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OPTIMAL CONTROL OF NONLINEAR DUFFING OSCILLATORS: LIE-GROUP APPROACHES

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Key words: Duffing oscillator, optimal control problem, Hamiltonian formulation, Lie-group method, Lie-group differential algebraic equation method.

ABSTRACT

In the optimal control theory, the Hamiltonian formulation is a famous one convenient to find an optimal designed control force. However, when the performance index is a complicated function of control force, the Hamiltonian method is not easy to find the optimal solution, because one may encounter a two-point boundary value problem of nonlinear differential algebraic equations (DAEs). In this paper we address this issue via a quite novel and effective approach, of which the optimally controlled vibration problem of Duffing oscillator is recast into a two-point nonlinear DAEs by identifying the unknown control force. We develop the corresponding *SL*(n, **R**) and *GL*(n, **R**) shooting methods, as well as a Lie-group differential algebraic equations (LGDAE) method to numerically solve the optimal control forces. Eight examples of a single Duffing oscillator and one coupled Duffing oscillators are used to test the performance of the present method.

I. INTRODUCTION

The purpose of this paper is to compute the control force u in the following equation of motion of a nonlinear Duffing oscillator:

$$\ddot{x} + \gamma \dot{x} + \alpha x + \beta x^3 = u(t) \tag{1}$$

as well as the control forces u_1 and u_2 in the following coupled Duffing oscillators:

$$\begin{aligned} &\ddot{q}_1(t) + \alpha_1 q_1(t) + \beta_1 q_1^3(t) + \alpha_2\left[q_1(t) - q_2(t)\right] \\ &\quad + \beta_2\left[q_1(t) - q_2(t)\right]^3 = u_1(t), \\ &\ddot{q}_2(t) - \alpha_2\left[q_1(t) - q_2(t)\right] - \beta_2\left[q_1(t) - q_2(t)\right]^3 = u_2(t). \end{aligned} \tag{2}$$

We select an optimal control force $\mathbf{u}(t)$ by satisfying the following minimization of a specified performance index J:

$$\min\{J = g\left(\mathbf{x}(t_f), t_f\right) + \int_{t_0}^{t_f} L\left(\mathbf{x}(t), \mathbf{u}(t), t\right) dt\}, \tag{3}$$

where $t \in \left[t_0, t_f\right]$ is a time interval we interest, $\mathbf{x} = \left(x_1, x_2\right)^T = \left(x, \dot{x}\right)^T$ is a state vector, and Eq. (1) can be written to be a vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{4}$$

with $f_1 = x_2$ and $f_2 = u(t) - \gamma x_2 - \alpha x_1 - \beta x_1^3$. Similarly, Eq. (2) can be written to be a vector form as that in Eq. (4) with

$$\mathbf{x} = \left(x_1, x_2, x_3, x_4\right)^T = \left(q_1, \dot{q}_1, q_2, \dot{q}_2\right)^T$$

and

$$\begin{aligned} f_1 &= x_2, \\ f_2 &= u_1(t) - \alpha_1 x_1 - \beta_1 x_1^3 - \alpha_2\left(x_1 - x_3\right) - \beta_2\left(x_1 - x_3\right)^3, \\ f_3 &= x_4, \\ f_4 &= u_2(t) + \alpha_2\left(x_1 - x_3\right) + \beta_2\left(x_1 - x_3\right)^3. \end{aligned}$$

The Duffing equation appeared in the literature for almost a century (Cvetićanin, 2013),with a wide range of applications in science and engineering from a nonlinear spring-mass system in mechanics to fault signal detection (Hu and Wen, 2003), and structures design (Suhardjo et al., 1992). The control of a Duffing oscillator has a seminal significance to be the control problems of nonlinear dynamic responses of structures such as beams, plates, and shells.

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Recently, Yue et al. (2014) have proposed an optimal scale polynomial interpolation method to find the periodic solutions of Duffing oscillator. Meanwhile, Dai et al. (2014) used a multiple scale time domain collocation method to solve Duffing equation, and Elgohary et al. (2014) have employed the radial basis function as trial function to solve Duffing equation by using the collocation method. Basically, the collocation method requires to solve nonlinear algebraic equations in a suitable time interval.

Sometimes, we are desired to control the response of a nonlinear structure remained within a specified limit for a safety reason, and thus we may encounter the problem that the external forces are not yet known, but service for a specific purpose of controlling the nonlinear structure to a desired state. Then the resulting problem is an optimal control problem. In this class of control problems, the control forces are intentionally designed such that a specified cost functional which weights the cost of control versus the allowed response is minimized. The control of nonlinear structural systems has gained much attention in the past several decades, and different controllers were proposed for the applications to different areas (Suhardjo, 1992; Agrawal et al., 1998). In the realm of nonlinear structural control, Davies (1972) has studied the time optimal control of Duffing oscillator. Van Dooren and Vlassenbroeck (1982) have introduced a direct method by the Chebyshev series expansion to solve the controlled problem of Duffing oscillator (El-Gindy et al., 1995; El-Kady et al., 2002). Razzaghi and Elnagar (1994) have applied a pseudospectral method to solve this problem, Garg et al. (2010) have provided a unified pseudospectral method to solve the optimal control problems, and Lakestani et al. (2006) have applied a semi-orthogonal spline wavelets to solve this problem. Recently, Rad et al. (2012) have used the radial basis functions method to solve the optimal control problem of Duffing oscillator, and Elgohary et al. (2014) have applied a simple collocation method together with the radial basis functions method to solve the optimal control problem of Duffing oscillator under a simple performance index only a function of control force. As a result, all the above methods required to solve a rather-complicated system of nonlinear algebraic equations. In this paper we will solve the optimal control problem of Duffing oscillators under a complex performance index without needing of the solution of nonlinear algebraic equations.

The remaining portions of this paper are arranged as follows. For the optimal control problem of undamped Duffing oscillator, we derive the governing equations and recast them into a Lie-type system of ordinary differential equations (ODEs) in Section 2, and then an implicit $SL(4, \mathbf{R})$ scheme is derived. In Section 3 we develop the $SL(4, \mathbf{R})$ shooting method for the undamped Duffing oscillator under three different given boundary conditions and four examples are evaluated. In Section 4 we develop the $GL(4, \mathbf{R})$ shooting method for the damped Duffing oscillator and coupled Duffing oscillators under complex performance index and the resulting nonlinear boundary value problems with constraint are solved by Lie-group differential algebraic equation (LGDAE) method, where four numerical examples are giving to test the performance. Finally, some conclusions are drawn in Section 5.

II. A HAMILTONIAN FORMULATION

Let H be the Hamiltonian:

$$H = L\left(\mathbf{x}(t), \mathbf{u}(t), t\right) + \boldsymbol{\lambda}^T \mathbf{f}, \tag{5}$$

where we can rewrite the augmented performance index by

$$\min\{J = g\left(\mathbf{x}\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} \left(H - \boldsymbol{\lambda}^T \dot{\mathbf{x}}\right) dt\}. \tag{6}$$

The variation of the above performance index expressed in terms of the variations of $\mathbf{x}$, $\boldsymbol{\lambda}$ and $\mathbf{u}$ is

$$\begin{aligned} \delta J &= \left(g_{\mathbf{x}} - \boldsymbol{\lambda}\right)^T \delta\mathbf{x}\Big|_{t_f} + \boldsymbol{\lambda}^T \delta\mathbf{x}\Big|_{t_0} \\ &+ \int_{t_0}^{t_f} \left[\left(H_{\mathbf{x}} + \dot{\boldsymbol{\lambda}}\right)^T \delta\mathbf{x} + \left(H_{\boldsymbol{\lambda}} - \dot{\mathbf{x}}\right)^T \delta\boldsymbol{\lambda} + H_{\mathbf{u}}{}^T \delta\mathbf{u}\right] dt, \end{aligned} \tag{7}$$

where the subscript denotes the partial differential. Thus, the minimization of Eq. (6) leads to a triple Euler-Lagrange equation:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}, \tag{8}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}, \tag{9}$$

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0}. \tag{10}$$

Depending on what are prescribed for the end states of $\mathbf{x}(t_0)$ and $\mathbf{x}\left(t_f\right)$, some complementary boundary conditions for $\boldsymbol{\lambda}$ at t_0 and t_f can be obtained from the vanishing of $\left(g_{\mathbf{x}} - \boldsymbol{\lambda}\right)^T \delta\mathbf{x}\Big|_{t_f} + \boldsymbol{\lambda}^T \delta\mathbf{x}\Big|_{t_0}$ in Eq. (7).

For most cases Eq. (10) renders an explicit form of u in terms of state and co-state variables, which is thus being inserted into Eqs. (8) and (9), obtaining a set of two-point boundary value problems. However, if the above statement is not true, Eqs. (8)-(10) constitute a two-point boundary value problem of nonlinear differential algebraic equations (DAEs), which is more difficult to be solved. We will solve this type problem in Section 4.

1. A Lie-Group Approach

In order to demonstrate the Lie-group scheme, let us consider an undamped Duffing equation with the following Hamiltonian:

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 \left(u - \alpha x_1 - \beta x_1^3\right), \tag{11}$$

where $x_1 = x$ and $x_2 = \dot{x}$, λ_1 and λ_2 are two Lagrange multipliers. Then we can derive

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha x_1 - \beta x_1^3 - \lambda_2, \\ \dot{\lambda}_1 &= \left(\alpha + 3\beta x_1^2\right)\lambda_2, \\ \dot{\lambda}_2 &= -\lambda_1, \end{aligned} \tag{12}$$

$$u = -\lambda_2 . \tag{13}$$

For more than one century, the Lie groups have played a decisive role in our understanding of the geometric structure of differential equations, which within their wider terminology and machinery of differential geometry are very useful in devising superior numerical methods to integrate ODEs, and to retain the invariant property of dynamical system. By sharing the geometric structure and invariant with the original ODEs, the new methods are more accurate, more stable and more effective than conventional numerical methods. In the last few years there has been a substantial development in the geometric integrators of ODEs evolving on the Lie groups as shown by (Munthe-Kaas, 1999; Iserles et al., 2000; Hochbruck and Ostermann, 2010). The Lie-group schemes apply to the problem of finding numerical approximations to the solution of

$$\dot{\mathbf{Y}} = \mathbf{A}\left(\mathbf{Y}, t\right)\mathbf{Y},\ \mathbf{Y}\left(0\right) = \mathbf{Y}_0, \tag{14}$$

where the exact solution $\mathbf{Y}$ evolves in a matrix Lie group with $\mathbf{A}$ a matrix function on the associated Lie algebra.

Inspired by Eq. (14) we can rewrite Eq. (12) as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix}, \tag{15}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha - \beta x_1^2 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha + 3\beta x_1^2 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{16}$$

whose resulting Lie-group is $SL(4, \mathbf{R})$, due to $\mathrm{tr}(\mathbf{A}) = 0$.

2. A Lie-Group Scheme

In this section we derive a Lie-group scheme for the solution of Eq. (15). Suppose that $(x_1, x_2, \lambda_1, \lambda_2)$ at the kth step are already known, and we will develop numerical method from Eq. (15) to compute the solution of $(x_1, x_2, \lambda_1, \lambda_2)$ at the $(k + 1)$th step, where $t_{k+1} = t_{k+h}$ with h a small time stepsize.

Accordingly, we can develop an implicit scheme based on $SL(4, \mathbf{R})$ for the integration of Eqs. (15) and (16), of which one can view

$$\overline{\mathbf{A}}_k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\alpha - \beta\left(\overline{x}_1^k\right)^2 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha + 3\beta\left(\overline{x}_1^k\right)^2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{17}$$

as a constant matrix, where $\overline{x}_1^k = \left(1 - \theta\right)x_1^k + \theta x_1^{k+1}$ is assumed to be constant within a small time step. The corresponding state transition matrix $\mathbf{G}_k = \exp\left(h\overline{\mathbf{A}}_k\right)$ is derived in the Appendix.

This Lie-group scheme is implicit because it also depends on x_1^{k+1}, which requires an iteration to determine the value of $(x_1, x_2, \lambda_1, \lambda_2)$ at the next time step.

(i) Give $0 \le \theta \le 1$.
(ii) Give initial values at an initial time $t = 0$ and a time stepsize h.
(iii) For $k = 0, 1, \ldots$, we repeat the following computations to a specified terminal time t_f. We first apply the forward Euler method to integrate Eq. (12):

$$\begin{aligned} x_1^{k+1} &= x_1^k + hx_2^k, \\ x_2^{k+1} &= x_2^k - h\left[\alpha x_1^k + \beta\left(x_1^k\right)^3 + \lambda_2^k\right], \\ \lambda_1^{k+1} &= \lambda_1^k + h\left[\alpha + 3\beta\left(x_1^k\right)^2\right]\lambda_2^k, \\ \lambda_2^{k+1} &= \lambda_2^k - h\lambda_1^k. \end{aligned} \tag{18}$$

with the above $\left(x_1^{k+1}, x_2^{k+1}, \lambda_1^{k+1}, \lambda_2^{k+1}\right)$ generated from an Euler step as an initial guess we iteratively solve the new $\left(x_1^{k+1}, x_2^{k+1}, \lambda_1^{k+1}, \lambda_2^{k+1}\right)$ by

$$\overline{x}_1^k = \left(1 - \theta\right)x_1^k + \theta x_1^{k+1},$$

Compute G_k by Eq. (A11) with $t = h$,

$$\begin{bmatrix} \hat{y}_1^{k+1} \\ \hat{y}_2^{k+1} \\ \hat{z}_1^{k+1} \\ \hat{z}_2^{k+1} \end{bmatrix} = \mathbf{G}_k \begin{bmatrix} x_1^k \\ x_2^k \\ \lambda_1^k \\ \lambda_2^k \end{bmatrix} \tag{19}$$

If $\left(\hat{y}_1^{k+1}, \hat{y}_2^{k+1}, \hat{z}_1^{k+1}, \hat{z}_2^{k+1}\right)$ converges according to a given stopping criterion, such that,

$$\sqrt{\left(\hat{y}_1^{k+1}-\hat{x}_1^{k+1}\right)^2+\left(\hat{y}_2^{k+1}-\hat{x}_2^{k+1}\right)^2+\left(\hat{z}_1^{k+1}-\lambda_1^{k+1}\right)^2+\left(\hat{z}_2^{k+1}-\lambda_2^{k+1}\right)^2}<\varepsilon \tag{20}$$

then go to (iii) for the next time step; otherwise, let $x_1^{k+1}=\hat{y}_1^{k+1}$, $x_2^{k+1}=\hat{y}_2^{k+1}$, $\lambda_1^{k+1}=\hat{z}_1^{k+1}$ and $\lambda_2^{k+1}=\hat{z}_2^{k+1}$, and go to Eq. (19).

III. *SL*(4, R) SHOOTING METHOD

In Section 2 we have constructed a Lie-group *SL*(4, **R**) method for Eq. (12), which maps $\mathbf{y}_k=\left(x_1^k, x_2^k, \lambda_1^k, \lambda_2^k\right)^T$ to $\mathbf{y}_{k+1}=\left(x_1^{k+1}, x_2^{k+1}, \lambda_1^{k+1}, \lambda_2^{k+1}\right)^T$ by

$$\mathbf{y}_{k+1}=\mathbf{G}_k\mathbf{y}_k \tag{21}$$

where $\mathbf{G}_k \in SL(4, \mathbf{R})$. By using the closure property of the Lie-group, there exists a Lie-group denoted by $\mathbf{G}(r)$ which maps

$$\mathbf{y}_0=\left(x_1(t_0), x_2(t_0), \lambda_1(t_0), \lambda_2(t_0)\right)^T$$

to

$$\mathbf{y}_f=\left(x_1(t_f), x_2(t_f), \lambda_1(t_f), \lambda_2(t_f)\right)^T$$

by

$$\mathbf{y}_f=\mathbf{G}(r)\mathbf{y}_0 \tag{22}$$

where **G**(*r*) is modified from Eq. (A11) by

$$\mathbf{G}(r)=\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14}\\ G_{21} & G_{22} & G_{23} & G_{24}\\ G_{31} & G_{32} & G_{33} & G_{34}\\ G_{41} & G_{42} & G_{43} & G_{44}\end{bmatrix}$$
$$=\begin{bmatrix} \cos\sqrt{a}\ell & \frac{\sin\sqrt{a}\ell}{\sqrt{a}} & \frac{\sin\sqrt{a}\ell}{(b-a)\sqrt{a}}+\frac{\sin\sqrt{b}\ell}{(a-b)\sqrt{b}} & \frac{\cos\sqrt{a}\ell}{a-b}+\frac{\cos\sqrt{b}\ell}{b-a}\\ -\sqrt{a}\sin\sqrt{a}\ell & \cos\sqrt{a}\ell & \frac{\cos\sqrt{a}\ell}{b-a}+\frac{\cos\sqrt{b}\ell}{a-b} & \frac{\sqrt{a}\sin\sqrt{a}\ell}{(b-a)}+\frac{\sqrt{b}\sin\sqrt{b}\ell}{(a-b)}\\ 0 & 0 & \cos\sqrt{b}\ell & \sqrt{b}\sin\sqrt{b}\ell\\ 0 & 0 & -\frac{\sin\sqrt{b}\ell}{\sqrt{b}} & \cos\sqrt{b}\ell \end{bmatrix} \tag{23}$$

in which

$$\begin{aligned} \ell &= t_f - t_0,\\ \overline{x}_1 &= (1-r)x_1^0 + r x_1^f,\\ a &= \alpha+\beta\overline{x}_1^2,\\ b &= \alpha+3\beta\overline{x}_1^2. \end{aligned} \tag{24}$$

and *r* is a parameter to be determined by matching the target equation.

In terms of $\mathbf{x}=(x_1, x_2)^T$, $\boldsymbol{\lambda}=(\lambda_1, \lambda_2)^T$ and

$$\mathbf{G}(r)=\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\\ \mathbf{0} & \mathbf{G}_3\end{bmatrix}. \tag{25}$$

Eq. (22) can be written as

$$\begin{bmatrix}\mathbf{x}^f\\ \boldsymbol{\lambda}^f\end{bmatrix}=\begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\\ \mathbf{0} & \mathbf{G}_3\end{bmatrix}\begin{bmatrix}\mathbf{x}^0\\ \boldsymbol{\lambda}^0\end{bmatrix}. \tag{26}$$

We consider the following problems by using the *SL*(4, **R**) shooting method.

1. Example 1

In this example we solve the optimal control problem of the undamped Duffing oscillator (Davies, 1972; van Dooren and Vlassenbroeck, 1982; El-Gindy et al., 1995; El-Kady, 2002; Lakestani et al., 2006; Liu, 2012), where an optimal control problem for Eq. (12) is under the following performance index and boundary conditions:

$$\begin{aligned} &J=\frac{1}{2}\int_{t_0}^{t_f}u^2dt,\\ &x(t_0)=A,\ \dot{x}(t_0)=B,\ x(t_f)=C,\ \dot{x}(t_f)=D. \end{aligned} \tag{27}$$

From Eq. (26) we can solve

$$\boldsymbol{\lambda}^0=\mathbf{G}_2^{-1}\mathbf{x}^f-\mathbf{G}_2^{-1}\mathbf{G}_1\mathbf{x}^0. \tag{28}$$

In terms of *r* we can obtain different initial values of $\boldsymbol{\lambda}^0$, and then we can integrate Eq. (12) from $t = t_0$ to $t = t_f$, where we select the best value of *r* to match the target equation:

$$\min_{r\in R}\left\{\sqrt{\left[x(t_f)-C\right]^2+\left[\dot{x}(t_f)-D\right]^2}\right\}. \tag{29}$$

Under the following parameters $t_0 = -2$, $t_f = 0$, $\alpha = 1$, $\beta = 0.15$, $A = 0.5$, $B = -0.5$, $C = D = 0$, we plot the error of mismatching in Fig. 1(a), and the responses of x_1, x_2 and the control force *u* are plotted in Fig. 1(b). The computed results are

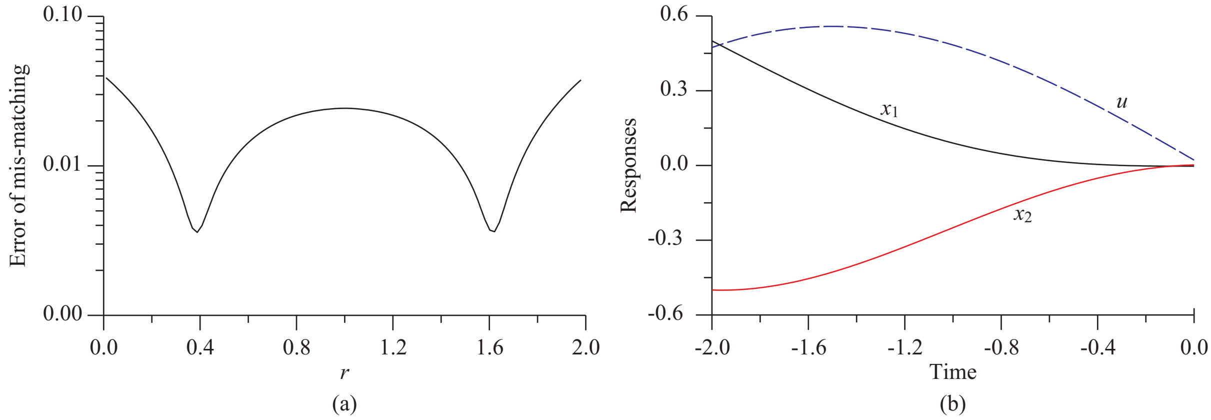

Fig. 1. For the optimal control of an undamped Duffing equation with $\beta = 0.15$ in example 1, (a) the error of mis-matching, and (b) the time histories of responses and control force.

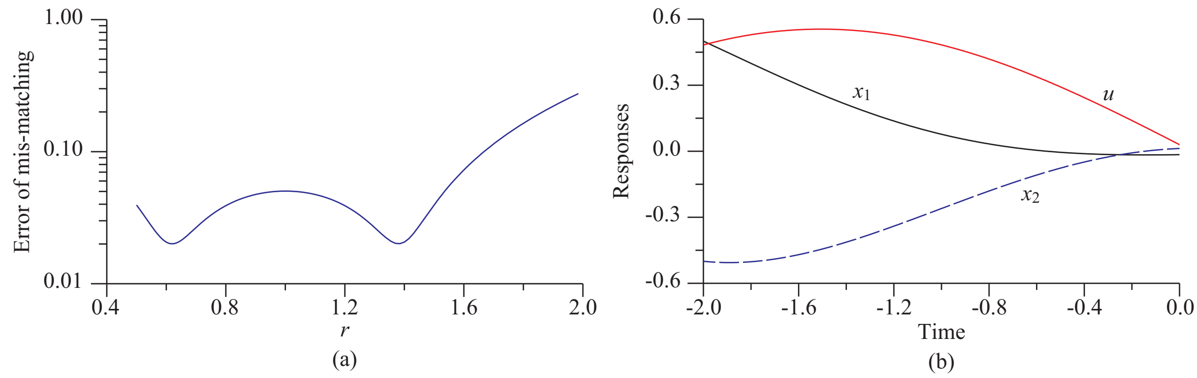

Fig. 2. For the optimal control of an undamped Duffing equation with $\beta = 0.75$ in example 1, (a) the error of mis-matching, and (b) the time histories of responses and control force.

$\lambda_1(t_0) = 0.271022$, $\lambda_2(t_0) = -0.4858358$, and $J = 0.1858713$. It can be seen that the curves of displacement and velocity both match the terminal conditions $x(t_f) = \dot{x}(t_f) = 0$ very well. It deserves to note that the value of J we obtained is slightly smaller than 0.1874, which was obtained by other methods (van Dooren and Vlassenbroeck, 1982; Razzaghi and Elnagar, 1994; Lakestani et al., 2006). It shows that the present method can achieve a better control strategy than other methods.

Under the following parameters $\alpha = 1$, $\beta = 0.75$, $A = 0.5$, $B = -0.5$, $C = D = 0$, we plot the error of mis-matching in Fig. 2(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 2(b). The computed results are $\lambda_1(t_0) = 0.2902921$, $\lambda_2(t_0) = -0.48281117$, and $J = 0.18910203$. The above value of J is close to $J = 0.1879$ obtained by Liu (2012), but is smaller than $J = 0.1979$ obtained by Razzaghi and Elnagar (1994). It shows that the present method can achieve a better optimal control force than the method of Razzaghi and Elnagar (1994).

2. Example 2

We consider the following performance index for the undamped Duffing oscillator:

$$J = \frac{1}{2}x^2\left(t_f\right) + \frac{1}{2}\dot{x}^2\left(t_f\right) + \frac{1}{2}\int_{t_0}^{t_f} u^2 dt\,, \tag{30}$$

where we fix $t_0 = 0, t_f = 2, x\left(t_0\right) = 0.5$ and $\dot{x}\left(t_0\right) = 0.5$.

From Eq. (26) with $\mathbf{x}^f = \boldsymbol{\lambda}^f$ we can solve

$$\boldsymbol{\lambda}^0 = \left(\mathbf{G}_3 - \mathbf{G}_2\right)^{-1} \mathbf{G}_1 \mathbf{x}^0\,. \tag{31}$$

In terms of r we can obtain different initial values of $\boldsymbol{\lambda}^0$ through some iterations, and then we can integrate Eq. (12) from $t = t_0$ to $t = t_f$, where we select the best value of r to match the target equation:

$$\min_{r\in R}\left\{\sqrt{\left[x\left(t_f\right) - \lambda_1\left(t_f\right)\right]^2 + \left[\dot{x}\left(t_f\right) - \lambda_2\left(t_f\right)\right]^2}\right\} \tag{32}$$

Under the following parameters $\alpha = 1$, $\beta = 0.9$, we plot the error of mis-matching in Fig. 3(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 3(b). The computed results are $\lambda_1(t_0) = 0.293177$, $\lambda_2(t_0) = 0.343054$, and $J = 0.15044$.

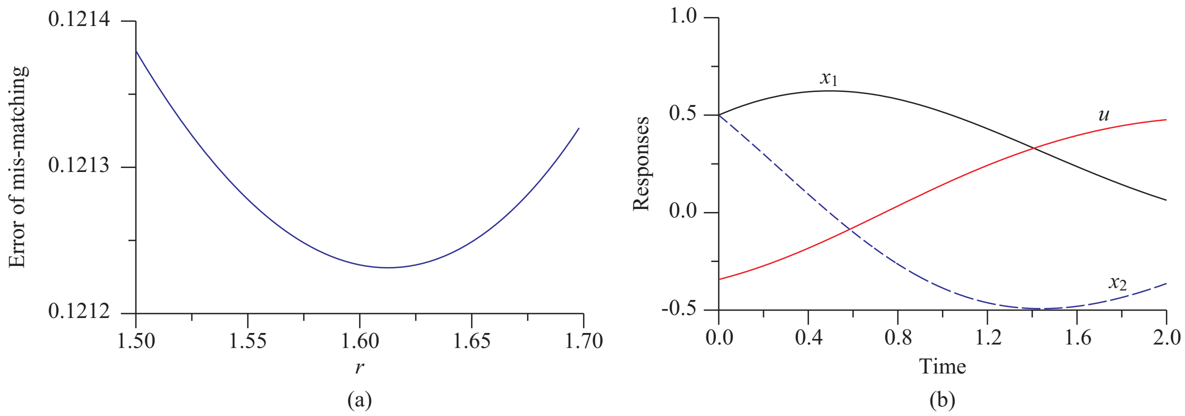

Fig. 3. For the optimal control of an undamped Duffing equation in example 2, (a) the error of mis-matching, and (b) the time histories of responses and control force.

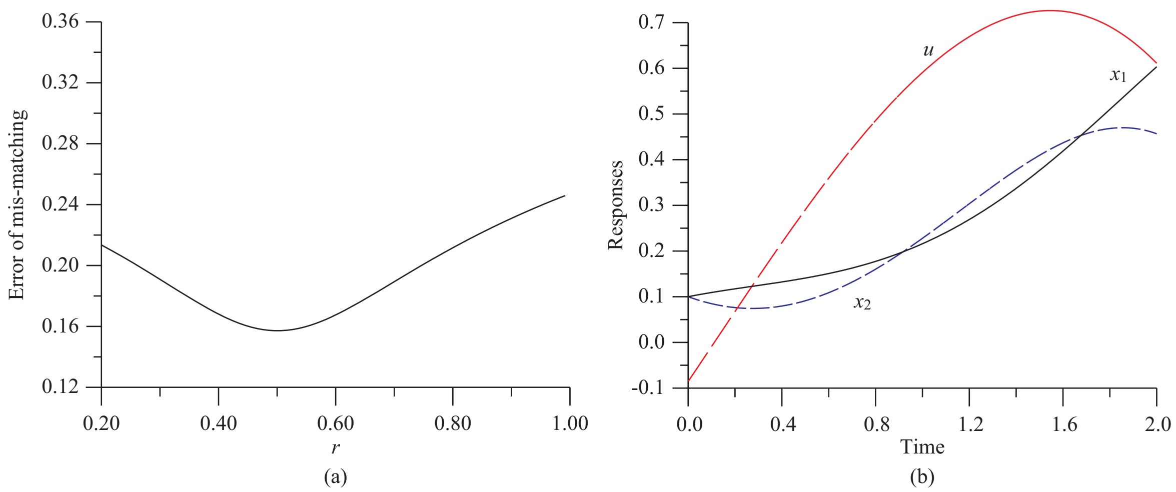

Fig. 4. For the optimal control of an undamped Duffing equation in example 3, (a) the error of mis-matching, and (b) the time histories of responses and control force.

3. Example 3

We consider the following performance index for the undamped Duffing oscillator (Elgohary et al., 2014):

$$J = \frac{1}{2}\left\|\mathbf{x}\left(t_f\right) - \mathbf{q}\right\|^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2 dt \tag{33}$$

where $\mathbf{q}$ is the desired final state at a specified final time. Here we fix $t_0 = 0$, $t_f = 2$, $x(t_0) = \dot{x}(t_0) = 0.1$ and $q_1 = q_2 = 1$.

From Eq. (26) with $\boldsymbol{\lambda}^f = \mathbf{x}^f - \mathbf{q}$ we can solve

$$\boldsymbol{\lambda}^0 = \left(\mathbf{G}_3 - \mathbf{G}_2\right)^{-1}\mathbf{G}_1\mathbf{x}^0 - \left(\mathbf{G}_3 - \mathbf{G}_2\right)^{-1}\mathbf{q}\,. \tag{34}$$

In terms of r we can obtain different initial values of $\boldsymbol{\lambda}^0$ through some iterations, and then we can integrate Eq. (12) from $t = t_0$ to $t = t_f$, where we select the best value of r to match the target equation:

$$\min_{r\in R}\left\{\sqrt{\left[x\left(t_f\right) - \lambda_1\left(t_f\right) - q_1\right]^2 + \left[\dot{x}\left(t_f\right) - \lambda_2\left(t_f\right) - q_2\right]^2}\right\}. \tag{35}$$

Under the following parameters $\alpha = 1$, $\beta = 0.9$, we plot the error of mis-matching in Fig. 4(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 4(b). The computed results are $\lambda_1(t_0) = 0.762934$, $\lambda_2(t_0) = 0.085703$, and $J = 0.5191889$.

4. Example 4

In this example we solve the optimal control problem of the damped Duffing oscillator under an external periodic force (Rad et al., 2012) with the performance index given by Eq. (27), of which the governing equations are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\gamma x_2 - \alpha x_1 - \beta x_1^3 - \lambda_2 + f_0 \cos\omega t, \\
\dot{\lambda}_1 &= \left(\alpha + 3\beta x_1^2\right)\lambda_2, \\
\dot{\lambda}_2 &= -\lambda_1 + \gamma\lambda_2.
\end{aligned} \tag{36}$$

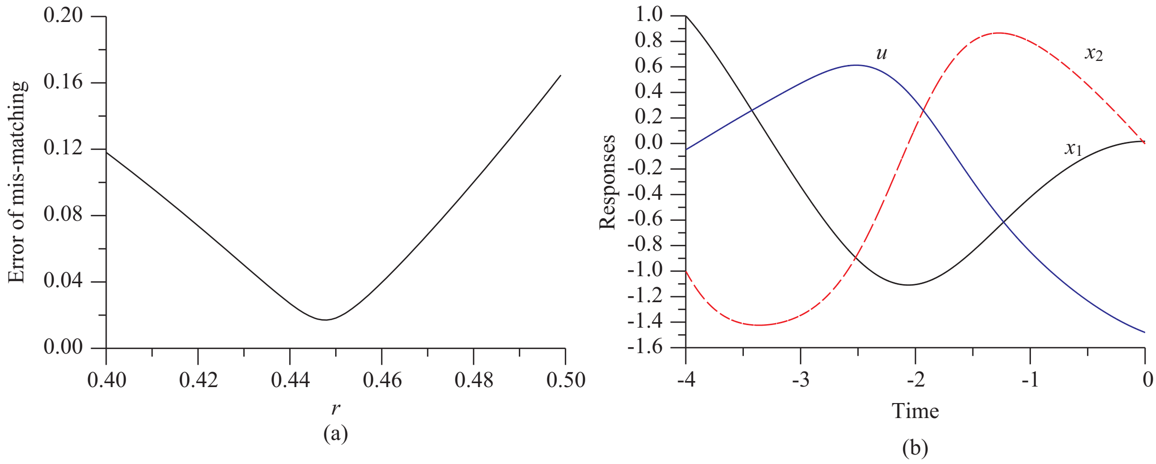

Fig. 5. For the optimal control of a damped and forced Duffing equation in example 4, (a) the error of mis-matching, and (b) the time histories of responses and control force.

Under the two-point boundary conditions $x_1(t_0) = 1$, $x_1(t_0) = -1$, $x_1(t_f) = x_2(t_f) = 0$, and under the following parameters $t_0 = -4$, $t_f = 0$, $\alpha = 0.49$, $\beta = 1$, $\gamma = 0.2$, $f_0 = 0.5$ and $\omega = 1$, we plot the error of mis-matching in Fig. 5(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 5(b). The computed results are $\lambda_1(t_0) = 0.536554$, $\lambda_2(t_0) = 0.049429$, and $J = 1.034594$. It can be seen that the final time conditions $x_1(t_f) = x_2(t_f) = 0$ are matched very well.

IV. *GL*(4, R) SHOOTING METHOD

In the above we have developed the $SL(4, \mathbf{R})$ shooting method for the optimal control problem of Duffing oscillators under relatively simple performance indices. However, when the performance index is complex, it is hard to derive the corresponding Lie-group $\mathbf{G} \in SL(4, \mathbf{R})$. In this section we derive a more general $GL(4, \mathbf{R})$ shooting method for the optimal problem of Duffing oscillators under relatively complex performance indices.

1. Example 5

First we solve the optimal control problem of the damped Duffing oscillator under a more complex performance index:

$$J = \frac{1}{2}\int_{t_0}^{t_f}\left[x^2(t) + \dot{x}^2(t) + u^2(t)\right]dt, \tag{37}$$

which is subjected to the initial conditions with $x(t_0) = A_0 = 0.5$, $\dot{x}(t_0) = B_0 = -0.5$ and the ends are free.

In the Hamiltonian formulation, we can derive

$$\begin{aligned}
&\dot{x}_1 = x_2,\ x_1(t_0) = A_0,\\
&\dot{x}_2 = u - \gamma x_2 - \left(\alpha x_1 + \beta x_1^2\right)x_1,\ x_2(t_0) = B_0,\\
&\dot{\lambda}_1 = \left(\alpha + 3\beta x_1^2\right)\lambda_2 - x_1,\ \lambda_1(t_f) = 0,\\
&\dot{\lambda}_2 = \gamma\lambda_2 - x_2 - \lambda_1,\ \lambda_2(t_f) = 0,
\end{aligned} \tag{38}$$

where

$$u = -\lambda_2 .$$

For the nonlinear ODEs system:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t),\ \mathbf{y} \in \mathbf{R}^n, \tag{39}$$

Liu (2013a) has derived the following implicit GL(n, $\mathbf{R}$) algorithm:

$$\mathbf{y}_{k+1} = \mathbf{G}_k\mathbf{y}_k,\ \mathbf{G}_k \in GL(n, \mathbf{R}), \tag{40}$$

where

$$\begin{aligned}
&\mathbf{a}_k = \frac{\mathbf{f}(\bar{\mathbf{y}}_k, \bar{t}_k)}{\|\bar{\mathbf{y}}_k\|},\\
&\eta_k = \frac{1}{\|\mathbf{a}_k\|}\left(\sinh\left(\|\mathbf{a}_k\|h\right) + \left[\cosh\left(\|\mathbf{a}_k\|h\right) - 1\right]\frac{\mathbf{a}_k \cdot \mathbf{y}_k}{\|\mathbf{a}_k\|\ \|\mathbf{y}_k\|}\right),\\
&\mathbf{G}_k = \mathbf{I}_n + \eta_k\frac{\mathbf{a}_k\mathbf{y}_k^T}{\|\mathbf{y}_k\|}.
\end{aligned} \tag{41}$$

in which $\bar{\mathbf{y}}_k = (1-\theta)\mathbf{y}_k + \theta\mathbf{y}_{k+1}$, and $\bar{t}_k = (1-\theta)t_k + \theta t_{k+1}$.

Accordingly, we can develop a GL(n, $\mathbf{R}$) shooting method to solve Eq. (38). $\mathbf{G}$ is computed by

$$\begin{aligned}
&\bar{\mathbf{y}} = (1-r)\mathbf{y}_0 + r\mathbf{y}_f,\\
&\bar{t} = (1-r)t_0 + rt_f,\\
&\mathbf{a} = \frac{\mathbf{f}(\bar{\mathbf{y}}, \bar{t})}{\|\bar{\mathbf{y}}\|},\\
&\eta = \frac{1}{\|\mathbf{a}\|}\left(\sinh\left(\|\mathbf{a}\|\bar{t}\right) + \left[\cosh\left(\|\mathbf{a}\|\bar{t}\right) - 1\right]\frac{\mathbf{a} \cdot \mathbf{y}_0}{\|\mathbf{a}\|\ \|\mathbf{y}_0\|}\right),\\
&\mathbf{G} = \mathbf{I}_n + \eta\frac{\mathbf{a}\mathbf{y}_0^T}{\|\mathbf{y}_0\|}.
\end{aligned} \tag{42}$$

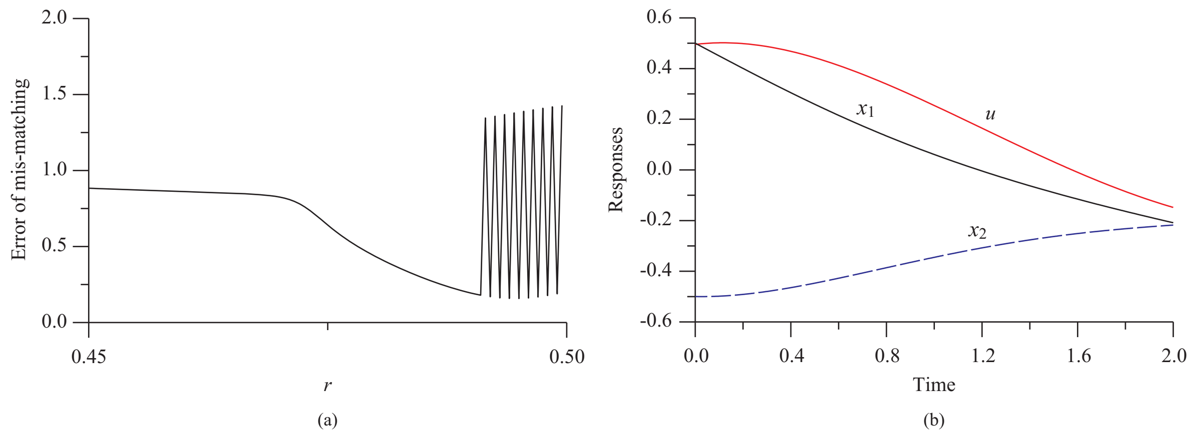

Fig. 6. For the optimal control of a damped Duffing equation in example 5, (a) the error of mis-matching, and (b) the time histories of responses and control force.

where

$$\mathbf{y}_0 = \left(x_1(t_0), x_2(t_0), \lambda_1(t_0), \lambda_2(t_0)\right)^T,$$

$$\mathbf{y}_f = \left(x_1(t_f), x_2(t_f), \lambda_1(t_f), \lambda_2(t_f)\right)^T.$$

In $\mathbf{y}_0$, $(\lambda_1(t_0), \lambda_2(t_0))^T$ are unknown and in $\mathbf{y}_f$, $(x_1(t_f), x_2(t_f))^T$ are unknown, which can be iteratively solved by

$$\begin{aligned} \boldsymbol{\lambda}^0 &= -\mathbf{G}_4^{-1}\mathbf{G}_3\mathbf{x}^0, \\ \mathbf{x}^f &= \left(\mathbf{G}_1 - \mathbf{G}_2\mathbf{G}_4^{-1}\mathbf{G}_3\right)\mathbf{x}^0 \end{aligned} \tag{43}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_3 & \mathbf{G}_4 \end{bmatrix}. \tag{44}$$

Under the following parameters $\gamma = 0.02$, $\alpha = 1$, $\beta = 0.15$, $t_0 = 0$ and $t_f = 2$, we plot the error of mis-matching in Fig. 6(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 6(b). The computed results are $\lambda_1(t_0) = 0.605366$, $\lambda_2(t_0) = -0.4952084$, and $J = 0.28393986$.

2. Example 6

Then we consider an optimal control problem of the damped Duffing oscillator under a complex performance index:

$$J = \frac{1}{2}\int_{t_0}^{t_f}\left[x^2(t) + \dot{x}^2(t) + \exp\left(u^2(t)\right)\right]dt, \tag{45}$$

which is subjected to the initial conditions with $x(t_0) = A_0 = 0.5$, $\dot{x}(t_0) = B_0 = -0.5$ and the ends are free.

We can derive

$$\begin{aligned} &\dot{x}_1 = x_2,\ x_1(t_0) = A_0, \\ &\dot{x}_2 = u - \gamma x_2 - \left(\alpha x_1 + \beta x_1^2\right)x_1,\ x_2(t_0) = B_0, \\ &\dot{\lambda}_1 = \left(\alpha + 3\beta x_1^2\right)\lambda_2 - x_1,\ \lambda_1(t_f) = 0, \\ &\dot{\lambda}_2 = \gamma\lambda_2 - x_2 - \lambda_1,\ \lambda_2(t_f) = 0, \end{aligned} \tag{46}$$

where u is solved from

$$\frac{\partial H}{\partial u} = u\exp\left(u^2\right) + \lambda_2 = 0. \tag{47}$$

It is difficult to express u as a function of λ_2, and hence many methods based on the Hamiltonian formulation cannot be applied to solve this problem. Indeed, Eqs. (46) and (47) are differential algebraic equations (DAEs) equipped with two-point boundary values. We solve this problem by the above $GL(n, \mathbf{R})$ shooting method together with the Lie-group differential algebraaic equation (LGDAE) method (Liu, 2013b; Liu, 2014a; Liu, 2014b; Liu, 2014c; Liu, 2014d; Liu, 2015a; Liu, 2015b; Liu, 2015c; Liu et al., 2017).

Eqs. (46) and (47) constitute a system of two-point nonlinear differential algebraic equations (DAEs). Hereby, we give a general setting to treat the DAEs which governthe evolution of $n + q$ variables x_i, $i = 1, \ldots, n$ and y_j, $j = 1, \ldots, q$, with n nonlinear ordinary differential equations (ODEs) and q nonlinear algebraic equations (NAEs):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, t),\ \mathbf{x}(0) = \mathbf{x}_0,\ t \in \mathbf{R},\ \mathbf{x} \in \mathbf{R}^n,\ \mathbf{y} \in \mathbf{R}^q, \tag{48}$$

$$\mathbf{F}(\mathbf{x}, \mathbf{y}, t) = \mathbf{0},\ \mathbf{F} \in \mathbf{R}^q. \tag{49}$$

Inspired by Eq. (14), we give a new form of Eq. (48) to fit the Lie-group equation from the $GL(n, \mathbf{R})$ Lie-group structure. The vector field $\mathbf{f}$ on the right-hand side of Eq. (48) can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{50}$$

where

$$\mathbf{A} = \frac{\mathbf{f}}{\|\mathbf{x}\|} \otimes \frac{\mathbf{x}}{\|\mathbf{x}\|}. \tag{51}$$

is the coefficient matrix. The symbol $\otimes$ in $\mathbf{u} \otimes \mathbf{y}$ denotes the dyadic operation of u and y, i.e., $(\mathbf{u} \otimes \mathbf{y})\mathbf{z} = \mathbf{y} \cdot \mathbf{z}\mathbf{u}$.

Within a small time step we can suppose that the variables y_j. $j = 1, \ldots, m$ are constant in the interval of $t_k < t < t_{k+1}$. Consequently, we can develop the following implicit scheme for solving the ODEs in Eq. (48) where y at the kth time step, denoted by $\mathbf{y}_k$, is viewed as a parameter:

(i) Give $0 \leq \theta \leq 1$.
(ii) Give an initial values $\mathbf{x}_0$ at an initial time $t = t_0$ and a time stepsize h.
(iii) For $k = 0, 1, \ldots$, we repeat the following computations to a terminal time:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h\mathbf{f}_k \tag{52}$$

where $\mathbf{f}_k = \mathbf{f}(\mathbf{x}_k, \mathbf{y}_k, t_k)$. With the above $\mathbf{x}_{k+1}$ generated from an Euler step as an initial guess we can iteratively solve the new $\mathbf{x}_{k+1}$ by

$$\begin{aligned} \bar{t}_k &= t_k + \theta h, \\ \bar{\mathbf{x}}_k &= (1-\theta)\mathbf{x}_k + \theta \mathbf{x}_{k+1}, \end{aligned} \tag{53}$$

$$\begin{aligned} \bar{\mathbf{f}}_k &= \mathbf{f}(\bar{\mathbf{x}}_k, \mathbf{y}_k, \bar{t}_k) \\ \mathbf{a}_k &= \frac{\bar{\mathbf{f}}_k}{\|\bar{\mathbf{x}}_k\|}, \mathbf{b}_k = \frac{\bar{\mathbf{x}}_k}{\|\bar{\mathbf{x}}_k\|}, \\ c_k &= \mathbf{a}_k \cdot \mathbf{b}_k, d_k = \mathbf{x}_k \cdot \mathbf{b}_k, \\ \eta_k &= \frac{\exp(c_k h) - 1}{c_k} \\ \mathbf{z}_{k+1} &= \mathbf{x}_k + \eta_k d_k \mathbf{a}_k. \end{aligned} \tag{54}$$

If $\mathbf{z}_{k+1}$ converges according to a given stopping criterion, such that,

$$\|\mathbf{z}_{k+1} - \mathbf{x}_{k+1}\| < \varepsilon. \tag{55}$$

then go to (iii) to the next time step; otherwise, let $\mathbf{x}_{k+1} = \mathbf{z}_{k+1}$ and go to the computations in Eqs. (53) and (54) again. In all the computations given below we will fix $\theta = 1/2$.

Now, we turn our attention to the DAEs defined in Eqs. (48) and (49). Within a small time step we can suppose that the variables y_j. $j = 1, \ldots, m$ are constant in the interval of $t_k < t < t_{k+1}$. We give an initial guess of y_j, $j = 1, \ldots, m$, and insert them into Eq. (48). Then we apply the above implicit scheme to find the next $\mathbf{x}_{k+1}$, supposing that $\mathbf{x}_k$ is already obtained in the previous time step. When $\mathbf{x}_{k+1}$ are available we insert them into Eq. (49), and then apply the Newton iterative scheme to solve $\mathbf{y}_{k+1}$ by

$$\mathbf{y}_{k+1}^{\ell+1} = \mathbf{y}_{k+1}^{\ell} - \mathbf{B}^{-1}\mathbf{F}(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}^{\ell}, t_{k+1}), \tag{56}$$

till the following convergence criterion is satisfied:

$$\|\mathbf{y}_{k+1}^{\ell+1} - \mathbf{y}_{k+1}^{\ell}\| < \varepsilon_1. \tag{57}$$

Otherwise, go to Eq. (53). In the above the component B_{ij} of the Jacobian matrix $\mathbf{B}$ is given by $\partial F_i / \partial y_j$.

We solve the unknown initial values of $\boldsymbol{\lambda}^0 = (\lambda_1(t_0), \lambda_2(t_0))^T$ by Eq. (43), and then the resulting DAEs by the LGDAE. Under the following parameters $\gamma = 0.02$, $\alpha = 1$, $\beta = 0.15$, $t_0 = 0$ and $t_f = 2$, we plot the error of mis-matching in Fig. 7(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 7(b). The computed results are $\lambda_1(t_0) = 0.610798$, $\lambda_2(t_0) = -0.506856$, and $J = 1.29008$.

When $\gamma = 0$ we can obtain $J = 1.29299$, which is better than $J = 1.466$ obtained by Liu (2012).

3. Example 7

Then we consider an optimal control problem of the damped Duffing oscillator under a more complex performance index:

$$J = \frac{1}{2}x^2(t_f) + \frac{1}{2}\dot{x}^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[x^2(t) + \dot{x}^2(t) + u^2(t) + \exp(u^2(t))\right]dt, \tag{58}$$

where we fix $t_0 = 0$, $t_f = 2$, $x(t_0) = A_0 = 0.5$ and $\dot{x}(t_0) = B_0 = -0.5$.

For this problem we have

$$\begin{aligned} \dot{x}_1 &= x_2, x_1(t_0) = A_0, \\ \dot{x}_2 &= u - \gamma x_2 - (\alpha x_1 + \beta x_1^2)x_1, x_2(t_0) = B_0, \\ \dot{\lambda}_1 &= (\alpha + 3\beta x_1^2)\lambda_2 - x_1, \lambda_1(t_f) = x_1(t_f), \\ \dot{\lambda}_2 &= \gamma\lambda_2 - x_2 - \lambda_1, \lambda_2(t_f) = x_2(t_f), \\ \frac{\partial H}{\partial u} &= u + u\exp(u^2) + \lambda_2 = 0. \end{aligned} \tag{59}$$

By using $\mathbf{x}^f = \boldsymbol{\lambda}^f$, we can solve

$$\boldsymbol{\lambda}^0 = (\mathbf{G}_4 - \mathbf{G}_2)^{-1}(\mathbf{G}_1 - \mathbf{G}_3)\mathbf{x}^0. \tag{60}$$

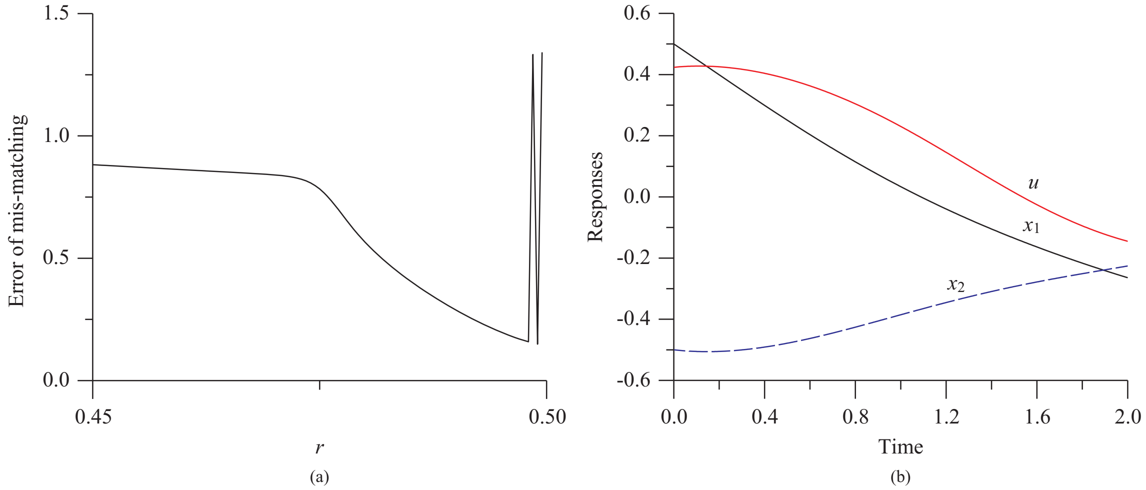

Fig. 7. For the optimal control of a damped Duffing equation under a complex performance index in example 6, (a) the error of mis-matching, and (b) the time histories of responses and control force.

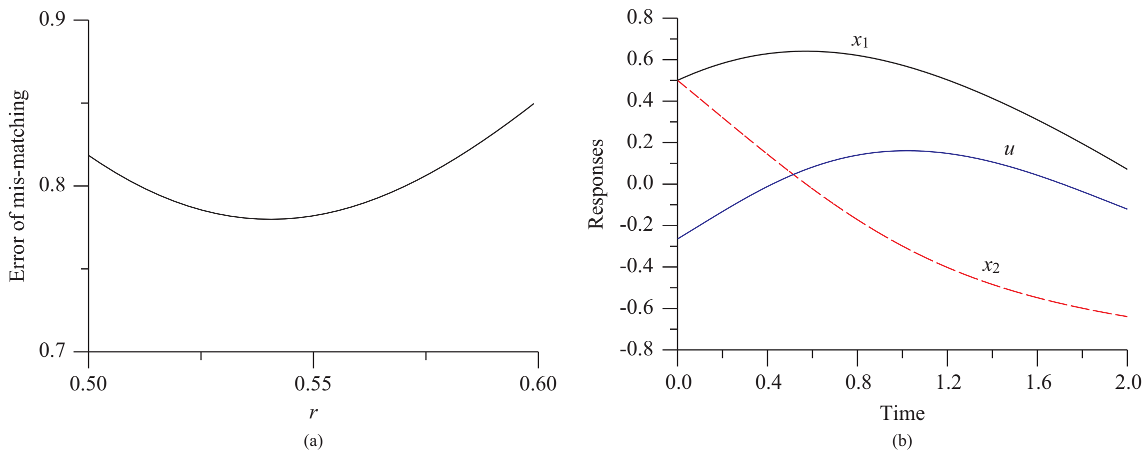

Fig. 8. For the optimal control of a damped Duffing equation under a complex performance index of example 7, (a) the error of mis-matching, and (b) the time histories of responses and control force.

In terms of r we can obtain different initial values of $\boldsymbol{\lambda}^0$ through some iterations, and then we can integrate Eq. (59) from $t = t_0$ to $t = t_f$, where we select the best value of r to match the target equation:

$$\min_{r\in R}\left\{\sqrt{\left[x\left(t_f\right)-\lambda_1\left(t_f\right)\right]^2+\left[\dot{x}\left(t_f\right)-\lambda_2\left(t_f\right)\right]^2}\right\} \tag{61}$$

Under the following parameters $\gamma = 0.02$, $\alpha = 1$, $\beta = 0.9$, $t_0 = 0$ and $t_f = 2$, we plot the error of mis-matching in Fig. 8(a), and the responses of x_1, x_2 and the control force u are plotted in Fig. 8(b). The computed results are $\lambda_1(t_0) = 0.979338$, $\lambda_2(t_0) = 0.5496034$, and $J = 1.6374$.

4. Example 8

Finally we consider an optimal control problem of two coupled Duffing oscillators as shown in Eq. (2) under a complex performance index:

$$J=\frac{1}{2}\int_{t_0}^{t_f}\left[q_1^2(t)+\dot{q}_1^2(t)+q_2^2(t)+\dot{q}_2^2(t)+\exp\left(u_1^2(t)+u_2^2(t)\right)\right]dt. \tag{62}$$

We fix $q_1(0) = q_2(0) = 0.5$ and $\dot{q}_1(0) = \dot{q}_2(0) = -0.5$. For this problem we have

$$\begin{aligned}
&\dot{x}_1 = x_2,\ \dot{x}_2 = u_1-\left(\alpha x_1+\beta x_1^2\right)x_1-\alpha_2\left(x_1-x_3\right)-\beta_2\left(x_1-x_3\right)^3,\\
&\dot{x}_3 = x_4,\ \dot{x}_4 = u_2+\alpha_2\left(x_1-x_3\right)+\beta_2\left(x_1-x_3\right)^3,\\
&\dot{\lambda}_1 = -x_1+\lambda_2\left(\alpha_1-3\beta_1x_1^2\right)-\left[\alpha_2+3\beta_2\left(x_1-x_3\right)^2\right]\left(\lambda_4-\lambda_2\right),\\
&\dot{\lambda}_2 = -x_2-\lambda_1,\ \dot{\lambda}_3 = -x_3+\left[\alpha_2+3\beta_2\left(x_1-x_3\right)^2\right]\left(\lambda_4-\lambda_2\right),\\
&\dot{\lambda}_2 = -x_4-\lambda_3,\\
&\frac{\partial H}{\partial u_1} = u_1\exp\left(u_1^2+u_2^2\right)+\lambda_2 = 0,\ \frac{\partial H}{\partial u_2} = u_2\exp\left(u_1^2+u_2^2\right)+\lambda_4 = 0.
\end{aligned} \tag{63}$$

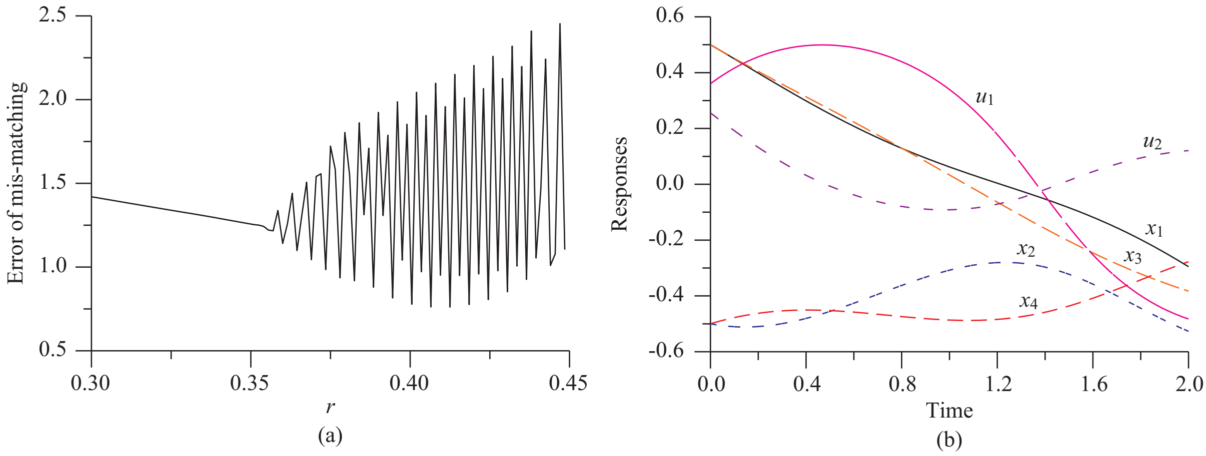

Fig. 9. For the optimal control of two coupled Duffing equations under a complex performance index of example 8, (a) the error of mis-matching, and (b) the time histories of responses and control forces.

We solve the unknown initial values of $\boldsymbol{\lambda}^0 = (\lambda_1(t_0), \lambda_2(t_0), \lambda_3(t_0), \lambda_4(t_0))^T$ by Eq. (43), and then the resulting DAEs by the LGDAE based on the Lie-group $GL(8, \mathbf{R})$.

Under the following parameters $\alpha_1 = 1$, $\beta_1 = 0.2$, $\alpha_2 = 2$, $\beta_2 = 0.3$, $t_0 = 0$ and $t_f = 2$, we plot the error of mis-matching in Fig. 9(a), and the responses of x_i, $i = 1, \ldots, 4$ and the control force u_1 and u_2 are plotted in Fig. 9(b). The computed results are $\lambda_1(t_0) = 1.32828$, $\lambda_2(t_0) = -0.437784$, $\lambda_3(t_0) = -0.2447005$, $\lambda_4(t_0) = -0.3104954$, and $J = 1.65577$.

V. CONCLUSIONS

For an optimally controlled problem of nonlinear Duffing oscillator to find the optimal control forces, we have transformed the Hamiltonian equations into a two-point boundary value problem equipped with constraint. The corresponding $SL(n, \mathbf{R})$ and $GL(n, \mathbf{R})$ shooting methods, as well as a Lie-group differential algebraic equation (LGDAE) method have been developed to numerically find the optimal control force. The present method can handle the minimization problem with a complex performance index, where the control force can be solved accurately. Numerical examples disclosed that the new method can obtain a better value of the performance index than other methods.

APPENDIX

In this Appendix we derive the state transition matrix **G** corresponding to **A** given in Eq. (17). Upon letting

$$a = \alpha + \beta\left(\bar{x}_1^k\right)^2, \tag{A1}$$

$$b = \alpha + 3\beta\left(\bar{x}_1^k\right)^2, \tag{A2}$$

we can obtain the following ODEs system:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -az_1 - z_4, \\ \dot{z}_3 &= bz_4, \\ \dot{z}_4 &= -z_3. \end{aligned} \tag{A3}$$

Through some derivations we can obtain

$$\frac{d^4 z_1}{dt^4} + (a+b)\ddot{z}_1 + abz_1 = 0, \tag{A4}$$

whose general solution is

$$z_1 = k_1 \cos\sqrt{a}t + k_2 \sin\sqrt{a}t + k_3 \cos\sqrt{b}t + k_4 \sin\sqrt{b}t. \tag{A5}$$

Similarly, we can derive

$$z_2 = -\sqrt{a}k_1 \sin\sqrt{a}t + \sqrt{a}k_2 \cos\sqrt{a}t - \sqrt{b}k_3 \sin\sqrt{b}t + \sqrt{b}k_4 \cos\sqrt{b}t, \tag{A6}$$

$$z_3 = (b-a)\sqrt{b}k_3 \sin\sqrt{b}t - (b-a)\sqrt{b}k_4 \cos\sqrt{b}t, \tag{A7}$$

$$z_4 = (b-a)k_3 \cos\sqrt{b}t + (b-a)k_4 \sin\sqrt{b}t. \tag{A8}$$

In terms of the following matrix:

$$\mathbf{H}(t) = \begin{bmatrix} \cos\sqrt{a}t & \sin\sqrt{a}t & \cos\sqrt{b}t & \sin\sqrt{b}t \\ -\sqrt{a}\sin\sqrt{a}t & \sqrt{a}\cos\sqrt{a}t & -\sqrt{b}\sin\sqrt{b}t & \sqrt{b}\cos\sqrt{b}t \\ 0 & 0 & (b-a)\sqrt{b}\sin\sqrt{b}t & (a-b)\sqrt{b}\cos\sqrt{b}t \\ 0 & 0 & (b-a)\cos\sqrt{b}t & (b-a)\sin\sqrt{b}t \end{bmatrix}, \tag{A9}$$

Eqs. (A5)-(A8) can be written as

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} = \mathbf{H}(t) \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \tag{A10}$$

Finally, the state transition matrix can be obtained as

$$\mathbf{G}(t) = \mathbf{H}(t)\mathbf{H}^{-1}(0)$$

$$= \begin{bmatrix} \cos\sqrt{at} & \frac{\sin\sqrt{at}}{\sqrt{a}} & \frac{\sin\sqrt{at}}{(b-a)\sqrt{a}} + \frac{\sin\sqrt{bt}}{(a-b)\sqrt{b}} & \frac{\cos\sqrt{at}}{(a-b)} + \frac{\cos\sqrt{bt}}{(b-a)} \\ -\sqrt{a}\sin\sqrt{at} & \cos\sqrt{at} & \frac{\cos\sqrt{at}}{(b-a)} + \frac{\cos\sqrt{bt}}{(a-b)} & \frac{\sqrt{a}\sin\sqrt{at}}{(b-a)} + \frac{\sqrt{b}\sin\sqrt{bt}}{(a-b)} \\ 0 & 0 & \cos\sqrt{bt} & \sqrt{b}\sin\sqrt{bt} \\ 0 & 0 & -\frac{\sin\sqrt{bt}}{\sqrt{b}} & \cos\sqrt{bt} \end{bmatrix} \tag{A11}$$

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