Journal of Marine Science and Technology

Volume 28 | Issue 3 Article 3

CONSTRAINT-TYPE FICTITIOUS TIME INTEGRATION METHOD FOR SOLVING NON-LINEAR MULTI-DIMENSIONAL ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Recommended Citation

Chen, Yung-Wei; Liu, Chein-Shan; Chang, Yen-Shen; and Chang, Jiang-Ren (2020) "CONSTRAINT-TYPE FICTITIOUS TIME INTEGRATION METHOD FOR SOLVING NON-LINEAR MULTI-DIMENSIONAL ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS," *Journal of Marine Science and Technology*: Vol. 28: Iss. 3, Article 3.
DOI: 10.6119/JMST.202006_28(3).0003
Available at: https://jmstt.ntou.edu.tw/journal/vol28/iss3/3

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Acknowledgements

The first author greatly appreciates the financial support that was provided by the Ministry of Science and Technology, Taiwan, ROC under contract number MOST 108-2221-E-019-015.

Journal of Marine Science and Technology, Vol. 28, No. 3, pp. 168-178 (2020)
DOI: 10.6119/JMST.202006_28(3).0003

CONSTRAINT-TYPE FICTITIOUS TIME INTEGRATION METHOD FOR SOLVING NON-LINEAR MULTI-DIMENSIONAL ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Key words: algebraic equations, ordinary differential equation, fictitious time integration method, Newton's method, iterative scheme.

ABSTRACT

In this paper, we propose a constraint-type fictitious time integration method (FTIM) for solving multi-dimensional non-linear elliptic-type partial differential equations. Based on the variable transformation of FTIM, the original governing equation is transformed into a new parabolic equation of an evolution type by introducing a space-time variable, and a new time integration direction is obtained. However, the space-time variable depends on the governing equation, boundary condition and fictitious time variable, especially due to the nonlinear effect. Previous studies have not discussed the definition of these nonlinear parameter problems, which may result in severe numerical instability and inaccuracy. To completely overcome this nonlinear parameter problem, a space-time variable with a minimum fictitious time size is introduced into the algorithm. By imposing a constraint condition that involves the system energy in the space domain and the minimum fictitious time step, the proposed scheme can absolutely satisfy the stringent convergence criterion and can quickly approach the true solution, even under a very small time step. More importantly, the convergence speed depends only on a space-time variable. The accuracy and efficiency of the scheme are evaluated by comparing the estimation results with those of previous studies. The obtained results demonstrate that the proposed method efficiently finds the true solution and can significantly improve both the accuracy and convergence.

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I. INTRODUCTION

Partial differential equations (PDEs), such as the Sturm-Liouville equation, the Fredholm integral equation, the Laplace equation, the heat conduction equation, the wave equation and the Helmholtz equation, are widely applied in many fields of engineering and science, for example, to sloshing problems in oil tanks (Chen and Chiang, 1999; Chen et al., 2019), underwater acoustic problems (Schenck, 1969; Sayhi and Ousset, 1981; Chen and Ginsberg, 1995), heat conduction problems (Raghu Kumar et al., 1998; Chen, 2016, 2018, 2019) and wave propagation (Chang et al., 2013). According to the number of real characteristic lines of physical phenomena, the parabolic and hyperbolic types are classified into evolutionary PDEs. A non-evolutionary PDE is typically referred to as an elliptic-type PDE because no real characteristic line exists. For solving PDEs, numerical methods, such as the finite difference method (FDM), the finite element method (FEM), the boundary element method (BEM), and the meshless (or meshfree) method are currently the most popular tools. Among these methods, the FDM was the earliest to be developed; it is easily combined with discrete techniques to solve engineering problems. However, for linear or nonlinear problems with noisy disturbances, the conventional FDM typically requires special numerical techniques for finding the solution.

Over the past years, many studies have addressed elliptic boundary value problems (BVPs); for example, Chen and Zhou (2000) presented iteration methods for quasilinear elliptic BVPs, such as the mountain iteration algorithm, the scaling iterative algorithm, the monotone iterative algorithm, and the direct iterative algorithm. A sequence of iterations is generated by various methods, and they are not typically guaranteed to converge to the true solution. Additionally, regarding the numerical solutions of linear elliptic BVPs, many studies have been conducted; for example, Zhu et al. (1998) and Atluri and Zhu (1998) proposed meshless methods, namely, the meshless local boundary integral equation method and the meshless local Petrov-Galerkin method, respectively, for solving nonlinear problems. Cheng et al. (2003) developed the multiquadric and Gaussian radial basis function for solving PDEs. Additionally, Cho et al. (2004) presented a Trefftz method for solving a class of second-order time-dependent PDEs, which include equations of parabolic, hyperbolic and parabolic-hyperbolic types. Jin (2004) applied the method of fundamental solutions for the solution of the Laplace and biharmonic equations. Hu et al. (2005) applied radial basis collocation methods for elliptic boundary value problems. To overcome an ill-posed matrix, Liu (2007) modified the T-complete function of the Trefftz method for solving elliptic BVPs. Hu and Chen (2008) combined the radial basis collocation method and quasi-Newton iteration to address nonlinear elliptic problems. Lasanen et al. (2018) used variational methods and Cameron–Martin space techniques to address elliptic BVPs with Gaussian white noise loads. Colbrook et al. (2018) used the Fokas method (unified transform method) to solve elliptic problems and to increase the rate of convergence. Milewski (2018) proposes a stochastic approach that is based on the Monte Carlo method with a random walk technique for analysing second-order elliptic PDEs. Based on the unified transform method, in conjunction with domain decomposition techniques, Grylonakis et al. (2019) proposed a hybrid approach for solving linear elliptic PDEs. These numerical methods are effective for linear problems. However, for nonlinear problems with noisy disturbances, low computational efficiency and a high number of iterations hinder the application of these methods.

A variable transformation of a time integration method, namely, the fictitious time integration method (FTIM), was proposed by Liu and Atluri (2008). The FTIM was used to solve linear or nonlinear algebraic equations by introducing the fictitious time and using it to derive a system of nonautonomous first-order ordinary differential equations that is equivalent to the original algebraic equations in n-dimensional space. The roots for the original algebraic equations are obtained by applying numerical integrations on the resulting ordinary differential equation, which do not require information on the derivatives of the nonlinear algebraic equations or their inverses. Furthermore, Ku et al. (2008) introduced a time-like function into the FTIM to accelerate the convergence. Tsai et al. (2010) further applied an FTIM in combination with the method of fundamental solution to solve Poisson-type nonlinear PDEs. Chang (2010) applied an FTIM for multi-dimensional backward heat conduction problems. To accelerate the convergence rate and avoid the selection of the parameters, Liu et al. (2009) and Ku et al. (2010) proposed a scalar-based homotopy method that does not calculate the inverse of the Jacobian matrix for solving a system of nonlinear algebraic equations (NAEs). However, the convergence of this scalar-based homotopy method is very slow, namely, it remains difficult to satisfy the tough convergence criterion. After that, Chen (2014), Chen et al. (2014) and Ku et al. (2015) applied the manifold-based exponentially convergent algorithm (MBECA), the residual-norm-based algorithm (RNBA), and the dynamic Jacobian-inverse-free method (DJIFM), which are used to solve the NAEs and BVPs. However, selection of the parameters is difficult for these approaches, such as the viscosity-damping coefficient, the fictitious time step, the convergence criterion, and the fictitious termination time. More importantly, the convergence criterion of their algorithms is specified by the fictitious termination time, which is impossible to determine at the initial time, namely, the conventional FTIM does not guarantee that the solution can be determined when solving nonlinear complex problems. It is inefficient and unstable when the algorithms must utilize a trial-and-error approach in the fictitious time domain.

Recently, Chen (2016, 2018) applied the FTIM for multi-dimensional backward heat conduction problems and successfully overcame the fictitious time step and convergence criterion problems. However, for solving parabolic-type PDEs via the FTIM, the initial guess value depends on the final condition, and the problem of determining the space-time coefficients remains unresolved. Unfortunately, it will produce multipole solutions for elliptic PDEs due to the lack of an initial condition or termination conditions. The space-time coefficient, fictitious time step and initial guess value will result in coupling, especially for nonlinear problems. In this paper, a constraint-type FTIM is proposed, which is an extension of the work of Liu and Atluri (2008), and a space-time constraint condition, which includes the computational domain, viscosity-damping coefficient and fictitious time step, is proposed and applied. A space-time variable with minimum fictitious time size can satisfy the NAEs simultaneously to avoid the selection of the parameters. The remainder of this paper is organized as follows: Section 2 describes the mathematical formulation of the FTIM. In Section 3, we evaluate the proposed method in several numerical examples. Finally, the conclusions of this study are presented in Section 4

II. NONLINEAR AND NONHOMOGENEOUS ELLIPTIC EQUATION

Let us consider the following equations:

$$\Delta u(x,y,z) = S(x,y,z,u,u_x,u_y,u_z \ldots), \quad (x,y,z) \in \Omega , \tag{1}$$

$$u(x,y,z) = H(x,y,z) \text{ on } \Gamma , \tag{2}$$

where Δ denotes the three-dimensional Laplacian operator; Γ is the boundary of the problem domain Ω ; u is a scalar field; x, y and z are spatial variables; u_x , u_y and u_z represent derivatives of u with respect to x, y and z, respectively; and S and H are specified functions.

1 Transformation into an evolutionary PDE and semi-discretization

First, we apply the following variable transformation:

$$X(x,y,z,\tau)=(1+\tau)u(x,y,z), \tag{3}$$

where τ is a fictitious time that differs from the real time *t*. As τ approaches to zero in Eq. (3), $X \cong u$. Therefore, τ can be assigned a very small value but cannot be zero. The main objective is to avoid integration in the spatial direction. A numerical integral method can avoid numerical divergence. Here, we multiply Eq. (1) by a space-time coefficient $\nu > 0$:

$$0=-\nu\Delta u+\nu S(x,y,z,u,u_x,u_y,u_z\ldots). \tag{4}$$

By multiplying the above equation by $1+\tau$ and applying Eq. (3), we obtain

$$0=-\nu\Delta X+\nu(1+\tau)S(x,y,z,u,u_x,u_y,u_z\ldots). \tag{5}$$

Since $\partial X/\partial\tau=u(x,y,z)$, from Eq. (3), we can add it on both sides of the above equation as follows:

$$\frac{\partial X}{\partial\tau}=-\nu\Delta X+\nu(1+\tau)S(u)+u. \tag{6}$$

Finally, by setting $u=X/(1+\tau)$, $u_x=X_x/(1+\tau)$, $u_y=X_y/(1+\tau)$ and $u_z=X_z/(1+\tau)$, Eqs. (1) and (2) can be transformed into a parabolic-type PDE of the evolutionary type:

$$\frac{\partial X}{\partial\tau}=-\nu\Delta X+\nu(1+\tau)S\left(x,y,z,\frac{X}{1+\tau},\frac{X_x}{1+\tau},\frac{X_y}{1+\tau},\frac{X_z}{1+\tau},\ldots\right)+\frac{X}{1+\tau}\quad(x,y,z)\in\Omega, \tag{7}$$

$$X(x,y,z,\tau)=(1+\tau)H(x,y,z),\ (x,y,z)\in\Gamma. \tag{8}$$

Applying a finite-difference discretization procedure to Eq. (7) yields a coupled system of ODEs:

$$\begin{aligned}\dot{X}_{i,j,k}=&-\frac{\nu}{(\Delta x)^2}\left[X_{i+1,j,k}-2X_{i,j,k}+X_{i-1,j,k}\right]\\&-\frac{\nu}{(\Delta y)^2}\left[X_{i,j+1,k}-2X_{i,j,k}+X_{i,j-1,k}\right]\\&-\frac{\nu}{(\Delta z)^2}\left[X_{i,j,k+1}-2X_{i,j,k}+X_{i,j,k-1}\right]\\&+\nu(1+\tau)S\left(\begin{array}{l}x_i,y_j,z_k,\frac{X_{i,j,k}}{1+\tau},\frac{X_{i+1,j,k}-X_{i-1,j,k}}{2\Delta x(1+\tau)}\\,\frac{X_{i,j+1,k}-X_{i,j-1,k}}{2\Delta y(1+\tau)},\frac{X_{i,j,k+1}-X_{i,j,k-1}}{2\Delta z(1+\tau)}\ldots\end{array}\right)\\&+\frac{X_{i,j,k}}{1+\tau},\end{aligned} \tag{9}$$

where Δx, Δy and Δz are the uniform spatial lengths in the *x*-, *y*- and *z*- directions; $X_{i,j,z}(\tau)=X(x_i,y_j,z_k,\tau)$; and $\dot{X}$ denotes derivative of X with respect to τ.

2 Convergence criterion

We use the fourth-order Runge-Kutta method (RK4) to integrate Eq. (9) starting from $\tau=0$. In the numerical integration process, we can examine the convergence of $X_{i,j,k}$ at steps *m* and $m+1$ via

$$\sqrt{\sum_{i,j,k=1}^{N_1}\left[X_{i,j,k}^{m+1}-X_{i,j,k}^{m}\right]^2}\le\varepsilon, \tag{10}$$

where ε is the selected criterion, N_1 is the number of grid points in each spatial direction and m is the iteration number in the fictitious time direction.

3 Constraint condition of the space-time variable

To avoid numerical divergence, a constraint condition of the space-time variable is imposed:

$$\frac{1}{\Delta\tau}>\nu, \tag{11}$$

$$\nu=\left(\frac{1}{\Delta\tau\cdot E_{xyz}}\right),\ 0<\Delta\tau<1,\ E_{xyz}>0, \tag{12}$$

where E_{xyz} is a constant. In this paper, E_{xyz} is defined as the system energy, which is based on the maximum mesh-grid numbers and the computational domain. If E_{xyz} increases, ν will decrease. If $\Delta\tau$, ν and E_{xyz} satisfy Eqs. (11) and (12), the FTIM can stably approach solutions. For the convenience of description, in this paper, E_{xyz} is set to 10^4 in all examples.

III. NUMERICAL EXAMPLES

We apply the FTIM, in combination with RK4, to the calculations of the PDEs in numerical examples. We are interested in the stability of our approach when the input measured data are subject to random noise for various problems. We can evaluate the stability by increasing the random noise levels in the boundary condition:

$$\hat{u}=u[1+\sigma R]\text{ on }\Gamma, \tag{13}$$

where u is the boundary condition. We use the function RANDOM_NUMBER in MATLAB to generate the noise data R, which are random numbers between [-1, 1], and σ is the level of absolute noise. Then, the boundary condition with the noise data $\hat{u}$ is employed in the calculations.

Table 1. Summary of the maximum absolute errors for various values of N_1, $\Delta\tau$ and E_{xyz}.

$\mathbf{U}_{i,j}=1$		Number of iterations	Maximum absolute error
$N_1=21$, $\Delta\tau=10^{-14}$	$E_{xyz}=10^4$, $\nu=10^{10}$	1.421×10^4	4.694×10^{-5}
	$E_{xyz}=10^5$, $\nu=10^9$	1.303×10^5	2.309×10^{-4}
$N_1=21$, $\Delta\tau=10^{-140}$	$E_{xyz}=10^4$, $\nu=10^{136}$	1.4207×10^4	4.53×10^{-5}
	$E_{xyz}=10^5$, $\nu=10^{135}$	1.303×10^5	4.53×10^{-5}
$N_1=41$, $\Delta\tau=10^{-14}$	$E_{xyz}=10^4$, $\nu=10^{10}$	1.458×10^4	1.308×10^{-5}
	$E_{xyz}=10^5$, $\nu=10^9$	1.339×10^5	2.439×10^{-4}
$N_1=41$, $\Delta\tau=10^{-140}$	$E_{xyz}=10^4$, $\nu=10^{136}$	1.456×10^4	1.136×10^{-5}
	$E_{xyz}=10^5$, $\nu=10^{135}$	1.339×10^5	1.136×10^{-5}

Table 2. Summary of the maximum absolute errors for various initial guess values.

$N_1=41$ $\Delta\tau=10^{-14}$ $E_{xyz}=10^4$ ($\nu=10^{10}$)	Number of iterations	Maximum absolute error
$\mathbf{U}_{i,j}=10$	1.606×10^4	1.136×10^{-5}
$\mathbf{U}_{i,j}=100$	1.729×10^4	1.136×10^{-5}
$\mathbf{U}_{i,j}=1000$	1.848×10^4	1.136×10^{-5}

1. Example 1

To evaluate the numerical accuracy and stability of FTIM under steady-state conditions, an analytical solution of the Laplace equation is considered:

$$u(x,y)=e^x\cos y\,. \tag{14}$$

The domain is specified by $\Omega=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$. Here, we set the parameters as $N_1=41$, $\Delta\tau=10^{-200}$, and $\varepsilon=10^{-14}$, and start with an initial value of $\mathbf{U}_{i,j}=1$. Fig. 1 presents a convergence plot. The proposed scheme converges within 1.4547×10^4 iterations. The absolute errors are plotted in Fig. 2. The maximum error of the numerical solution is

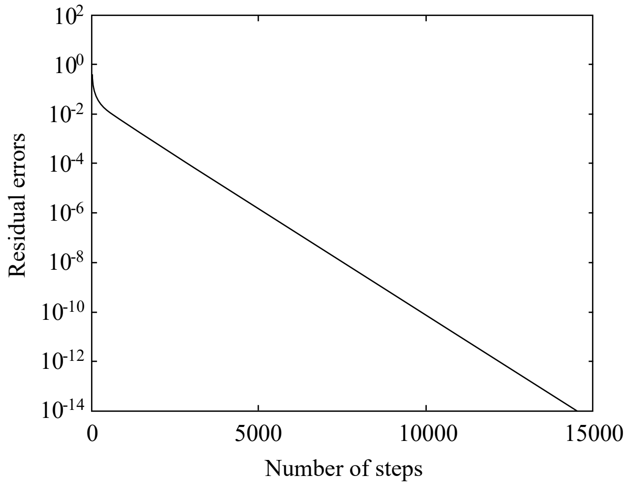

Fig. 1. Plot of the residual errors of Example 1.

smaller than 1.2×10^{-5} in the proposed scheme and 4.9×10^{-5} in the conventional FTIM by Liu (2008). To evaluate the effects of the parameters N_1, $\Delta\tau$, and E_{xyz}, as shown in Table 1, the parameters are set as specified above and the initial value is set to $\mathbf{U}_{i,j}=1$. The order of the numerical errors converges in the range of 10^{-4} to 10^{-5} as the number of discretization points increases. Additionally, from Table 2, the numerical result is insensitive to the initial guess value. For a more stringent evaluation, we consider the application of a noise level of σ=5%. The convergence plot and absolute errors are presented in Fig. 3. Even with the noise, the maximum error of the numerical solution is smaller than 0.15 and convergence occurs within 1.4548×10^4 iterations. Hence, this scheme provides high numerical accuracy and stability in the solution of this problem.

2. Example 2

To evaluate the effect of the external force in the elliptic equation, an analytical solution is considered:

$$u(x,y)=x^3+2xy \tag{15}$$

of a linear Poisson equation:

$$\Delta u=6x\,. \tag{16}$$

The domain is specified by $\Omega=\{(x,y)\,|\,-1\le x\le 1, -1\le y\le 1\}$. Liu et al. (2006) have solved this problem via a Trefftz method and using the SVD regularization technique; however, the numerical results are unsatisfactory, with an error that is on the order of 10^0. Here, we set the parameters as $N_1=41$, $\Delta\tau=10^{-300}$, and $\varepsilon=10^{-14}$, and we start from an initial value of $\mathbf{U}_{i,j}=1$. Fig. 4 shows the convergence of the residual errors within 5.6944×10^4 iterations. According to the numerical result, by using a small $\Delta\tau$, the computational efficiency can

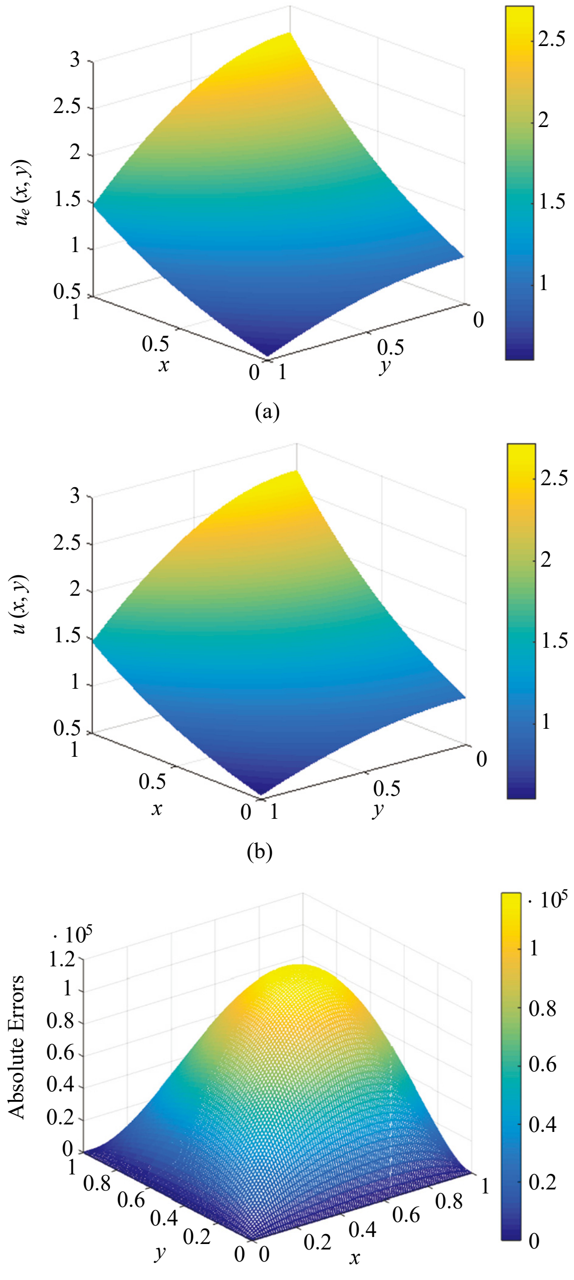

Fig. 2. Plots of (a) the exact solution, (b) the numerical solution, and (c) the numerical errors

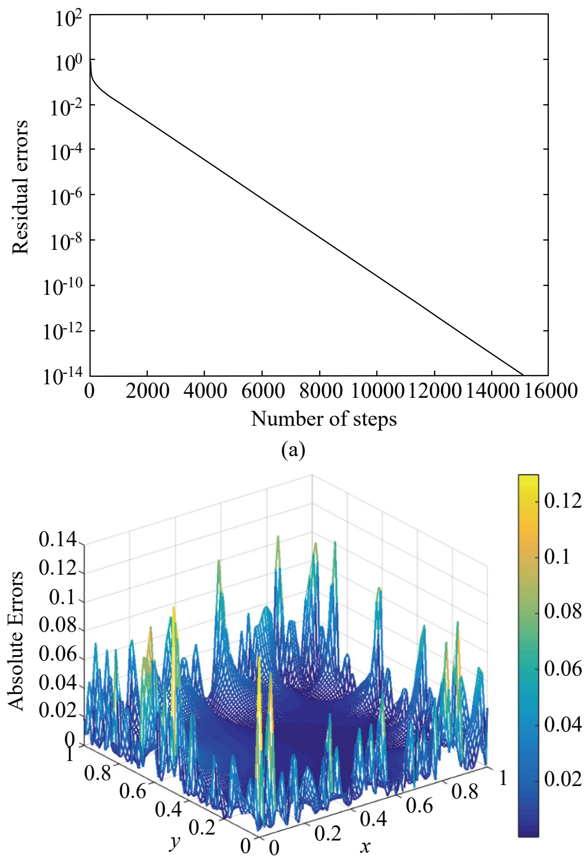

Fig. 3. For Example 1, plots of (a) the residual errors and (b) the numerical errors under the relative random noise level of σ=5%.

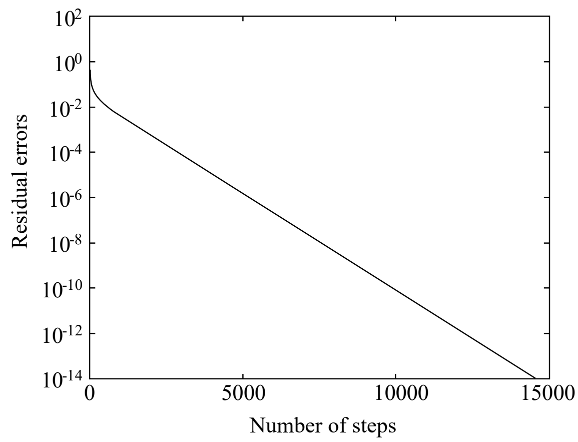

Fig. 4. Plot of the residual errors of Example 2.

be increased. The numerical result and absolute errors are plotted in Fig. 5. The maximum error of the numerical solution is smaller than 1.2×10^{-12}, which is better than the maximum error of 2.2×10^{-7} that was realized by Liu (2008). With a noise level of σ=5%, the convergence speed and absolute errors are plotted in Fig. 6. The maximum error of the numerical solution is still smaller than 0.33, and the residual errors converge within 5.6939×10^{4} iterations. Hence, the proposed method can efficiently avoid the noise effect, and the error is very small.

3. Example 3

For the nonlinear effect, the numerical accuracy and stability of the proposed scheme are evaluated. Consider the following nonlinear elliptic equation:

$$\Delta u + u^2 + 0.001u^3 = P \tag{17}$$

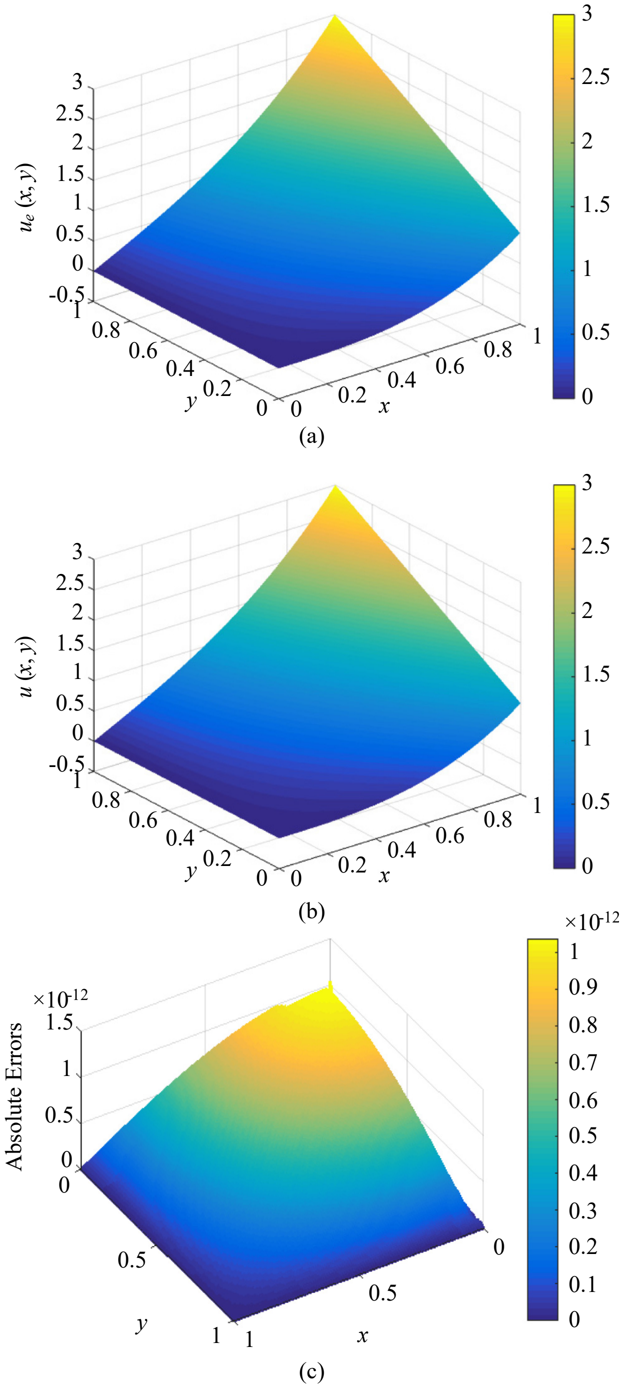

Fig. 5. Plots of the (a) exact solution, (b) the numerical solution, and (c) the numerical errors

The analytical solution of Eq. (17) is

$$u(x,y)=\frac{-5}{6}\left(x^3+y^3\right)+3\left(x^2y+xy^2\right). \tag{18}$$

The exact value of P can be obtained via Eqs. (17) and (18). The domain is specified by $\Omega=\left\{(x,y)\middle|0\le x\le 1,\ 0\le y\le 1\right\}$.

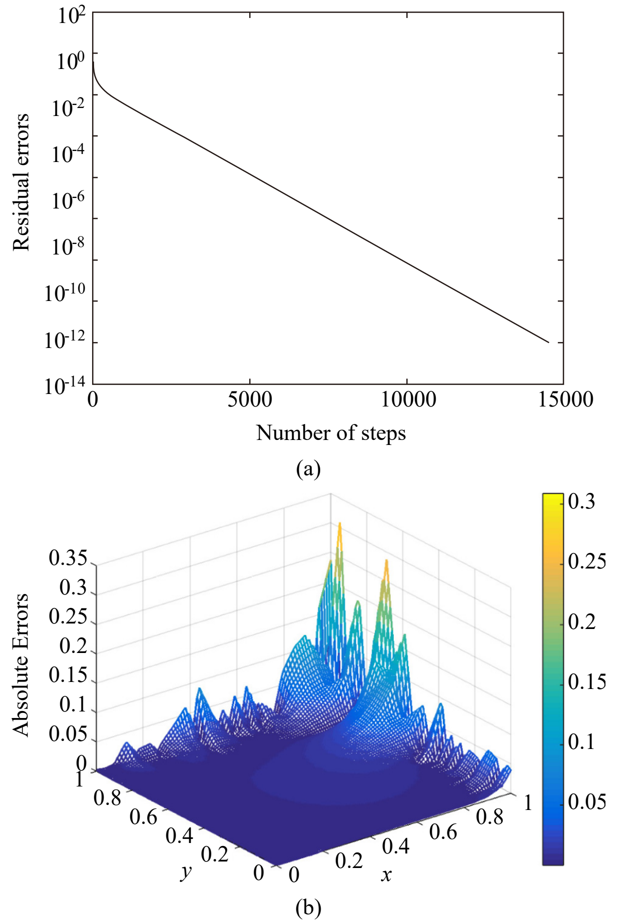

Fig. 6. For Example 2, plots of (a) the residual errors and (b) the numerical errors under the relative random noise level of $\sigma=5\%$.

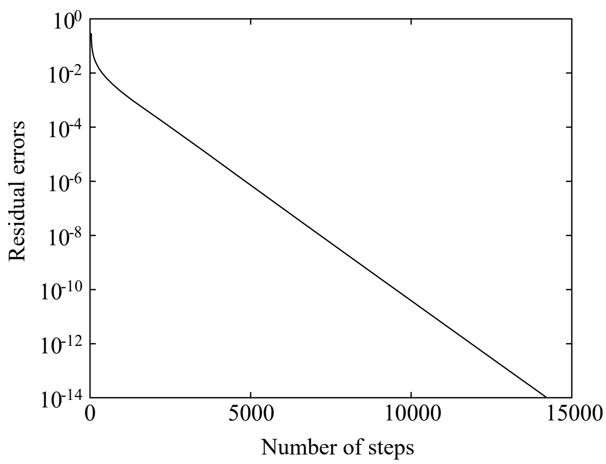

Fig. 7. Plot of the residual errors of Example 3.

The parameters are selected as $N_1=21$, $\Delta\tau=10^{-300}$, and $\varepsilon=10^{-14}$ and the initial value is $\mathbf{U}_{i,j}=1$. Fig. 7 shows the convergence plot. The results demonstrate that the residual errors are reduced to 10^{-14} in 1.4194×10^4 iterations. The

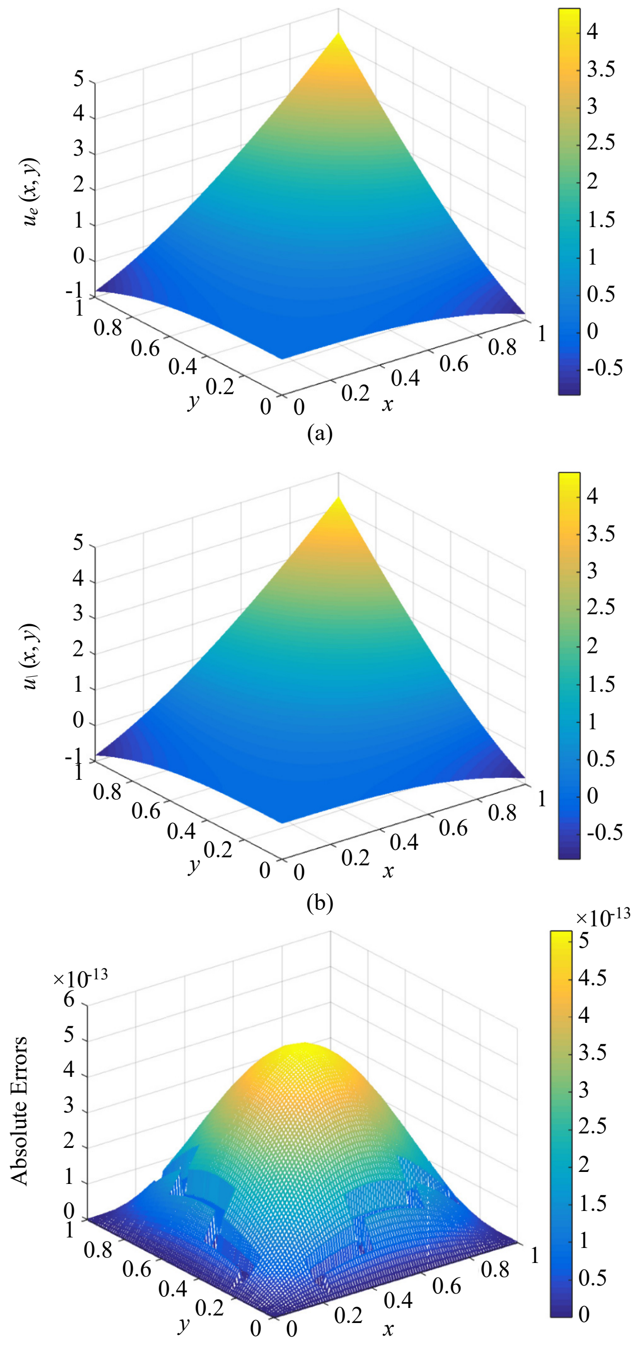

Fig. 8. Plots of the (a) exact solution, (b) the numerical solution, and (c) the numerical errors

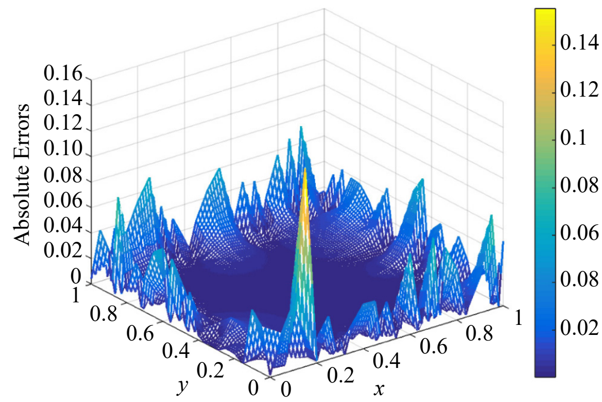

Fig. 9. ot of the numerical errors under a random noise level of $\sigma = 5\%$.

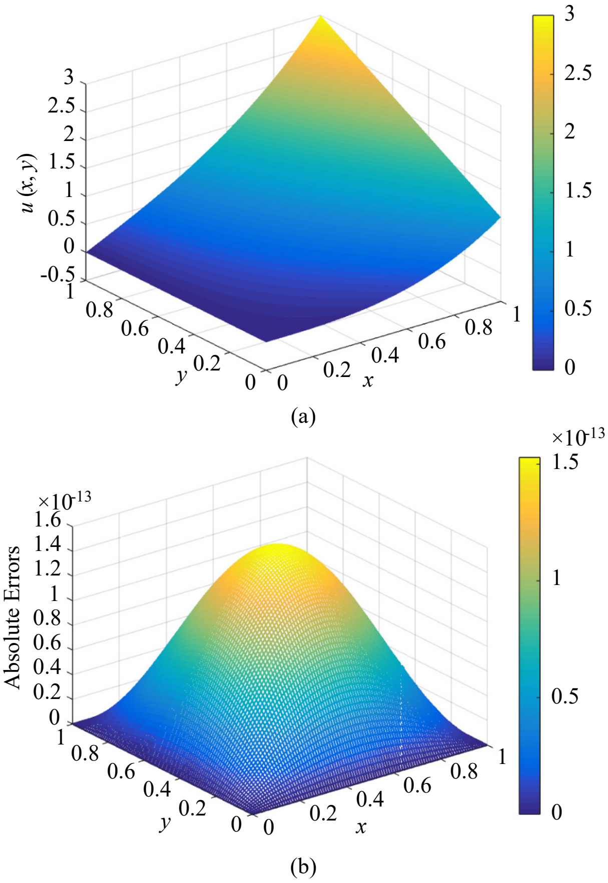

Fig. 10. Plots of (a) the exact solution and (b) the numerical errors.

numerical results and absolute errors are plotted in Fig. 8. The figure shows that the maximum error in the numerical solution is smaller than 1.884×10^{-13}, which is better than that of Atluri and Zhu (1998), namely, 10^{-5}, and that of Liu and Atluri (2008), namely, 4.4×10^{-6}. Additionally, to evaluate the stability of the proposed algorithm, a random noise (σ=5%) is considered. The residual errors are reduced to 10^{-14} in 1.4196×10^{4} iterations. Even with the noise disturbance, the maximum error in the numerical solution is less than 0.155, as shown in Fig. 9. From the obtained results, we find that the proposed method is highly effective and accurate for solving non-linear problems with random noise disturbances. and (c) the numerical errors.

4. Example 4

To evaluate the numerical stability of the FTIM for an elliptic

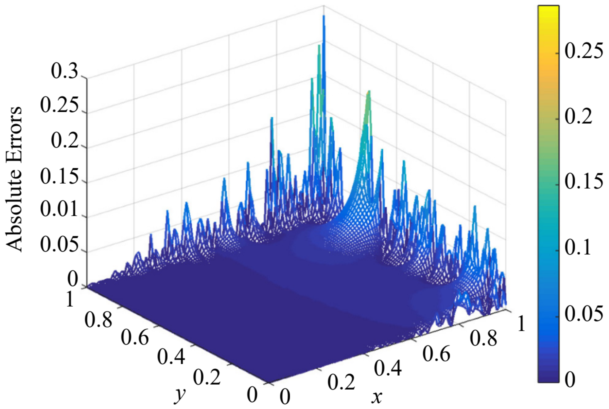

Fig. 11. Plot of the numerical errors under a random noise level of $\sigma = 5\%$.

equation with a nonlinear external force, a two-dimensional non-linear Poisson equation is considered:

$$\Delta u = u^2 + 6x - x^6 - 4x^4 y - 4x^2 y^2 . \qquad (19)$$

The analytical solution of Eq. (19) is the same as in Example 2.

The domain is secified by $\Omega = \{(x, y) | -1 \le x \le 1, -1 \le y \le 1\}$. The parameters are $N_1 = 61$, $\Delta\tau = 10^{-300}$, and $\varepsilon = 10^{-14}$, and the initial value is $\mathbf{U}_{i,j} = 1$. The residual errors are reduced to 10^{-14} in 1.3686×10^4 iterations. The numerical results and absolute errors are plotted in Fig. 10. The figure shows that the maximum error in the numerical solution is smaller than 1.53×10^{-13}, which is better than the maximum error of 1.8×10^{-7} that was realized by Liu (2008).

When the traditional FTIM with different parameters (N_1 and $\Delta\tau$) fails, it shows a maximum error of 1.8.

Additionally, to evaluate the stability of the proposed algorithm, a random noise (σ=5%) is considered. The residual errors are reduced to 10^{-14} in 1.3675×10^4 iterations. Even with the noise disturbance, the maximum error in the numerical solution is less than 0.274, as shown in Fig. 11. Similarly, for the nonlinear case, highly accurate numerical results are also obtained.

5. Example 5

To evaluate the numerical stability, a non-linear Helmholtz equation that arises from wave propagation is considered:

$$\Delta u = k^2(u) u \qquad (20)$$

where $k^2(u)$ is set to $4u^2$. The analytical solution of Eq. (20) is

$$u(x, y) = \frac{1}{(x^2 + y^2 + 4)} \qquad (21)$$

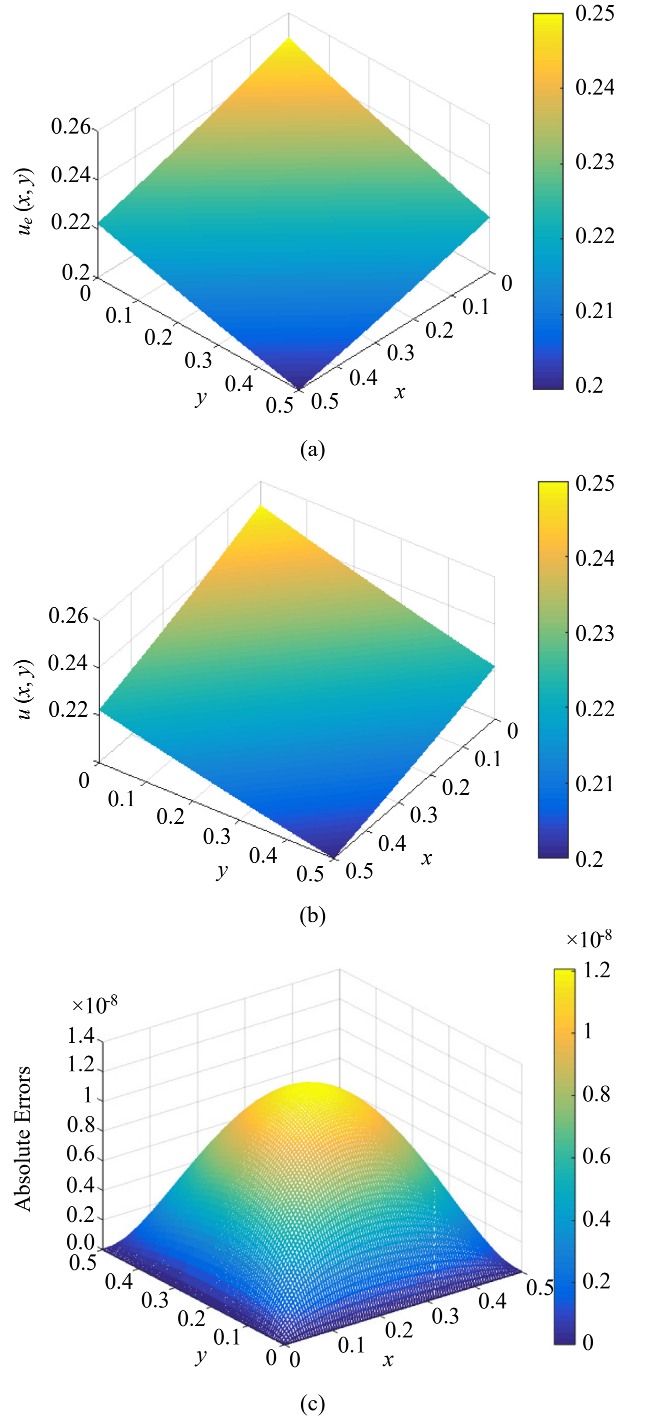

Fig. 12. Plots of the (a) exact solution, (b) the numerical solution, and (c) the numerical errors

The domain is specified by $\Omega = \{(x, y) | 0 \le x \le 0.5, 0 \le y \le 0.5\}$. We set the parameters as N_1 =31, $\Delta\tau = 10^{-300}$, and $\varepsilon = 10^{-14}$, and we start from an initial value of $\mathbf{U}_{i,j} = 1$. The residual errors are reduced to 10^{-14} in 3.808×10^3 iterations. The numerical results and absolute errors are plotted in

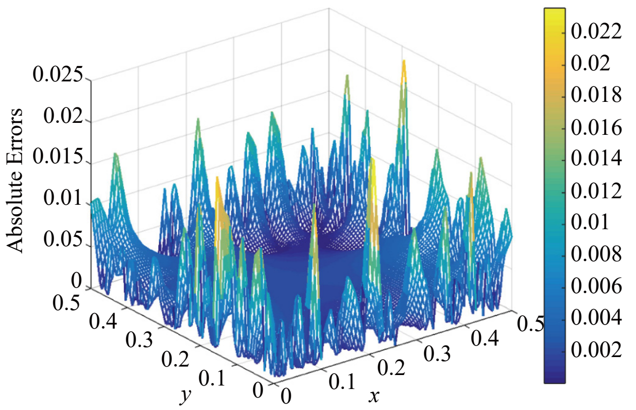

Fig. 13. Plot of the numerical errors under a random noise level $\sigma = 5\%$.

Fig. 12. The figure shows that the maximum error in the numerical solution is smaller than 1.206×10^{-8}. The numerical result is better than that of Tsai et al. (2010). Although the use of a time-like function can increase the number of discretization points, it cannot overcome the convergence problem. Under a random noise of σ=5%, the residual errors are reduced to 10^{-14} in 3.807×10^{3} iterations. Even with the noise disturbance, the maximum error in the numerical solution is less than 2.675×10^{-2}, as shown in Fig. 13.

6. Example 6

To evaluate the convergence speed and the numerical stability in a large computational domain, this scheme can be extended to a three-dimensional non-linear Helmholtz equation with a singularity solution:

$$\Delta u = k^2(u)u \tag{22}$$

where $k^2(u)$ is set as $6u^2$. The analytical solution of Eq. (22) is

$$u(x,y,z) = \frac{1}{(x+y+z+1)}. \tag{23}$$

The domain is specified by $\Omega = \{(x,y,z) | 0 \le x \le 9,\ 0 \le y \le 9,\ 0 \le z \le 9\}$. The singularity is on $x+y+z=-1$. The parameters are set to $\Delta\tau = 10^{-300}$ and $\varepsilon = 10^{-14}$, and the initial value is set to $\mathbf{U}_{i,j,k} = 1$. The residual errors are reduced to 10^{-14} in 5.35594×10^{5} iterations for $N_1 = 21$ and in 5.57978×10^{5} iterations for $N_1 = 41$. The numerical results and absolute errors on the z= 4.5 plane are plotted in Fig. 14. The figure shows that the maximum error of the numerical solution is smaller than 3.431×10^{-5} with $N_1 = 21$ and smaller than 8.84×10^{-6} with $N_1 = 41$. Comparing the distribution of the absolute errors with that of Ku et al. (2009), the numerical solution is more accurate. As the order of the residual errors is 10^{-6} in Ku et al. (2009), it is highly difficult to satisfy the stringent convergence condition, especially by increasing the number of discretization points. Despite the increased number of discretization points, the proposed scheme with a constraint condition is highly effective and stable for solving this highly nonlinear problem. Moreover, with a random noise level of σ=5%, the residual errors are reduced to 10^{-14} in 5.3554×10^{5} iterations. Even with the noise disturbance, the maximum error in the numerical solution is less than 2.432×10^{-2}, as shown in

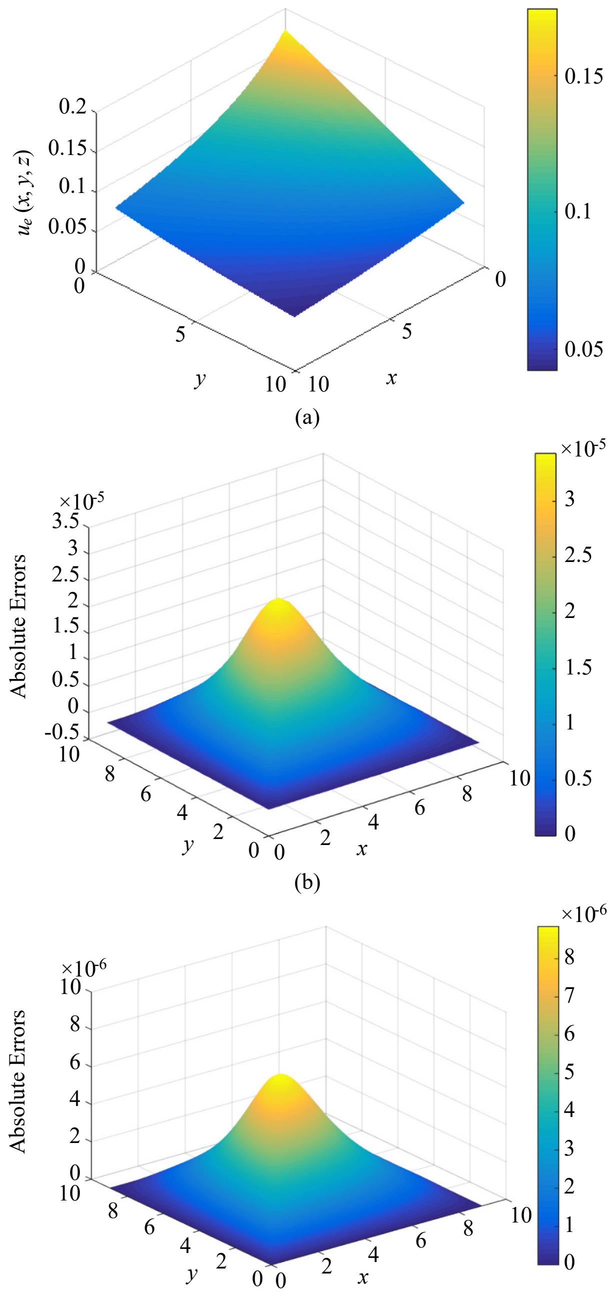

Fig. 14. Plots of (a) the exact solution and the numerical errors with (b) N_1=21 and (c) 41.

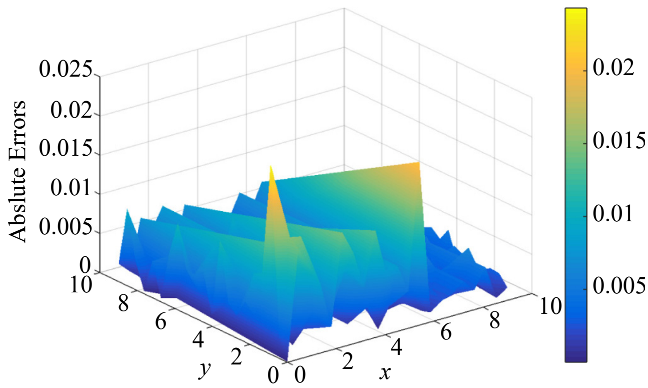

Fig. 15. Plot of the numerical errors under a random noise level of $\sigma = 5\%$.

Fig. 15. When considering $E_{xyz} = 10^2$, the residual errors are reduced to 10^{-14} in 6.585×10^3 iterations for $N_1 = 21$ and 41. The convergence, numerical results and absolute errors on the z= 4.5 plane are plotted in Fig. 16. The figure shows that the maximum error of the numerical solution is smaller than 8.4×10^{-6} with $N_1 = 21$ and 41. When using the minimum value of E_{xyz}, the convergence speed and numerical accuracy are independent from the mesh-grid numbers. Hence, the proposed scheme can successfully avoid the influences of the mesh-grid and the viscous damping, and the precision does not change with the discrete technique.

IV. CONCLUSIONS

In the paper, we have successfully developed a constraint-type FTIM for solving multi-dimensional non-linear elliptic-type PDEs. Previously, there were severe drawbacks to using the conventional FTIM, such as the selection of the parameters: the viscosity-damping coefficient, the fictitious time step, the convergence criterion, and the fictitious terminal time. To enhance the stability of the numerical integration of the discretized equations, a space-time variable with minimal fictitious time size is introduced into the algorithm. More importantly, this constraint condition of the space-time variable can avoid the selection of the space-time coefficient and the fictitious time step and preserve the time integration direction of the FTIM. Moreover, the numerical solution is in satisfactory agreement with the exact solution, even under a random noise disturbance. Additionally, it is surprising that by using a space-time variable with minimal fictitious time size, the proposed scheme can absolutely satisfy the stringent convergence criterion, and even under a very small time step, it can quickly approach the true solution. Six linear and non-linear numerical examples in two and three dimensions are evaluated, and the results demonstrate that the proposed scheme can operate more effectively and stably than the original scheme. The FTIM has excellent robustness to noise and does not require the calculation of the derivatives of the nonlinear algebraic equations or their inverses. For further practical engineering applications, this scheme can be combined with dimensionless techniques to increase the computing efficiency and does not require the selection of parameters. Hence, it is concluded that the proposed schemes are accurate, stable, effective, and insensitive to the boundary conditions, even under noise level disturbances.

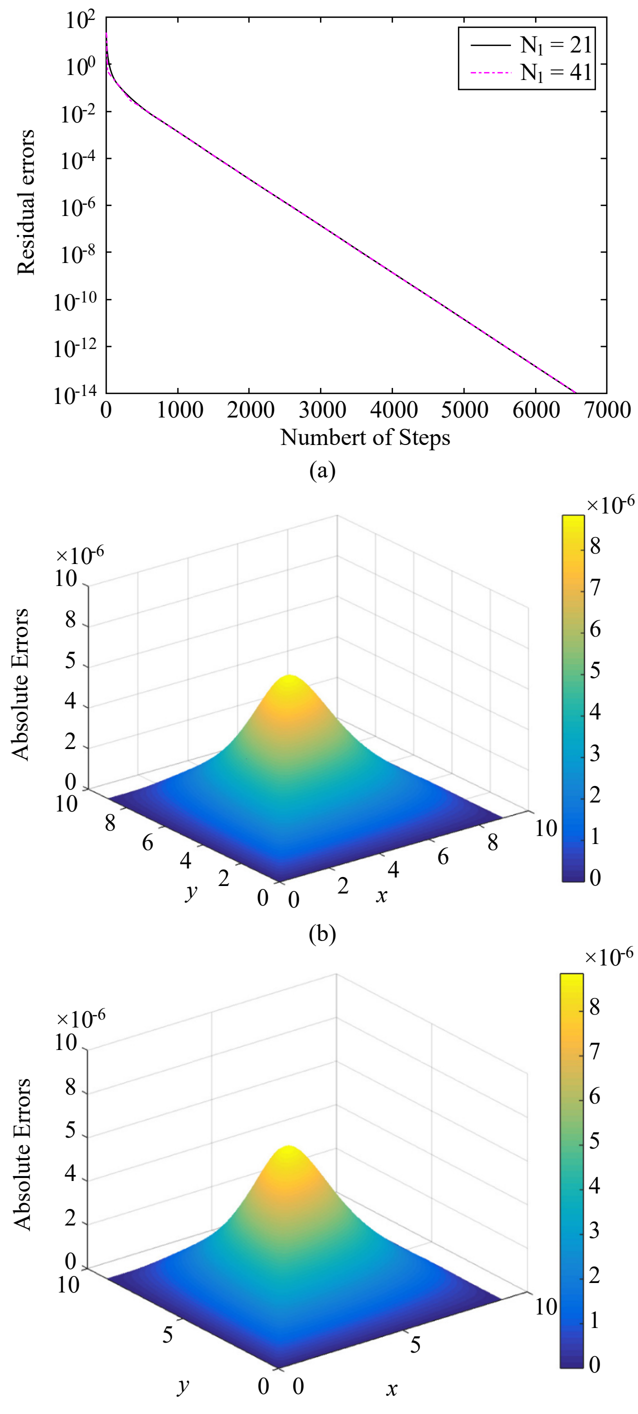

Fig. 16. For Example 6, plots of (a) the residual errors and the numerical errors with (b) N_1=21 and (c) 41

ACKNOWLEDGEMENTS

The first author greatly appreciates the financial support that was provided by the Ministry of Science and Technology, Taiwan, ROC under contract number MOST 108-2221-E-019-015.

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