the star Kochab, (25°00.0' N, 25°20.0' N), are lower than the lower bound (LB<sub>B</sub>), 31°24.2' N, and the Sumner method fails to determine the AVP. Consequently, the proposed ABT enforces the assumed latitudes (L<sub>K</sub>(<sub>T<sub>a</sub></sub>), L<sub>K</sub>(<sub>T<sub>a</sub></sub>)) of (31°24.2' N, 31°44.2' N) to fall within the adaptive boundaries (LB<sub>B</sub>, UB<sub>B</sub>) of (31°24.2' N, 63°03.0' N) in order to establish the Sumner line. Then, the AVP can be determined. This shows that whether the Sumner line can be established depends on the ABT but not on the DR position.

(2) Once the two Sumner lines are established, as shown in Table 2, the real AVP (39°00.0' N, 156°21.7' W) is accurately obtained when the iteration method is introduced.
Table 3. Needed information for solving the AVP in Example 2.

<table>
<thead>
<tr>
<th>Celestial body</th>
<th>Sun</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR</td>
<td>1975/05/31, ZT 12-24-00,</td>
<td>20°17.4' N 050°07.4' W</td>
</tr>
<tr>
<td>ZT</td>
<td>12-15-15</td>
<td>12-24-13</td>
</tr>
<tr>
<td>$H$</td>
<td>88°09.2'</td>
<td>87°42.8'</td>
</tr>
<tr>
<td>$d$</td>
<td>21°53.1' N</td>
<td>21°53.1' N</td>
</tr>
<tr>
<td>$G$</td>
<td>049°25.6'</td>
<td>051°40.1'</td>
</tr>
</tbody>
</table>


Table 4. Solving procedures in details by the MSM for Example 2.

<table>
<thead>
<tr>
<th>Step</th>
<th>CB1</th>
<th>Equations</th>
<th>Sun</th>
<th>CB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$DR(T_0)$</td>
<td>(9a)-(9c),(10a),(10b)</td>
<td>$20°19.0'$ N 050°09.6' W</td>
<td>$DR(T_0)$</td>
</tr>
<tr>
<td>2</td>
<td>$A_1(T_1)$</td>
<td>(9a)-(9c),(10a),(10b)</td>
<td>$20°17.4'$ N 050°07.3' W</td>
<td>$B_1(T_0)$</td>
</tr>
<tr>
<td>3</td>
<td>$A_2(T_1)$</td>
<td>(9a)-(9c),(10a),(10b)</td>
<td>$20°09.0'$ N 050°06.2' W</td>
<td>$B_2(T_1)$</td>
</tr>
<tr>
<td>4</td>
<td>$F(0)$</td>
<td>(7a),(7b)</td>
<td>$20°08.3'$ N 050°05.6' W</td>
<td>$F(0)$</td>
</tr>
<tr>
<td>5</td>
<td>$F(1)$</td>
<td>(8),(7a),(7b)</td>
<td>$20°08.8'$ N 050°05.1' W</td>
<td>(8),(7a),(7b)</td>
</tr>
<tr>
<td></td>
<td>$F(2)$</td>
<td>(8),(7a),(7b)</td>
<td>$20°08.2'$ N 050°05.6' W</td>
<td>(8),(7a),(7b)</td>
</tr>
<tr>
<td></td>
<td>$F(3)$</td>
<td>(8),(7a),(7b)</td>
<td>$20°08.1'$ N 050°05.7' W</td>
<td>(8),(7a),(7b)</td>
</tr>
<tr>
<td></td>
<td>$F(4)$</td>
<td>(8),(7a),(7b)</td>
<td>$20°08.0'$ N 050°05.7' W</td>
<td>(8),(7a),(7b)</td>
</tr>
<tr>
<td></td>
<td>$F(5)$</td>
<td>(8),(7a),(7b)</td>
<td>$20°08.0'$ N 050°05.7' W</td>
<td>(8),(7a),(7b)</td>
</tr>
</tbody>
</table>

Also, as shown in Fig. 7, the same results are reached by using the AVP-MSM Prog and validated in (Chen et al., 2003; Chen et al., 2014).

Example 2

On May 31, 1975, the ZT 12-24-00 the DR position of a vessel is L20°17.4’ N, $\lambda$050°07.4’ W. The ship is on course 127°, speed 18 knots. The navigator observes the lower limb of the Sun twice. The first observation is made at 12-15-15. The second observation is made at 12-24-13. The navigator records the needed information and further reduces it from the Nautical Almanac for sight reduction as shown in Table 3. (p. 569 of Bowditch (1984))

Required

Determine the AVP at ZT 12-24-00.
Table 5. Needed information for solving the AVP in Example 3.

<table>
<thead>
<tr>
<th>Celestial body</th>
<th>Deneb</th>
<th>Antares</th>
<th>Vega</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR</td>
<td>1973/02/25, ZT 06-16-00,</td>
<td>45°10.0'N</td>
<td>030°15.0'W</td>
</tr>
<tr>
<td>ZT</td>
<td>06-09-04</td>
<td>06-12-05</td>
<td>06-16-02</td>
</tr>
<tr>
<td>H</td>
<td>46°35.6'</td>
<td>18°46.9'</td>
<td>66°48.7'</td>
</tr>
<tr>
<td>d</td>
<td>45°10.9'N</td>
<td>26°22.5'S</td>
<td>38°45.2'N</td>
</tr>
<tr>
<td>G</td>
<td>327°11.5'</td>
<td>031°07.6'</td>
<td>000°03.4'</td>
</tr>
</tbody>
</table>


Solution

The AVP (20°08.0' N, 050°05.7' W) is determined by using the MSM and the AVP-MSM Prog, respectively. Results and detailed information are shown in Table 4 and Fig. 8.

Remarks

(1) As shown in Table 4, two Sumner points, \((A_1(T_A), A_2(T_A))\), obtained at \(T_A\) and another two Sumner points, \((B_1(T_B), B_2(T_B))\), obtained at \(T_B\), are translated to \(A_1(T_F), A_2(T_F)\), \(B_1(T_F)\) and \(B_2(T_F)\) at \(T_F\) by using middle-latitude sailing. Thus, by way of the four translated Sumner points, a set of two Sumner lines can be established to determine the AVP \((L_F, \lambda_F)\).

(2) After iteration, the real AVP, \((20°08.0' N, 050°05.7' W)\), is reached and validated in (Chen et al., 2014). Note that this example is a case of bodies at high altitudes; it shows that the proposed approach is quite accurate and widely applicable.

Example 3

On February 25, 1973, the ZT 06-16-00 the DR position of a ship is L45°10.0' N, 2030°15.0' W. The navigator observes Deneb, Antares, and Vega. The ship is on course 180° and speed 20 knots. Detailed information is recorded as shown in Table 5. (p.29 of NIMA (1981)).

Required

Determine the AVP at ZT 06-16-00.

Solution

The cocked hat is formed by three Sumner lines when the AVP-MSM Prog is run. Three points of the cocked hat are \((44°49.04' N, 030°16.65' W)\), \((44°49.04' N, 030°16.72' W)\) and \((44°49.13' N, 030°16.71' W)\). Results and detailed information are shown in Fig. 9.

Remarks

(1) As shown in Fig. 9, since star Antares is observed near the time of meridian passage, the determined Sumner line is very close to the parallel of latitude in which the Sumner method cannot be used. However, once the ABT of the MSM is introduced, as shown in Table 6, the proposed ABT enforces the assume latitudes \( (L_{A1}(T_A), L_{A2}(T_A)) \) of \((44°30.6' N, 44°50.6' N)\) to fall within the adaptive boundaries \( (L_{BA}, U_{BA}) \) of \((82°24.4' S, 44°50.6' N)\) in order to establish the Sumner line. Consequently, the proposed MSM effectively eliminates the shortcomings of the Sumner method in this example.

(2) Conventionally, the AVP is determined by the plotting method (NIMA, 1981) and thus requires tedious work, easily leading to errors; however, as shown in Fig. 10, the proposed MSM, using the developed AVP-MSM Prog can quickly and accurately calculate the AVP.

V. CONCLUSIONS

Based on the geometrical properties of celestial triangles,
Table 6. Solving procedures in details by the MSM for Example 3.

<table>
<thead>
<tr>
<th>Step</th>
<th>CB</th>
<th>Equations</th>
<th>Antares</th>
<th>Vega</th>
<th>Equations</th>
<th>CB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DR</td>
<td>(9a)-(9c)</td>
<td>45°11.3’ N</td>
<td>45°10.0’ N</td>
<td>(10a), (10b)</td>
<td>DR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(10a), (10b)</td>
<td>030°15.0’ W</td>
<td>030°15.0’ W</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>LB</td>
<td>(2a)</td>
<td>82°24.4’ S</td>
<td>15°33.9’ N</td>
<td>(4a)</td>
<td>LB</td>
</tr>
<tr>
<td></td>
<td>UB</td>
<td>(2b)</td>
<td>44°50.6’ N</td>
<td>61°56.5’ N</td>
<td>(4b)</td>
<td>UB</td>
</tr>
<tr>
<td></td>
<td>higher than UB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>L_{DR}(T_A) - I</td>
<td>(2c)</td>
<td>45°01.3’ N</td>
<td>45°00.0’ N</td>
<td>(4c)</td>
<td>L_{DR}(T_B) - I</td>
</tr>
<tr>
<td></td>
<td>L_{DR}(T_A) + I</td>
<td>(2c)</td>
<td>45°21.3’ N</td>
<td>45°20.0’ N</td>
<td>(4c)</td>
<td>L_{DR}(T_B) + I</td>
</tr>
<tr>
<td></td>
<td>L_A(T_A)</td>
<td>(2c)</td>
<td>44°30.6’ N</td>
<td>45°00.0’ N</td>
<td>(4c)</td>
<td>L_A(T_B)</td>
</tr>
<tr>
<td></td>
<td>L_A(T_B)</td>
<td>(2c)</td>
<td>44°50.6’ N</td>
<td>45°20.0’ N</td>
<td>(4c)</td>
<td>L_A(T_B)</td>
</tr>
<tr>
<td></td>
<td>t_A</td>
<td>(3a)</td>
<td>00°31.5’ W</td>
<td>030°12.0’ E</td>
<td>(5a)</td>
<td>t_B</td>
</tr>
<tr>
<td></td>
<td>L_A</td>
<td>(3b)</td>
<td>00°00.0’</td>
<td>030°09.4’ E</td>
<td>(5b)</td>
<td>t_B</td>
</tr>
<tr>
<td></td>
<td>A_i(T_A)</td>
<td>(3c)</td>
<td>44°30.6’ N</td>
<td>023°36.1’ W</td>
<td>(5c)</td>
<td>B_i(T_B)</td>
</tr>
<tr>
<td></td>
<td>A_i(T_B)</td>
<td>(3d)</td>
<td>44°50.6’ N</td>
<td>031°07.6’ W</td>
<td>(5d)</td>
<td>B_i(T_B)</td>
</tr>
<tr>
<td>3</td>
<td>A_i(T_A)</td>
<td>(9a)-(9c)</td>
<td>44°29.3’ N</td>
<td>45°00.0’ N</td>
<td>(9a)-(9c)</td>
<td>A_i(T_A)</td>
</tr>
<tr>
<td></td>
<td>A_i(T_B)</td>
<td>(9a)-(9c)</td>
<td>44°49.3’ N</td>
<td>45°20.0’ N</td>
<td>(9a)-(9c)</td>
<td>A_i(T_B)</td>
</tr>
<tr>
<td></td>
<td>A_A(T_A)</td>
<td>(9a)-(9c)</td>
<td>023°36.1’ W</td>
<td>030°12.8’ W</td>
<td>(9a)-(9c)</td>
<td>A_A(T_B)</td>
</tr>
<tr>
<td></td>
<td>A_A(T_B)</td>
<td>(9a)-(9c)</td>
<td>031°07.6’ W</td>
<td>030°12.8’ W</td>
<td>(9a)-(9c)</td>
<td>A_A(T_B)</td>
</tr>
<tr>
<td>4</td>
<td>F(0)</td>
<td>(7a), (7b)</td>
<td>(44°47.06’ N, 030°17.16’ W)</td>
<td>(7a), (7b)</td>
<td>F(0)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>F(7)</td>
<td>(8), (7a), (7b)</td>
<td>(44°49.04’ N, 030°16.72’ W)</td>
<td>(8), (7a), (7b)</td>
<td>F(7)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fix</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The ABT has been successfully derived. The Sumner method together with the ABT and the iteration method (MSM) effectively overcome the disadvantages of the Sumner method. Importantly, it is found that whether the Sumner line is successfully established is dependent on the ABT but not on the DR position. Due to the inclusion of the iteration method, the

Fig. 10. Comparison of the AVPs determined by AVP-MSM Prog and in the literature.
The proposed approach is more accurate. Furthermore, when the non-simultaneous sights occur, middle-latitude sailing and the running fix are adopted, extending the application of the proposed approach. The three examples validate the applicability of the proposed method.

ACKNOWLEDGMENTS
The corresponding author would like to express the gratitude to the National Taiwan Ocean University for financial support under contract number: NTOU-105-001.

REFERENCES