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ROBUST CONTROL OF DISCRETE-TIME UNCERTAIN STOCHASTIC SYSTEMS SUBJECT TO MIXED H_2/*PASSIVITY* PERFORMANCE

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Key words: mixed performance, stochastic systems, passivity theory, H_2 scheme, extended LMI.

ABSTRACT

This paper investigates a mixed H_2/*Passivity* performance control problem of uncertain stochastic systems. Based on Itô stochastic equation, the considered system is described by a linear difference equation with multiplicative noise term. To minimize output energy and guarantee asymptotical stability, the H_2 scheme is employed. Moreover, passivity theory is applied to constrain the effect of external disturbance on the system. According to the passivity theory, a general and flexible mixed performance controller design method is proposed. Based on Lyapunov function, some sufficient conditions are derived into extended Linear Matrix Inequality (LMI) form which reduces conservatism of finding the feasible solutions. Furthermore, the derived conditions can be directly solved by convex optimization algorithm to establish a controller such that asymptotical stability and mixed H_2/*Passivity* performance of the uncertain stochastic system are achieved. At last, an inverted pendulum system is used to show effectiveness and applicability of the proposed method.

I. INTRODUCTION

In most real environment, many researches for stochastic systems have been received much attention (Øksendal 2003; Eli et al., 2005) during the past two decades. Generally, stochastic behaviors were described as traditional additive noise which is also called as external disturbance effect. Different from the traditional additive noise, multiplicative noise is more practical because it allows statistical description of noise to be unknown. Therefore, the stochastic systems were represented by Itô stochastic equation (Ghaoui 1995; Rami et al., 2001; Jiao et al., 2006; Ku and Chen, 2018) that is structured as normal difference equation with the multiplicative noise term. Based on the Itô stochastic equation, many stability criteria for the multiplicative noised system have been discussed and investigated. Thus, Riccati equation (Willems 1971), LMI (Boyd et al., 1994), fuzzy control (Chang and Shing, 2004) and observer design approaches (Chang et al., 2011) for deterministic systems have been extended to discuss stability of the multiplicative noised systems. In addition, according to realist property of multiplicative noise in engineering field, control performance of the stochastic systems has become an important issue. In this paper, a robust stabilization problem of the multiplicative noised systems is discussed to achieve the required mixed performance.

In the existing mixed performance schemes (Khargonekar and Rotea, 1991; Kim 2001; Qiu 2008; Shi and Yu, 2011; Mao et al., 2012; Zhang et al., 2015; Li et al., 2016; Ku and Chen, 2017), both H_2 and H_∞ control technologies were used such that output energy is minimized and external disturbance effect is constrained at the same time. For constraining the external disturbance, the passivity theory provides more general and flexible approach than H_∞ scheme. Referring to some literature (Xie et al., 1998; Li et al., 2000; Lozano et al., 2000; Ma and Chen, 2006; Ku et al., 2010), through setting power supply rate function, the passivity theory covers some performance indices, such as H_∞ performance, positive realness and passivity constraints. On the other hand, the passivity theory has been extensively applied to several engineering problems according to the practical application as that various systems need to be passive to elevate disturbance. Thus, a general mixed performance is proposed by substituting the passivity theory for H_∞ scheme in this paper.

Concluding the above motivations, the robust stability issue of the uncertain stochastic systems is investigated subject to mixed H_2/*Passivity* performance in this paper. For practical control problem, uncertainty, external disturbance and stochastic behavior are considered, simultaneously. Under the existence of external disturbance, the proposed design method is to ensure that effect of external disturbance is constrained by the passivity theory and output energy is minimized by H_2 scheme (Peres and Geromel, 1993; Du 2006). On the other hand, the asymptotical stability of the system without external disturbance can be guaranteed by the proposed design method. For achieving the purposes, some sufficient conditions are derived by the Lyapunov function

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based on mean-square calculation (Troch 1998; Lazar 2009). Furthermore, those conditions are expressed as the extended LMI form which is less conservative than the standard LMI form (Pipeleers et al., 2009) for using the convex optimization algorithm (Boyd and Vandenberighe, 2004). According to the above illustrations, the advantages of this paper can be listed as follows:

(1) A more general robust stability criterion than the related work (Khargonekar and Retea, 1991; Zhang et al., 2015; Li et al., 2016) that is developed according to the consideration of stochastic behaviors.
(2) The existing mixed H_2/H_∞ performance controller design method can be regarded as a special case of the proposed method.
(3) The derived sufficient conditions are converted into the extended LMI form (Feng et al., 2010) to increase a relaxation of the proposed design method.

Via solving the sufficient conditions, a controller can be designed such that asymptotical stability and mixed H_2/*Passivity* performance of uncertain stochastic system are achieved. Finally, a control problem of inverted pendulum system (Gurumoorthy and Sanders, 1993; Iordanou and Surgenor, 1997; Mori et al., 1997; Chang et al., 2010) is discussed to demonstrate the effectiveness and usefulness of the proposed design method.

The paper is organized as follows: In Section II, a discrete-time uncertain stochastic linear system is introduced, and its problem formulation is also proposed. The mixed H_2/*Passivity* design technique for the system is developed in Section III. In Section IV, a numerical simulation is provided. Finally, some conclusions are given in Section V.

II. SYSTEM DESCRIPTIONS AND PROBLEM STATEMENTS

In this paper, an uncertain stochastic system is described as follows:

$$\begin{aligned}\boldsymbol{x}(k+1)&=(\boldsymbol{A}+\Delta\boldsymbol{A}(k))\boldsymbol{x}(k)+(\boldsymbol{B}+\Delta\boldsymbol{B}(k))\boldsymbol{u}(k)+\boldsymbol{E}\boldsymbol{v}(k)\\&\quad+\left((\bar{\boldsymbol{A}}+\Delta\bar{\boldsymbol{A}}(k))\boldsymbol{x}(k)+(\bar{\boldsymbol{B}}+\Delta\bar{\boldsymbol{B}}(k))\boldsymbol{u}(k)+\bar{\boldsymbol{E}}\boldsymbol{v}(k)\right)\boldsymbol{w}(k)\end{aligned} \tag{1a}$$

$$\boldsymbol{y}(k)=\boldsymbol{C}_1\boldsymbol{x}(k)+\boldsymbol{D}_1\boldsymbol{v}(k) \tag{1b}$$

$$\boldsymbol{z}(k)=\boldsymbol{C}_2\boldsymbol{x}(k)+\boldsymbol{D}_2\boldsymbol{u}(k) \tag{1c}$$

where $\boldsymbol{x}(k)\in R^n$ is the state vector, $\boldsymbol{y}(k)\in R^m$ is the measured output vector, $\boldsymbol{z}(k)\in R^p$ is the controlled output vector, $\boldsymbol{u}(k)\in R^q$ is the control input vector, $\boldsymbol{v}(k)\in R^m$ is the external disturbance input vector, and $\boldsymbol{w}(k)$ is a scalar discrete type Brownian motion which is defined on a compete probability space (Karatzas and Shreve, 1991). According to Karatzas and Shreve (1991), the properties as $E\{\boldsymbol{w}(k)\} = 0$, $E\{\boldsymbol{x}(k)\boldsymbol{w}(k)\} = 0$ and $E\{\boldsymbol{w}(k)\boldsymbol{w}(k)\} = 1$ can be found. $\boldsymbol{A}$, $\bar{\boldsymbol{A}}$, $\boldsymbol{B}$, $\bar{\boldsymbol{B}}$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{E}$ and $\bar{\boldsymbol{E}}$ are the known constant matrices with compatible dimensions. $\Delta\boldsymbol{A}(k)$, $\Delta\bar{\boldsymbol{A}}(k)$, $\Delta\boldsymbol{B}(k)$ and $\Delta\bar{\boldsymbol{B}}(k)$ represent the parameter uncertainties and satisfy the following combinations.

$$\begin{bmatrix}\Delta\boldsymbol{A}(k) & \Delta\boldsymbol{B}(k)\end{bmatrix}=\boldsymbol{M}_1\boldsymbol{Q}_1(k)\begin{bmatrix}\boldsymbol{N}_a & \boldsymbol{N}_b\end{bmatrix} \tag{2a}$$

and

$$\begin{bmatrix}\Delta\bar{\boldsymbol{A}}(k) & \Delta\bar{\boldsymbol{B}}(k)\end{bmatrix}=\boldsymbol{M}_2\boldsymbol{Q}_2(k)\begin{bmatrix}\bar{\boldsymbol{N}}_a & \bar{\boldsymbol{N}}_b\end{bmatrix} \tag{2b}$$

where $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{N}_a$, $\boldsymbol{N}_b$, $\bar{\boldsymbol{N}}_a$ and $\bar{\boldsymbol{N}}_b$ are the known constant matrices. $\boldsymbol{Q}_1(k)$ and $\boldsymbol{Q}_2(k)$ are the time-varying functions satisfying $\boldsymbol{Q}_1^{\mathrm{T}}(k)\boldsymbol{Q}_1(k)\le\boldsymbol{I}$ and $\boldsymbol{Q}_2^{\mathrm{T}}(k)\boldsymbol{Q}_2(k)\le\boldsymbol{I}$. In which, $\boldsymbol{I}$ denotes an identity matrix.

For the stabilization problem of (1), the following state feedback controller is proposed.

$$\boldsymbol{u}(k)=\boldsymbol{F}\boldsymbol{x}(k) \tag{3}$$

where $\boldsymbol{F}\in \mathrm{R}^{q\times n}$ is the feedback gain matrix.

Substituting (3) into (1), the corresponding closed-loop system is inferred.

$$\begin{aligned}\boldsymbol{x}(k+1)&=\left(\boldsymbol{A}_f+\boldsymbol{M}_1\boldsymbol{Q}_1(k)\boldsymbol{N}_1\right)\boldsymbol{x}(k)+\boldsymbol{E}\boldsymbol{v}(k)\\&\quad+\left(\left(\bar{\boldsymbol{A}}_f+\boldsymbol{M}_2\boldsymbol{Q}_2(k)\boldsymbol{N}_2\right)\boldsymbol{x}(k)+\bar{\boldsymbol{E}}\boldsymbol{v}(k)\right)\boldsymbol{w}(k)\end{aligned} \tag{4a}$$

$$\boldsymbol{y}(k)=\boldsymbol{C}_1\boldsymbol{x}(k)+\boldsymbol{D}_1\boldsymbol{v}(k) \tag{4b}$$

$$\boldsymbol{z}(k)=\boldsymbol{C}_{2f}\boldsymbol{x}(k) \tag{4c}$$

where

$$\boldsymbol{A}_f=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{F},\ \bar{\boldsymbol{A}}_f=\bar{\boldsymbol{A}}+\bar{\boldsymbol{B}}\boldsymbol{F},\ \boldsymbol{N}_1=\boldsymbol{N}_a+\boldsymbol{N}_b\boldsymbol{F},$$

$$\boldsymbol{N}_2=\bar{\boldsymbol{N}}_a+\bar{\boldsymbol{N}}_b\boldsymbol{F}\ \text{and}\ \boldsymbol{C}_{2f}=\boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}.$$

For the considered mixed H_2/*Passivity* performance, the following definitions and lemma are provided. Based on the energy concept (Li et al., 2005), the passivity theory provides an useful and effective scheme to constrain the disturbance effect on system. Moreover, the power supply rate function (Lozano et al., 2000) can be described in the following definition.

Definition 1 (Lozano et al., 2000): If there exist the matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2>0$ and $\boldsymbol{S}_3$ to satisfy the following inequality, the closed-loop system (4) with external disturbance $\boldsymbol{v}(k)$ and measured output $\boldsymbol{y}(k)$ is called passive.

$$E\left\{2\sum_{k=0}^{k_p}\boldsymbol{y}^T(k)\boldsymbol{S}_1\boldsymbol{v}(k)\right\} > E\left\{\sum_{k=0}^{k_p}\boldsymbol{y}^T(k)\boldsymbol{S}_2\boldsymbol{y}(k)+\sum_{k=0}^{k_p}\boldsymbol{v}^T(k)\boldsymbol{S}_3\boldsymbol{v}(k)\right\} \quad (5)$$

for any terminal time $k_p > 0$ and $\boldsymbol{v}(k) \neq 0$. #

Without loss of generality, the inequality (5) can be reduced into several performance constraints by setting matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$. However, the same external disturbance in (1a) and (1b) is required to discuss the passivity performance. In this paper, the generalized power supply function (5) is proposed to be the constraint index.

For the case as zero disturbance $(\boldsymbol{v}(k)=0)$, the following definition for H_2 performance is applied to minimize the output energy and guarantee the asymptotical stability of (4).

Definition 2 (Kim 2001): The H_2 performance of system (4) can be ensured by satisfying the following inequality.

$$E\left\{\sum_{k=0}^{T_f}\boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k)\right\} < \alpha \quad (6)$$

where $T_f > 0$ is the terminal time of the control. The value of α is used to minimize output energy. In this paper, the asymptotical stability of the system (4) is required for achieving (6). #

The following lemma is introduced to deal with uncertainty in system (4).

Lemma 1 (Wang et al., 1992): Given real appropriate dimensions matrices $\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{\beta}$ and $\boldsymbol{S}$, and time-varying function $\boldsymbol{F}(k)$ satisfying $\boldsymbol{F}^{\mathrm{T}}(k)\boldsymbol{F}(k) \leq \boldsymbol{I}$, one has the following inequality for any $\boldsymbol{\beta} > 0$ and scalar $\varepsilon > 0$ such that $\boldsymbol{\beta} - \varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} > 0$.

$$\begin{aligned}&\left(\boldsymbol{A}+\boldsymbol{D}\boldsymbol{F}(k)\boldsymbol{S}\right)^{\mathrm{T}}\boldsymbol{\beta}^{-1}\left(\boldsymbol{A}+\boldsymbol{D}\boldsymbol{F}(k)\boldsymbol{S}\right)\\ &\leq \boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{\beta}-\varepsilon\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}\right)^{-1}\boldsymbol{A}+\varepsilon^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}\end{aligned} \quad (7)$$

#

Employing the above definitions and lemma, the proposed controller design method is developed to guarantee the asymptotical stability of the closed-loop system (4) subject to mixed H_2/*Passivity* performance.

III. MIXED H_2/*PASSIVITY* PERFORMANCE CONTROLLER DESIGN METHOD

In this section, some sufficient conditions are derived via the Lyapunov function. In order to apply the convex optimization algorithm, the derived conditions are converted into the extended LMI form. And then, the controller (3) can be established through solving those sufficient conditions.

Theorem 1: Given matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2 > 0$ and $\boldsymbol{S}_3$ satisfying $\boldsymbol{S}_3 - \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1 - \boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 + \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{D}_1 < 0$, asymptotical stability and mixed H_2/*Passivity* performance of the closed-loop system (4) are guaranteed if there exist positive definite matrix $\boldsymbol{P}$, feedback gain matrix $\boldsymbol{F}$, and positive scalars α, ε_1 and ε_2 to satisfy the following conditions.

$$\boldsymbol{\Xi} + \begin{bmatrix} \boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{C}_1 & * \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1 + \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{C}_1 & \boldsymbol{S}_3 - \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1 - \boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 + \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{D}_1 \end{bmatrix} < 0 \quad (8)$$

$$\boldsymbol{A}_f^{\mathrm{T}}\boldsymbol{R}_1^{-1}\boldsymbol{A}_f + \overline{\boldsymbol{A}}_f^{\mathrm{T}}\boldsymbol{R}_2^{-1}\overline{\boldsymbol{A}}_f - \boldsymbol{P} + \varepsilon_1^{-1}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \varepsilon_2^{-1}\boldsymbol{N}_2^{\mathrm{T}}\boldsymbol{N}_2 + \boldsymbol{C}_{2f}^{\mathrm{T}}\boldsymbol{C}_{2f} < 0 \quad (9)$$

$$x^{\mathrm{T}}(0)\boldsymbol{P}x(0) - \alpha < 0 \quad (10)$$

where

$$\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{A}_f^{\mathrm{T}}\boldsymbol{R}_1^{-1}\boldsymbol{A}_f + \overline{\boldsymbol{A}}_f^{\mathrm{T}}\boldsymbol{R}_2^{-1}\overline{\boldsymbol{A}}_f - \boldsymbol{P} + \varepsilon_1^{-1}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \varepsilon_2^{-1}\boldsymbol{N}_2^{\mathrm{T}}\boldsymbol{N}_2 & * \\ \boldsymbol{A}_f^{\mathrm{T}}\boldsymbol{R}_1^{-1}\mathrm{E} + \overline{\boldsymbol{A}}_f^{\mathrm{T}}\boldsymbol{R}_2^{-1}\overline{\mathrm{E}} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{R}_1^{-1}\boldsymbol{E} + \overline{\boldsymbol{E}}^{\mathrm{T}}\boldsymbol{R}_2^{-1}\overline{\boldsymbol{E}} \end{bmatrix},$$

$$\boldsymbol{R}_1 = \boldsymbol{P}^{-1} - \varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}},$$
$$\boldsymbol{R}_2 = \boldsymbol{P}^{-1} - \varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}}$$

and * denotes the transposed elements or matrices for symmetric position.

Proof:

Choose the following Lyapunov function.

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{x}(k) \quad (11)$$

Taking first forward difference and expectation of (11), one has

$$E\{\Delta V(\boldsymbol{x}(k))\} = E\{\boldsymbol{x}^{\mathrm{T}}(k+1)\boldsymbol{P}\boldsymbol{x}(k+1) - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{x}(k)\} \quad (12)$$

Based on the independent increment property of Brownian motion, the following equation can be obtained by substituting (4a) into (12).

$$\begin{aligned}E\{\Delta V(\boldsymbol{x}(k))\} = E\Big\{&\tilde{\boldsymbol{x}}^{\mathrm{T}}(k)\Big(\left(\boldsymbol{O}_1 + \boldsymbol{M}_1\boldsymbol{Q}_1(k)\boldsymbol{J}_1\right)^{\mathrm{T}}\boldsymbol{P}\\ &\times\left(\boldsymbol{O}_1 + \boldsymbol{M}_1\boldsymbol{Q}_1(k)\boldsymbol{J}_1\right) + \left(\boldsymbol{O}_2 + \boldsymbol{M}_2\boldsymbol{Q}_2(k)\boldsymbol{J}_2\right)^{\mathrm{T}}\\ &\times\boldsymbol{P}\left(\boldsymbol{O}_2 + \boldsymbol{M}_2\boldsymbol{Q}_2(k)\boldsymbol{J}_2\right)\Big)\tilde{\boldsymbol{x}}(k) - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{x}(k)\Big\}\end{aligned} \quad (13)$$

where

$$\boldsymbol{O}_1 = \left[\boldsymbol{A}_f \quad \boldsymbol{E}\right], \boldsymbol{O}_2 = \left[\bar{\boldsymbol{A}}_f \quad \bar{\boldsymbol{E}}\right],$$
$$\boldsymbol{J}_1 = [\boldsymbol{N}_1 \quad 0], \boldsymbol{J}_2 = [\boldsymbol{N}_2 \quad 0]$$

and

$$\tilde{\boldsymbol{x}}(k) = [\boldsymbol{x}^{\mathrm{T}}(k) \quad \boldsymbol{v}^{\mathrm{T}}(k)]^{\mathrm{T}}.$$

Applying Lemma 1, the following inequality can be found.

$$\begin{aligned}&\left(\boldsymbol{O}_1 + \boldsymbol{M}_1\boldsymbol{Q}_1(k)\boldsymbol{J}_1\right)^{\mathrm{T}} \boldsymbol{P}\left(\boldsymbol{O}_1 + \boldsymbol{M}_1\boldsymbol{Q}_1(k)\boldsymbol{J}_1\right)\\ &\leq \boldsymbol{O}_1^{\mathrm{T}}\left(\boldsymbol{P}^{-1} - \varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}}\right)^{-1}\boldsymbol{O}_1 + \varepsilon_1^{-1}\boldsymbol{J}_1^{\mathrm{T}}\boldsymbol{J}_1\end{aligned} \quad (14a)$$

and

$$\begin{aligned}&\left(\boldsymbol{O}_2 + \boldsymbol{M}_2\boldsymbol{Q}_2(k)\boldsymbol{J}_2\right)^{\mathrm{T}} \boldsymbol{P}\left(\boldsymbol{O}_2 + \boldsymbol{M}_2\boldsymbol{Q}_2(k)\boldsymbol{J}_2\right)\\ &\leq \boldsymbol{O}_2^{\mathrm{T}}\left(\boldsymbol{P}^{-1} - \varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}}\right)^{-1}\boldsymbol{O}_2 + \varepsilon_2^{-1}\boldsymbol{J}_2^{\mathrm{T}}\boldsymbol{J}_2\end{aligned} \quad (14b)$$

According to (14), one can find the following relation.

$$\begin{aligned}&E\left\{\Delta V\left(\boldsymbol{x}(k)\right)\right\}\\ &\leq E\left\{\tilde{x}^{\mathrm{T}}(k)\left(\boldsymbol{O}_1^{\mathrm{T}}\left(\boldsymbol{P}^{-1} - \varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}}\right)^{-1}\boldsymbol{O}_1 + \boldsymbol{O}_2^{\mathrm{T}}\left(\boldsymbol{P}^{-1} - \varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}}\right)^{-1}\right.\right.\\ &\left.\left.\times\boldsymbol{O}_2 + \varepsilon_1^{-1}\boldsymbol{J}_1^{\mathrm{T}}\boldsymbol{J}_1 + \varepsilon_2^{-1}\boldsymbol{J}_2^{\mathrm{T}}\boldsymbol{J}_2\right)\tilde{\boldsymbol{x}}(k) - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}\boldsymbol{x}(k)\right\}\\ &= E\left\{\tilde{\boldsymbol{x}}^{\mathrm{T}}(k)\boldsymbol{\Xi}\tilde{\boldsymbol{x}}(k)\right\}\end{aligned} \quad (15)$$

For nonzero external disturbance, let us define the following cost function with zero initial condition.

$$\begin{aligned}&\Gamma(x, v, k)\\ &= E\left\{\sum_{k=0}^{k_p}\left|\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{y}(k) + \boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{v}(k) - 2\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_1\boldsymbol{v}(k)\right|\right\}\\ &= E\left\{\sum_{k=0}^{k_p}\left(\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{y}(k) + \boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{v}(k) - 2\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_1\boldsymbol{v}(k)\right.\right.\\ &\left.\left.+\Delta V\left(\boldsymbol{x}(k)\right)\right) - V\left(\boldsymbol{x}\left(k_p + 1\right)\right)\right\}\\ &\leq E\left\{\sum_{k=0}^{k_p}\Psi(x, v, k)\right\}\end{aligned} \quad (16)$$

where

$$\begin{aligned}\Psi(x, \in v, k) = \boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{y}(k) + \boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{v}(k)\\ -2\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_1\boldsymbol{v}(k) + \Delta V\left(\boldsymbol{x}(k)\right)\end{aligned}. \quad (17)$$

Introducing (4b) and (15) into (17), one has

$$\Psi(x, v, k) = \tilde{\boldsymbol{x}}^{\mathrm{T}}(k)\boldsymbol{\Lambda}\tilde{\boldsymbol{x}}(k) \quad (18)$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{\Xi} + \begin{bmatrix} \boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{C}_1 & * \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1 + \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{C}_1 & \boldsymbol{S}_3 - \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1 - \boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 + \boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{D}_1 \end{bmatrix}. \quad (19)$$

Obviously, the matrix $\boldsymbol{\Lambda}$ is equal to the left-hand side of condition (8). Hence, if condition (8) holds, then one can find $\boldsymbol{\Lambda} < 0$ and $E\{\Psi(x, v, k)\} < 0$. From (16), $\Gamma(x, v, k) < 0$ can also be found due to $E\{\Psi(x, v, k)\} < 0$. It means that

$$\begin{aligned}&E\left\{2\sum_{k=0}^{k_p}\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_1\boldsymbol{v}(k)\right\}\\ &> E\left\{\sum_{k=0}^{k_p}\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{y}(k) + \sum_{k=0}^{k_p}\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{v}(k)\right\}\end{aligned}. \quad (20)$$

Because (20) is equal to (5) in Definition 1, one can conclude that the closed-loop system (4) is passive for all nonzero external disturbance. Next, assuming $\boldsymbol{v}(k) = 0$, the following equation can be directly inferred from (15).

$$\begin{aligned}&E\left\{\Delta V\left(\boldsymbol{x}(k)\right)\right\}\\ &\leq E\left\{\boldsymbol{x}^{\mathrm{T}}(k)\left(\boldsymbol{A}_f^{\mathrm{T}}\boldsymbol{R}_1^{-1}\boldsymbol{A}_f + \bar{\boldsymbol{A}}_f^{\mathrm{T}}\boldsymbol{R}_2^{-1}\bar{\boldsymbol{A}}_f - \boldsymbol{P} + \varepsilon_1^{-1}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \varepsilon_2^{-1}\boldsymbol{N}_2^{\mathrm{T}}\boldsymbol{N}_2\right)\boldsymbol{x}(k)\right\}\end{aligned} \quad (21)$$

Besides, the following inequality is inferred by pre- and post-multiplying (9) by $x^T(k)$ and $x(k)$, respectively.

$$x^{\mathrm{T}}(k)\left(\boldsymbol{A}_f^{\mathrm{T}}\boldsymbol{R}_1^{-1}\boldsymbol{A}_f + \bar{\boldsymbol{A}}_f^{\mathrm{T}}\boldsymbol{R}_2^{-1}\bar{\boldsymbol{A}}_f - \boldsymbol{P} + \varepsilon_1^{-1}\boldsymbol{N}_1^{\mathrm{T}}\boldsymbol{N}_1 + \varepsilon_2^{-1}\boldsymbol{N}_2^{\mathrm{T}}\boldsymbol{N}_2 + \boldsymbol{C}_{2f}^{\mathrm{T}}\boldsymbol{C}_{2f}\right)x(k) < 0 \quad (22)$$

Based on (21) and (4c), the following inequalities can be obtained by taking the expectation of (22).

$$E\left\{\Delta V\left(\boldsymbol{x}(k)\right) + \boldsymbol{z}^T(k)\boldsymbol{z}(k)\right\} < 0 \quad (23)$$

or

$$E\left\{\Delta V\left(\boldsymbol{x}(k)\right)\right\} < E\left\{-\boldsymbol{z}^T(k)\boldsymbol{z}(k)\right\} \quad (24)$$

From (24), $E\left\{\Delta V\left(\boldsymbol{x}(k)\right)\right\} < 0$ can be found according to $\boldsymbol{z}^T(k)\boldsymbol{z}(k) \geq 0$. It is easy to find that if (9) holds then the closed-loop system is asymptotically stable due to $E\left\{\Delta V\left(\boldsymbol{x}(k)\right)\right\} < 0$. Summarizing (24) from 0 to T_f, one has the following relation.

$$E\left\{\boldsymbol{x}^{\mathrm{T}}\left(T_f\right)\boldsymbol{P}\boldsymbol{x}\left(T_f\right)-\boldsymbol{x}^{\mathrm{T}}\left(0\right)\boldsymbol{P}\boldsymbol{x}\left(0\right)\right\}<E\left\{-\sum_{k=0}^{T_f}\boldsymbol{z}^{\mathrm{T}}\left(k\right)\boldsymbol{z}\left(k\right)\right\} \tag{25}$$

The following inequality is easily found from (25) according to $\boldsymbol{P} > 0$.

$$E\left\{\sum_{k=0}^{T_f}\boldsymbol{z}^{\mathrm{T}}\left(k\right)\boldsymbol{z}\left(k\right)\right\}<E\left\{\boldsymbol{x}^{\mathrm{T}}\left(0\right)\boldsymbol{P}\boldsymbol{x}\left(0\right)\right\} \tag{26}$$

By holding (9), one can find that $\boldsymbol{x}^{\mathrm{T}}\left(T_f\right)\boldsymbol{P}\boldsymbol{x}\left(T_f\right)\to 0$ as $T_f \to \infty$ because the asymptotical stability of the closed-loop system (4) is achieved. Furthermore, if (10) holds, $\boldsymbol{x}^{\mathrm{T}}\left(0\right)\boldsymbol{P}\boldsymbol{x}\left(0\right)\le \alpha$ is confirmed. From (26), the following inequality is obtained by holding (9) and (10) in the case $T_f \to \infty$.

$$E\left\{\sum_{k=0}^{T_f}\boldsymbol{z}^{\mathrm{T}}\left(k\right)\boldsymbol{z}\left(k\right)\right\}<\alpha \tag{27}$$

Because (27) is equal to (6), the upper bound of output energy is constrained by α. Thus, under the given initial condition, $\boldsymbol{P}$ and α are needed to be found to satisfy (10) such that the upper bound is minimized as well as H_2 performance is achieved. The proof of this theorem is complete. #

Theorem 1 provides some sufficient conditions to guarantee the asymptotical stability and mixed H_2/*Passivity* performance. However, the convex optimization algorithm cannot be applied to solve the conditions in Theorem 1. For applying the convex optimization algorithm, it is necessary to convert those conditions into the extended LMI form. In next theorem, the extended LMI stability conditions are proposed to find the feasible solutions to satisfy the sufficient conditions in Theorem 1.

Theorem 2: Given matrices $\boldsymbol{S}_1$, $\boldsymbol{S}_2 > 0$ and $\boldsymbol{S}_3$ satisfying $\boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1+\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_2\boldsymbol{D}_1<0$, if there exist positive definite matrix $\boldsymbol{P}$, feedback gain matrix $\boldsymbol{F}$, any matrix $\boldsymbol{G}$, and positive scalars α, ε_1 and ε_2 to satisfy the following conditions, asymptotically stability and mixed H_2/*Passivity* performance of the closed-loop system (4) are guaranteed.

$$\begin{bmatrix} \boldsymbol{X}-\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{G} & * & * & * & * & 0 & 0 \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 & * & * & * & 0 & 0 \\ \boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{D}_1 & -\boldsymbol{S}_2^{-1} & * & * & 0 & 0 \\ \boldsymbol{N}_a\boldsymbol{G}+\boldsymbol{N}_b\boldsymbol{K} & 0 & 0 & -\varepsilon_1\boldsymbol{I} & * & 0 & 0 \\ \bar{\boldsymbol{N}}_a\boldsymbol{G}+\bar{\boldsymbol{N}}_b\boldsymbol{K} & 0 & 0 & \boldsymbol{0} & -\varepsilon_2\boldsymbol{I} & 0 & 0 \\ \boldsymbol{A}\boldsymbol{G}+\boldsymbol{B}\boldsymbol{K} & \boldsymbol{E} & 0 & \boldsymbol{0} & 0 & -\boldsymbol{X}+\varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}} & 0 \\ \bar{\boldsymbol{A}}\boldsymbol{G}+\bar{\boldsymbol{B}}\boldsymbol{K} & \bar{\boldsymbol{E}} & 0 & \boldsymbol{0} & 0 & * & -\boldsymbol{X}+\varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}} \end{bmatrix}<0 \tag{28}$$

$$\begin{bmatrix} \boldsymbol{X}-\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{G} & * & * & * & * & * \\ \boldsymbol{C}_2\boldsymbol{G}+\boldsymbol{D}_2\boldsymbol{K} & -\boldsymbol{I} & * & * & * & * \\ \boldsymbol{N}_a\boldsymbol{G}+\boldsymbol{N}_b\boldsymbol{K} & 0 & -\varepsilon_1\boldsymbol{I} & * & * & * \\ \bar{\boldsymbol{N}}_a\boldsymbol{G}+\bar{\boldsymbol{N}}_b\boldsymbol{K} & 0 & 0 & -\varepsilon_2\boldsymbol{I} & * & * \\ \boldsymbol{A}\boldsymbol{G}+\boldsymbol{B}\boldsymbol{K} & 0 & 0 & 0 & -\boldsymbol{X}+\varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}} & * \\ \bar{\boldsymbol{A}}\boldsymbol{G}+\bar{\boldsymbol{B}}\boldsymbol{K} & 0 & 0 & 0 & 0 & -\boldsymbol{X}+\varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}} \end{bmatrix}<0 \tag{29}$$

$$\begin{bmatrix} -\alpha & * \\ \boldsymbol{x}\left(0\right) & -\boldsymbol{X} \end{bmatrix}<0 \tag{30}$$

where $\boldsymbol{X} = \boldsymbol{P}^{-1}$ and $\boldsymbol{K} = \boldsymbol{F}\boldsymbol{G}$.

Proof:

Applying Schur complement to (8), one has

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * & * & * \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1 & \boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 & * & * & * & * & * \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 & -\boldsymbol{S}_2^{-1} & * & * & * & * \\ \boldsymbol{N}_1 & 0 & 0 & -\varepsilon_1\boldsymbol{I} & * & * & * \\ \boldsymbol{N}_2 & 0 & 0 & 0 & -\varepsilon_2\boldsymbol{I} & * & * \\ \boldsymbol{A}_f & \boldsymbol{E} & 0 & 0 & 0 & -\boldsymbol{R}_1 & * \\ \bar{\boldsymbol{A}}_f & \bar{\boldsymbol{E}} & 0 & 0 & 0 & 0 & -\boldsymbol{R}_2 \end{bmatrix}<0 \tag{31}$$

Pre- and post-multiplying (31) by $diag\left\{\boldsymbol{G}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}\right\}$ and $diag\left\{\boldsymbol{G},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}\right\}$, respectively, one can obtain the following inequality.

$$\begin{bmatrix} -\boldsymbol{G}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{G} & * & * & * & * & * & * \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 & * & * & * & * & * \\ \boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{D}_1 & -\boldsymbol{S}_2^{-1} & * & * & * & * \\ \boldsymbol{N}_1\boldsymbol{G} & 0 & 0 & -\varepsilon_1\boldsymbol{I} & * & * & * \\ \boldsymbol{N}_2\boldsymbol{G} & 0 & 0 & 0 & -\varepsilon_2\boldsymbol{I} & * & * \\ \boldsymbol{A}_f\boldsymbol{G} & \boldsymbol{E} & 0 & 0 & 0 & -\boldsymbol{R}_1 & * \\ \bar{\boldsymbol{A}}_f\boldsymbol{G} & \bar{\boldsymbol{E}} & 0 & 0 & 0 & 0 & -\boldsymbol{R}_2 \end{bmatrix}<0 \tag{32}$$

According to $\boldsymbol{P} > 0$, the following facts can be found.

$$\left(\boldsymbol{P}^{-1}-\boldsymbol{G}^{\mathrm{T}}\right)^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{P}^{-1}-\boldsymbol{G}\right)\ge 0 \tag{33}$$

and

$$\boldsymbol{P}^{-1}-\boldsymbol{G}^{T}-\boldsymbol{G}\ge -\boldsymbol{G}^{T}\boldsymbol{P}\boldsymbol{G}\,. \tag{34}$$

Since (34), the following inequality is inferred from (32).

$$\begin{bmatrix} \boldsymbol{P}^{-1}-\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{G} & * & * & * & * & * & * \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 & * & * & * & * & * \\ \boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{D}_1 & -\boldsymbol{S}_2^{-1} & * & * & * & * \\ \boldsymbol{N}_1\boldsymbol{G} & 0 & 0 & -\varepsilon_1\boldsymbol{I} & * & * & * \\ \boldsymbol{N}_2\boldsymbol{G} & 0 & 0 & 0 & -\varepsilon_2\boldsymbol{I} & * & * \\ \boldsymbol{A}_f\boldsymbol{G} & \boldsymbol{E} & 0 & 0 & 0 & -\boldsymbol{R}_1 & * \\ \bar{\boldsymbol{A}}_f\boldsymbol{G} & \bar{\boldsymbol{E}} & 0 & 0 & 0 & 0 & -\boldsymbol{R}_2 \end{bmatrix} < 0 \tag{35}$$

Based on the definitions as

$$\boldsymbol{A}_f = \boldsymbol{A}+\boldsymbol{B}\boldsymbol{F}\,,\ \bar{\boldsymbol{A}}_f = \bar{\boldsymbol{A}}+\bar{\boldsymbol{B}}\boldsymbol{F}\,,\ \boldsymbol{N}_1 = \boldsymbol{N}_a+\boldsymbol{N}_b\boldsymbol{F}\,,$$

$$\boldsymbol{N}_2 = \bar{\boldsymbol{N}}_a+\bar{\boldsymbol{N}}_b\boldsymbol{F}\,,\ \boldsymbol{C}_{2f} = \boldsymbol{C}_2+\boldsymbol{D}_2\boldsymbol{F}\,,\ \boldsymbol{R}_1 = \boldsymbol{P}^{-1}-\varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}}\,,$$

$$\boldsymbol{R}_2 = \boldsymbol{P}^{-1}-\varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}}\,,\ \boldsymbol{X} = \boldsymbol{P}^{-1} \text{ and } \boldsymbol{K} = \boldsymbol{F}\boldsymbol{G}\,,$$

one has the following inequality.

$$\begin{bmatrix} \boldsymbol{X}-\boldsymbol{G}^{\mathrm{T}}-\boldsymbol{G} & * & * & * & * & 0 & 0 \\ -\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{S}_3-\boldsymbol{D}_1^{\mathrm{T}}\boldsymbol{S}_1-\boldsymbol{S}_1^{\mathrm{T}}\boldsymbol{D}_1 & * & * & * & 0 & 0 \\ \boldsymbol{C}_1\boldsymbol{G} & \boldsymbol{D}_1 & -\boldsymbol{S}_2^{-1} & * & * & 0 & 0 \\ \boldsymbol{N}_a\boldsymbol{G}+\boldsymbol{N}_b\boldsymbol{K} & 0 & 0 & -\varepsilon_1\boldsymbol{I} & * & 0 & 0 \\ \bar{\boldsymbol{N}}_a\boldsymbol{G}+\bar{\boldsymbol{N}}_b\boldsymbol{K} & 0 & 0 & 0 & -\varepsilon_2\boldsymbol{I} & 0 & 0 \\ \boldsymbol{A}\boldsymbol{G}+\boldsymbol{B}\boldsymbol{K} & \boldsymbol{E} & 0 & 0 & 0 & -\boldsymbol{X}+\varepsilon_1\boldsymbol{M}_1\boldsymbol{M}_1^{\mathrm{T}} & 0 \\ \bar{\boldsymbol{A}}\boldsymbol{G}+\bar{\boldsymbol{B}}\boldsymbol{K} & \bar{\boldsymbol{E}} & 0 & 0 & 0 & * & -\boldsymbol{X}+\varepsilon_2\boldsymbol{M}_2\boldsymbol{M}_2^{\mathrm{T}} \end{bmatrix} < 0 \tag{36}$$

Obviously, (36) is equal to (28). Hence, if condition (28) is satisfied then condition (8) is also held. Through the similar proof processes, (29) and (30) can be derived from conditions (9) and (10), respectively. Thus, the proofs of (29) and (30) are omitted here. Also, the proof of this theorem is completed. #

Remark 1: Referring to the literature (Pipeleers et al., 2009; Feng et al., 2010), the extended LMI form is more general and less conservative than the standard LMI form because an arbitrary matrix $\boldsymbol{G}$ is introduced. Moreover, the extended LMI form can be reduced as the standard LMI form via setting $\boldsymbol{G} = \boldsymbol{P}^{-1}$. For the reason, the conditions in Theorem 2 are converted into the extended LMI form. #

To demonstrate the effectiveness and applicability of the proposed design method, the control problem of inverted pendulum system is discussed in the following section subject to mixed H_2/*Passivity* performance.

IV. SIMULATION

Based on the works in Gurumoorth & Sanders (1993) and Iordanou & Surgenor (1997), the inverted pendulum system consists of an inverted pendulum on a cart system which is free to move on a horizontal plane. Applying force to the cart so that the pendulum is balanced and kept vertically upwards. For obtaining the linear model of inverted pendulum system, the sampling time is 0.01s as dictated by the speed limitations of the data acquisition board. In order to consider uncertainties, some perturbations are added in this simulation. Besides, external disturbance $\boldsymbol{v}(k)$ and multiplicative noise $\boldsymbol{w}(k)$ are added to express the practical operation condition. Hence, the discrete-time uncertain stochastic linear inverted pendulum system is described as follows:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \left(\boldsymbol{A}+\Delta\boldsymbol{A}(k)\right)\boldsymbol{x}(k)+\left(\boldsymbol{B}+\Delta\boldsymbol{B}(k)\right)\boldsymbol{u}(k)+\boldsymbol{E}\boldsymbol{v}(k) \\ &\quad +\left(\left(\bar{\boldsymbol{A}}+\Delta\bar{\boldsymbol{A}}(k)\right)\boldsymbol{x}(k)+\left(\bar{\boldsymbol{B}}+\Delta\bar{\boldsymbol{B}}(k)\right)\boldsymbol{u}(k)+\bar{\boldsymbol{E}}\boldsymbol{v}(k)\right)\boldsymbol{w}(k) \end{aligned} \tag{37a}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}_1\boldsymbol{x}(k)+\boldsymbol{D}_1\boldsymbol{v}(k) \tag{37b}$$

$$\boldsymbol{z}(k) = \boldsymbol{C}_2\boldsymbol{x}(k)+\boldsymbol{D}_2\boldsymbol{u}(k) \tag{37c}$$

where $\boldsymbol{x}(k) = \begin{bmatrix} \boldsymbol{x}_1(k) & \boldsymbol{x}_2(k) & \boldsymbol{x}_3(k) & \boldsymbol{x}_4(k) \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{x}_1(k)$ is the cart position, $\boldsymbol{x}_2(k)$ is the cart velocity, $\boldsymbol{x}_3(k)$ is payload angle, $\boldsymbol{x}_4(k)$ is payload angle velocity and $\boldsymbol{u}(k)$ is applied force and $\boldsymbol{v}(k)$ is a zero-mean white noise with 0.5 variance. The matrices in (37) are presented as

$$\boldsymbol{E} = \begin{bmatrix} 0 & 0 & 0.01 & 0 \end{bmatrix}^{\mathrm{T}},\ \bar{\boldsymbol{E}} = \begin{bmatrix} 0 & 0 & 0.001 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{C}_1 = \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},\ \boldsymbol{D}_1 = \boldsymbol{D}_2 = 1,$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.0087 & 0 & 0 \\ 0 & 0.7515 & 0 & 0 \\ 0 & -0.0111 & 1.0015 & 0.0100 \\ 0 & -2.1235 & 0.3052 & 0.9999 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0.0027 \\ 0.5219 \\ 0.0234 \\ 4.4593 \end{bmatrix},$$

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0.035 & 0 \\ 0 & 0.02 & 0 & 0.015 \end{bmatrix},\ \bar{\boldsymbol{B}} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.05 \\ 0.06 \end{bmatrix},$$

$$\Delta\boldsymbol{A} = \begin{bmatrix} 0.002 & 0.005 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0.0021 & 0.0035 & 0 \\ 0 & 0.002 & 0.004 & 0.001 \end{bmatrix} sin(k),$$

$$\Delta\boldsymbol{B} = \begin{bmatrix} 0.001 \\ 0 \\ 0.0014 \\ 0.003 \end{bmatrix} sin(k),\ \Delta\bar{\boldsymbol{B}} = \begin{bmatrix} 0.0016 \\ 0 \\ 0.0006 \\ 0.0045 \end{bmatrix} cos(k) \text{ and}$$

$$\Delta\bar{\boldsymbol{A}} = \begin{bmatrix} 0.0024 & 0.0032 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0.0006 & 0.0024 & 0 \\ 0 & 0.0009 & 0.0018 & 0.0054 \end{bmatrix} cos(k).$$

According to (2), the uncertainties in (37) can be furtherly described by the following matrices.

$$\boldsymbol{M}_1=\begin{bmatrix}0.01&0&0&0\\0&0.005&0&0\\0&0&0.007&0\\0&0&0&0.01\end{bmatrix},$$

$$\boldsymbol{M}_2=\begin{bmatrix}0.008&0&0&0\\0&0.005&0&0\\0&0&0.006&0\\0&0&0&0.009\end{bmatrix},$$

$$\boldsymbol{N}_a=\begin{bmatrix}0.2&0.5&0&0\\0&0.2&0&0\\0&0.3&0.5&0\\0&0.2&0.4&0.1\end{bmatrix},\ \boldsymbol{N}_b=\begin{bmatrix}0.1\\0\\0.2\\0.3\end{bmatrix},$$

$$\bar{\boldsymbol{N}}_a=\begin{bmatrix}0.3&0.4&0&0\\0&0.2&0&0\\0&0.1&0.4&0\\0&0.1&0.2&0.6\end{bmatrix},\ \bar{\boldsymbol{N}}_b=\begin{bmatrix}0.2\\0\\0.1\\0.5\end{bmatrix},\ \boldsymbol{Q}_1(k)=sin(k)$$

and

$$\boldsymbol{Q}_2(k)=cos(k).$$

Under the consideration of H_∞ performance which is included in the passivity performance, the constants $\boldsymbol{S}_1 = 0$, $\boldsymbol{S}_2 = 1$ and $\boldsymbol{S}_3 = -2$ are given. One can find the following feasible solutions through solving the conditions in Theorem 2 with initial condition $\boldsymbol{x}(0)=\begin{bmatrix}0.5&0&0.872&0\end{bmatrix}^{\mathrm{T}}$.

$$\boldsymbol{P}=\begin{bmatrix}21.0061&20.9955&-10.8128&-2.5511\\20.9955&89.5541&-40.3883&-10.9192\\-10.8128&-40.3883&39.2954&5.2042\\-2.5511&-10.9192&5.2042&1.3621\end{bmatrix},$$

$$\boldsymbol{G}=\begin{bmatrix}0.0626&-0.0205&0.0048&-0.0657\\-0.0206&0.5891&-0.0367&4.8473\\0.0048&-0.0350&0.0551&-0.4799\\0.3122&1.6260&-2.1088&-0.2805\end{bmatrix},$$

$$\boldsymbol{F}=\begin{bmatrix}0.3122&1.6260&-2.1088&-0.2805\end{bmatrix},$$

$$\varepsilon_1=1.6741,\ \varepsilon_2=129.3020\ \text{and}\ \alpha=27.2951$$

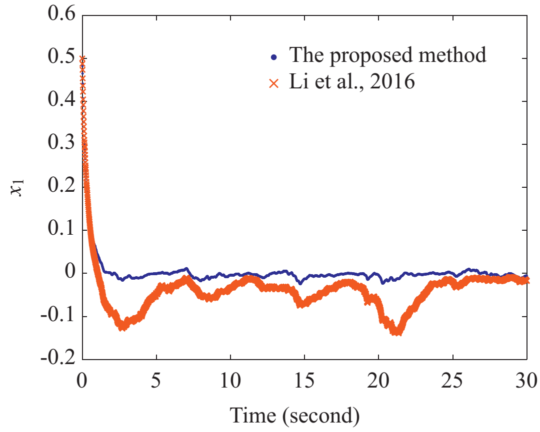

Fig. 1. Responses of state $x_1(k)$.

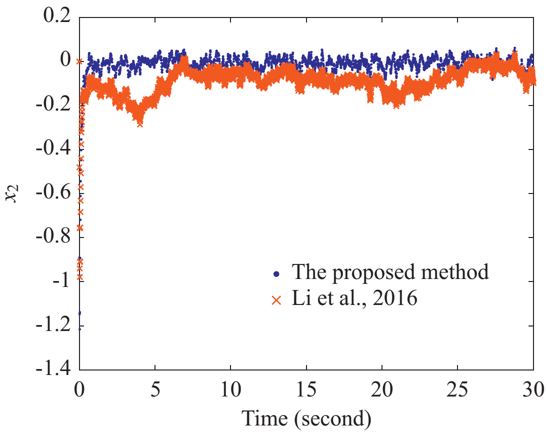

Fig. 2. Responses of state $x_2(k)$.

With the obtained feasible solutions, the following controller can be established to stabilize system (37).

$$u(t)=\boldsymbol{F}x(t) \tag{38}$$

Based on the designed controller (38), the simulation results are shown in Figs. 1-4 with the initial condition. To ensure the achievement of the performances, the following equations arranged from (5) and (6) are proposed via introducing the simulation results.

$$\frac{\sum_{k=0}^{30}\boldsymbol{y}^{\mathrm{T}}(k)\boldsymbol{S}_2\boldsymbol{y}(k)}{\sum_{k=0}^{30}\boldsymbol{v}^{\mathrm{T}}(k)\boldsymbol{S}_3\boldsymbol{v}(k)}=0.249 \tag{39}$$

and

$$\sum_{k=0}^{30}\boldsymbol{z}^{\mathrm{T}}(k)\boldsymbol{z}(k)=1.175 \tag{40}$$

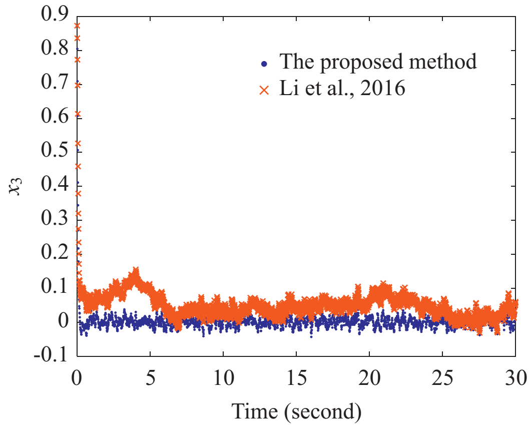

Fig. 3. Responses of state $x_3(k)$.

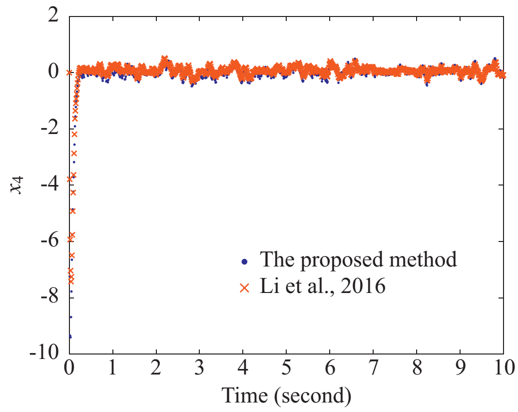

Fig. 4. Responses of state $x_4(k)$.

Based on the above results, one can find that the ratio value of (39) is smaller than one. Thus, the controlled system (37) is passive according to Definition 1. Besides, the value of (40) is smaller than the obtained $\alpha = 27.2951$. Also, the H_2 performance in Definition 2 is achieved. Therefore, it is easy to show that the proposed controller design approach can guarantee the asymptotical stability and mixed H_2/*Passivity* performance of uncertain stochastic inverted pendulum system.

In order to show the importance of considering stochastic behavior, a comparison with Li et al. (2016) is proposed. In Li et al. (2016), a mixed H_2/H_∞ performance controller design method was proposed for deterministic systems. By applying the method in Li et al. (2016), the following controller can be obtained.

$$u(t) = \boldsymbol{F}x(t) \tag{41}$$

where

$$\boldsymbol{F} = \begin{bmatrix} -0.0095 & -0.0788 & -1.0358 & -0.0142 \end{bmatrix}.$$

Applying (41), the responses of (37) are also stated in Figs. 1-4 with the same initial condition. Based on the responses, the following output energy can be calculated as follows:

$$\sum_{k=0}^{30} z^{\mathrm{T}}(k) z(k) = 1.479 \tag{42}$$

From (42), one can easily find that the output energy minimized by (41) is bigger than one minimized by (38). It is found that the proposed design method is less conservative than the method of Li et al. (2016). Besides, referring to Figs. 1-4, the settling time and maximum overshoot of (37) driven by (41) are bigger than one driven by (38). Obviously, the proposed design method provides some improvements to the method of Li et al. (2016) in controlling the uncertain stochastic systems (37).

V. CONCLUSIONS

This paper has proposed a mixed H_2/*Passivity* controller design method to deal with stability and stabilization problem of the discrete-time uncertain stochastic system. Through the proposed design method, the output energy can be minimized and the asymptotical stability can be guaranteed. Furthermore, the effect of external disturbance can be constrained under the assigned power supply rate function. According to the passivity theory, the proposed design method is more general and flexible than the existing mixed performance design method. Moreover, some sufficient conditions derived by Lyapunov theory was converted into the extended LMI form which reduces the conservatism of the derived sufficient conditions. At last, the stabilization problem of uncertain stochastic inverted pendulum system has been discussed via the proposed design method subject to mixed H_2/*Passivity* performance. In the future, the design method will be extended to discuss stability and stabilization problem of delayed stochastic systems.

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REFERENCES

Boyd, S. and L. Vandenberghe (2004) Convex Optimization, Cambridge Cambridge University Press: Cambridge, UK.

Boyd, S., L. E. Ghaoui, E. Feron and V. Balakrishnan (1994). Linear Matrix Inequalities in System and Control Theory. Philadelphia: SIAM.

Chang, W. J., C. H. Chang and C. C. Ku (2010). Fuzzy control with relaxed nonquadratic stability conditions for inverted pendulum robot system with multiplicative noise. Proceedings. of the IEEE International Conference on Control and Automation, 1091-1024, Xiamen, China.

Chang, W. J. and C. C. Shing (2004). LMI-based fuzzy controller design with minimum upper bound input energy and state variance constraints for discrete fuzzy stochastic systems. Journal of Intelligent and Fuzzy Systems 15, 115-124.

Chang, W. J., W. Y. Wu and C. C. Ku (2011). H_∞ constrained fuzzy control via state observer feedback for discrete-time Takagi-Sugeno fuzzy systems with multiplicative noises. ISA Transactions 50, 37-43.

Du, D. (2006). Generalized H_2 output feedback controller design for uncertain discrete-time switched systems via switched Lyapunov functions. Nonlinear Analysis: Theory, Methods & Applications 65, 2135-2146.

Eli, G., S. Uri and Y. Isaac (2005). H_∞ Control and Estimation of State-multiplicative Linear Systems. London: Springer.

Feng, Y., M. Yagoubi and P. Chevrel (2010). Dilated LMI characterisations for linear time-invariant singular systems. International Journal of Control 83, 2276-2284.

Ghaoui, L. E. (1995). State-feedback control of systems with multiplicative noise via linear matrix inequalities. Systems & Control Letters 24, 223-228.

Gurumoorthy, R. and S. R. Sanders (1993). Controlling non-minimum phase nonlinear systems-the inverted pendulum on a cart example. Proceedings of the American Control Conference, 680-685. San Francisco, USA.

Iordanou, H. N. and B. W. Surgenor (1997). Experimental evaluation of the robustness of discrete sliding mode control versus linear quadratic control. IEEE Transactions on Control Systems Technology 5, 254-260.

Jiao, W., S. Yang and S. Qian (2006). Homomorphic transform based BSS algorithm for multiplicative noise reduction. Journal of Zhejiang University (Engineering Science) 40, 49-52.

Karatzas, I. and S. E. Shreve (1991). Brownian Motion and Stochastic Calculus, New York: Springer.

Khargonekar, P. P. and M. A. Rotea (1991). Mixed H_2/H_∞ control: a convex optimization approach. IEEE Transactions on Automation Control 36, 824-837.

Kim, J. H. (2001). Robust mixed H_2/H_∞ control of time-varying delay systems. International Journal of Systems Science 32, 1345-1351.

Ku, C. C., P. H. Huang and W. J. Chang (2010). Passive fuzzy controller design for nonlinear systems with multiplicative noises. Journal of The Franklin Institute 347, 732-750.

Ku, C. C. and R. W. Chen (2017). Mixed H_2/*Passivity* performance control of discrete-time linear stochastic systems. Journal of Marine Science and Technology 25, 424-434.

Ku, C. C. and G. W. Chen (2018). Robust H_∞ control of uncertain stochastic systems with time-varying interval delays. International Journal of Control, Automation and Systems 16, 243-255.

Lazar, M. (2009). Lyapunov functions, stability and input-to-state stability subtleties for discrete-time discontinuous systems. IEEE Control Systems Society 54, 2421-2425.

Li, C. G., H. B. Zhang and X. F. Liao (2005). Passivity and passification of uncertain fuzzy systems. IEE Proceedings Circuits Devices Systems 152, 649-653.

Lozano, R., B. Brogliato, O. Egeland and B. Maschke (2000). Dissipative Systems Analysis and Control Theory and Applications. London: Springer.

Li, Z., C. Zang, P. Zeng, H. Yu, H. Li, L. Gao and X. Bai, (2016). Robust mixed H_2/H_∞ based optimal controller design for single-phase grid-connected converter. IEEE Chinese Control and Decision Conference (CCDC), 1230-1234.

Ma, M. and H. Chen (2006). Constrained H_2 control of active suspensions using LMI optimization." Proceedings of the Chinese Control Conference, 702-707. Harbin, China.

Mao, W., F. Deng and A. Wan (2012). Robust mixed H_2/H_∞ filtering for uncertain stochastic systems with interval time-varying delays. Proceedings of the Chinese Control Conference, 1676-1683. Hefei, China.

Mori, S., H. Nishihara and K. Furuta (1976). Control of unstable mechanical system control of pendulum. International Journal of Control 23, 673-692.

Øksendal, B. (2003). Stochastic Differential Equations: An Introduction with Applications. New York: Springer.

Peres, P. L. D. and J. C. Geromel (1993). H_2 control for discrete-time systems optimality and robustness. Automatica 29, 225-228.

Pipeleers, G., B. Demeulenaere, J. Swevers and L. Vandenberghe (2009). Extended LMI characterizations for stability and performance of linear systems. Systems & Control Letters 58, 510-518.

Qiu, J. (2008). Robust mixed H_2/H_∞ filtering design for discrete-time switched polytopic linear systems. IET Control Theory & Applications 2, 420-430.

Rami, M. A., X. Chen, J. B. Moore and X. Y. Zhou (2001). Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls. IEEE Transactions on Automatic Control 46, 428-440.

Shi, Y. and B. Yu (2011). Robust mixed H_2/H_∞ control of networked control systems with random time delays in both forward and backward communication links. Automatica 47, 754-760.

Troch, I. (1998). Solving the discrete Lyapunov equation using the solution of the corresponding continuous Lyapunov equation and vice versa. IEEE Control Systems Society 33, 944-946.

Wang, Y., L. Xie and C. E. D Souza (1992). Robust control of a class of uncertain nonlinear systems. Systems & Control Letters 19, 139-149.

Willems, J. C. (1971). Least squares stationary optimal control and the algebraic Riccati equation. IEEE Transactions on Automatic Control 16, 621-634.

Xie, S., L. Xie and C. E. De Souza (1998). Robust dissipative control for linear systems with dssipative uncertainty. International Journal of Control 70, 169-191.

Zhang, S., J. J. Yang and G. G. Zhu (2015). LPV modeling and mixed H_2/H_∞ constrained control of an electronic throttle. IEEE/ASME Transactions on Mechatronics 20, vol. 20, 2120-2132.